Linear preservers of stable and Lorentzian polynomials and deformations of hyperbolicity cones

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Let $h$ be hyperbolic with hyperbolicity cone $\Lambda_+$. If $v \in \Lambda_+$, then the **directional derivative**

$$D_v h = v_1 \frac{\partial h}{\partial x_1} + \cdots + v_n \frac{\partial h}{\partial x_n}$$

is hyperbolic with hyperbolicity cone $\Lambda_+^{(1)} \supseteq \Lambda_+$. Hence we get a sequence of relaxations

$$\Lambda_+ \subseteq \Lambda_+^{(1)} \subseteq \Lambda_+^{(2)} \subseteq \cdots \subseteq \Lambda_+^{(d-1)}, \quad d = \deg h.$$  

The map $h \mapsto D_v h$ defines a linear operator which preserves hyperbolicity, under which hyperbolicity cones behave "nicely".

**Questions.** Are there other such preservers? Can we determine how they deform hyperbolicity cones?
Stable polynomials

- Let $K = \mathbb{R}$ or $\mathbb{C}$, and $\mathbf{x} = (x_1, \ldots, x_n)$ a tuple of variables.
- A polynomial $f \in K[\mathbf{x}]$ is stable if
  \[
  \text{Im}(z_j) > 0 \text{ for all } j \implies f(\mathbf{z}) \neq 0.
  \]
- We also consider the identically zero polynomial to be stable.
- $x_1 - 2 + 5i, -1 + x_1 + 5x_2 + 3x_3$ and $1 - 2x_1x_2$ are stable.
- A polynomial $f \in \mathbb{R}[x_1]$ is stable iff $f$ is real-rooted.
- Let
  \[
  h = x_0^d f(x_1/x_0, \ldots, x_n/x_0).
  \]
- **Lemma.** Let $f \in \mathbb{R}[\mathbf{x}]$. Then $f$ is stable iff $h$ is hyperbolic with respect to $(0, 1, 1, \ldots, 1)$, and $\Lambda_+(F) \supseteq \{0\} \times \mathbb{R}^{n}_{\geq 0}$.
- **Question.** Which linear operators preserve stability?
Brief history

- RH is equivalent to that
  \[
  \Xi(t) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s), \quad s = \frac{1}{2} + it,
  \]
  may be approximated, uniformly on compacts, by real-rooted polynomials.
- This motivated Hermite, Laguerre, Jensen, Pólya, Schur, De Bruijn, ... to study linear operators preserving real-rootedness.
- Lee and Yang (1952) used ideas and results of Pólya to prove their celebrated Lee-Yang theorem of multivariate stability of the partition function of the Ising model.
- Choe-Oxley-Sokal-Wagner, Gurvits, B., Borcea-B.-Ligget studied stability (preservers) from a combinatorial point of view.
- Marcus, Srivastava, Spielman used stability preservers in their work on Ramanujan graphs and the Kadison-Singer problem.
- Anari and Oveis Gharan used stability in computer science (e.g. for Traveling salesman problem).
Stability preservers

- **Example.** \( T = \frac{\partial}{\partial x_j} \) preserves stability (Gauss-Lucas theorem).

- For \( \kappa \in \mathbb{N}^n \), let

  \[
  K_\kappa[x] = \{ f \in K[x] : \deg_{x_j}(f) \leq \kappa_j \text{ for all } j \}.
  \]

- **The symbol** of a linear operator \( T : K_\kappa[x] \rightarrow K[x] \) is

  \[
  G_T = T((x + y)^\kappa) = \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} T(x^\alpha) y^{\kappa - \alpha},
  \]

  where \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( \binom{\kappa}{\alpha} = \binom{\kappa_1}{\alpha_1} \cdots \binom{\kappa_n}{\alpha_n} \).

- \[
  G_{\frac{\partial}{\partial x_1}} = \kappa_1(x_1 + y_1)^{\kappa_1 - 1}(x_2 + y_2)^{\kappa_2} \cdots (x_n + y_n)^{\kappa_n}
  \]
Pólya-Schur master theorem

- Pólya and Schur (1914) characterized diagonal operators $x^n \to \lambda_n x^n$ preserving real-rootedness.

- **Theorem** (Borcea, B., 2009). Let $T : \mathbb{C}_\kappa[x] \to \mathbb{C}[x]$ be a linear operator of rank $> 1$. Then $T$ preserves stability iff $G_T$ is stable.

- $G_{\frac{\partial}{\partial x_1}} = \kappa_1(x_1 + y_1)^{\kappa_1-1}(x_2 + y_2)^{\kappa_2} \cdots (x_n + y_n)^{\kappa_n}$ is stable.

- **Theorem** (Borcea, B., 2009). Let $T : \mathbb{R}_\kappa[x] \to \mathbb{R}[x]$ be a linear operator of rank $> 2$. Then $T$ preserves real stability iff
  - $G_T(x, y)$ is stable, or
  - $G_T(-x, y)$ is stable.

- **Example.** $T(f)(x) = f(-x)$ preserves real stability, and $G_T(-x, y) = (x + y)^\kappa$. 
Transcendental characterization

- The Laguerre–Pólya class, $\mathcal{L}–\mathcal{P}_n$, is the class of entire functions which are limits, uniformly on compacts, of real stable polynomials.

Example.

\[
e^{-x \cdot y} = e^{-(x_1 y_1 + \cdots + x_n y_n)} = \lim_{k \to \infty} \left( 1 - \frac{x_1 y_1}{k} \right)^k \cdots \left( 1 - \frac{x_n y_n}{k} \right)^k
\]

- The symbol of a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is

\[
\tilde{G}_T(x, y) = T(e^{-x \cdot y}) = \sum_{\alpha \in \mathbb{N}^n} T(x^\alpha) \frac{(-y)^\alpha}{\alpha!}.
\]

Theorem (Borcea, B., 2009). Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator of rank $> 2$. Then $T$ preserves real stability iff

- $\tilde{G}_T(x, y) \in \mathcal{L}–\mathcal{P}_{2n}$, or
- $\tilde{G}_T(-x, y) \in \mathcal{L}–\mathcal{P}_{2n}$.
Transcendental characterization

▶ Example. If $T = -1 + 2 \frac{\partial}{\partial x_1} + 3 \frac{\partial}{\partial x_2}$, then

$$\bar{G}_T(x, y) = T(e^{-x \cdot y}) = (-1 + 2y_1 + 3y_2)e^{-x \cdot y} \in \mathcal{L}-\mathcal{P}_4.$$ 

▶ Corollary. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator of rank $> 2$. Then $T$ preserves real-rootedness iff

▶ $\bar{G}_T(x, y) \in \mathcal{L}-\mathcal{P}_2$, or
▶ $\bar{G}_T(-x, y) \in \mathcal{L}-\mathcal{P}_2$.

▶ Question. Is there a transcendental Helton–Vinnikov theorem? If $f(x, y) \in \mathcal{L}-\mathcal{P}_2$, then

$$f(x, y) = \det(A + xB + yC),$$

where $A, B, C??$ and $\det??$
Discrete convexity

- A finite subset $M$ of $\mathbb{Z}^n$ is a **polymatroid** (or **M-convex**) if
  \[
  \alpha, \beta \in M \text{ and } \alpha_i > \beta_i \implies \exists j \text{ such that } \beta_j > \alpha_j \text{ and } \alpha - e_i + e_j \in M.
  \]

- The **support** of a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} a(\alpha)x^\alpha$ is
  \[
  \text{supp}(f) = \{ \alpha \in \mathbb{N}^n : a(\alpha) \neq 0 \}.
  \]

- **Theorem**[Choe-Oxley-Sokal-Wagner, 2004]. The support of a homogeneous and stable polynomial is a polymatroid.

- **Theorem**[B., 2007]. The set of bases of the Fano matroid $F_7$ is **not** the support of any stable polynomial.
Lorentzian (CLC, SLC) polynomials

- Consider a quadratic $f$ written as $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j$, where $A = (a_{ij})_{i,j=1}^{n}$ is a symmetric matrix with nonnegative entries.

- **Lemma.** $f$ is stable iff $A$ has exactly one positive eigenvalue.

- **Definition.** A homogeneous degree $d$ polynomial $f \in \mathbb{R}[x]$ with positive coefficients is **strictly Lorentzian** if for all $i_1, i_2, \ldots, i_{d-2}$, the quadratic

  \[
  \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f
  \]

  has Lorentz signature $(+, -, -, \ldots)$, i.e., exactly one positive eigenvalue and $n - 1$ negative eigenvalues.

- **Definition.** A polynomial is **Lorentzian** if it is the limit of strictly Lorentzian polynomials.
Lorentzian polynomials

- **Theorem** [B., Huh, 2019]. A homogeneous degree $d$ polynomial $f \in \mathbb{R}_{\geq 0}[x]$ is **Lorentzian** if
  - $\text{supp}(f)$ is a polymatroid, and
  - For all $i_1, i_2, \ldots, i_{d-2}$, the quadratic
    $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$
    is stable.

- **Example.** A polynomial $\sum_{k=M}^{N} a_k x^k y^{d-k}$, $a_k > 0$, is Lorentzian iff
  $\frac{a_k^2}{\left(\begin{array}{c} d \\ k \end{array}\right)^2} \geq \frac{a_{k-1}}{\left(\begin{array}{c} d \\ k-1 \end{array}\right)} \cdot \frac{a_{k+1}}{\left(\begin{array}{c} d \\ k+1 \end{array}\right)}$, $M < k < N$. 

Lorentzian polynomials

- **Example.** If $M$ is a polymatroid, then $\sum_{\alpha \in M} \frac{x^\alpha}{\alpha!}$, is Lorentzian.

- **Example.** If $r : 2^{[n]} \to \mathbb{N}$ is the rank function of a matroid and $0 < q \leq 1$, then

$$\sum_{A \subseteq [n]} q^{-r(A)} x_0^{n-|A|} \prod_{i \in A} x_i$$

is Lorentzian.

- **A matrix** $A \in \mathbb{R}^{n \times n}$ is an $M$-matrix if all off-diagonal entries are nonpositive and all principal minors are nonnegative.

- **Theorem** [B., Huh, 2019]. If $A$ is an $M$-matrix, then

$$\det(x_0 I + \text{diag}(x_1, \ldots, x_n) A) = \sum_{S \subseteq [n]} A(S)x_0^{n-|S|} \prod_{i \in S} x_i$$

is Lorentzian.
Lorentzian preservers

- **Question.** Which linear operators preserve the Lorentzian property?

- **Theorem (B., Huh, 2019+).** Let $T : \mathbb{R}_\kappa[x] \to \mathbb{R}[x]$. If $G_T$ is Lorentzian, then $T$ preserves the Lorentzian property.

- **Corollary.** If $G_T$ is homogeneous and stable, then $T$ preserves the Lorentzian property.

- **Example.** Let $\alpha \leq \beta \in \mathbb{N}^n$. The operator

$$T \left( \sum_{\gamma \in \mathbb{N}^n} a(\gamma)x^\gamma \right) = \sum_{\alpha \leq \gamma \leq \beta} a(\gamma)x^\gamma$$

preserves the Lorentzian property (but not stability).

- **Nonexample.** The operator $T : \mathbb{R}_\kappa[x] \to \mathbb{R}_\kappa[x]$ defined by $T(f)(x) = x^\kappa f(1/x_1, \ldots, 1/x_n)$ preserves real stability but **not** the Lorentzian property.
How do zeros move under preservers?

- The **symmetric additive convolution** of two univariate polynomials \( p, q \in \mathbb{R}_d[x] \) is

\[
(f \boxplus_d g)(x) = \frac{1}{d!} \sum_{k=0}^{d} f^{(k)}(0) \cdot g^{(d-k)}(x).
\]

- Let \( \lambda_{\text{max}}(f) \) be the largest zero of a real-rooted polynomial \( f \).
- For \( \alpha \geq 0 \), let \( U_\alpha = 1 - \alpha \frac{d}{dx} \).
- **Theorem** (Marcus, Srivastava, Spielman, 2015).

\[
\lambda_{\text{max}}(U_\alpha(f \boxplus_d g)) \leq \lambda_{\text{max}}(U_\alpha(f)) + \lambda_{\text{max}}(U_\alpha(g)) - d\alpha.
\]

- **Theorem** (Leake, Ryder, 2018).

\[
\lambda_{\text{max}}(f \boxplus_d g \boxplus_d h) + \lambda_{\text{max}}(h) \leq \lambda_{\text{max}}(g \boxplus_d h) + \lambda_{\text{max}}(f \boxplus_d h).
\]
What is the max-root of a real stable polynomial $f$?

**Definition.** The **hyperbolicity set** of $f$ is

$$C[f] = \{ x \in \mathbb{R}^n : f(x)f(y) > 0 \text{ for all } y \geq x \} = \Lambda_{++}(h) \cap \{ x_0 = 1 \},$$

where $h = x_0^d f(x_1/x_0, \ldots, x_n/x_0)$.

If $\lambda$ is the largest zero of $f(x_1)$, then $C[f] = \{ x \in \mathbb{R} : x > \lambda \}$.

For $f, g \in \mathbb{R}_\kappa[x]$, let

$$(f \boxplus_\kappa g)(x) = \frac{1}{\kappa!} \sum_{\alpha \leq \kappa} (\partial^\alpha f)(0) \cdot (\partial^{\kappa-\alpha} g)(x),$$

where

$$\partial^\alpha = \prod_{i=1}^n \left( \frac{\partial}{\partial x_i} \right)^{\alpha_i}.$$
Lemma. $\boxplus_\kappa$ preserves stability on $\mathbb{C}_\kappa[x] \times \mathbb{C}_\kappa[x]$.

Proof. Fix stable $g$ and consider $T(f) = f \boxplus_\kappa g$. Then

$$G_T = (x + y)^\kappa \boxplus_\kappa g = g(x + y)$$

is stable.

Lemma. If $f$ is stable and $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$, then $f - D_{\mathbf{w}} f$ is stable, where $D_{\mathbf{w}} = w_1 \frac{\partial}{\partial x_1} + \cdots + w_n \frac{\partial}{\partial x_n}$.

Proof. Let $T = 1 - D_{\mathbf{w}}$. Then

$$\bar{G}_T = T(e^{-x \cdot y}) = (1 + w_1 y_1 + \cdots + w_n y_n)e^{-x \cdot y} \in \mathcal{L}_-\mathcal{P}_n.$$
Theorem (B., Marcus, 2019+). If $f, g \in \mathbb{R}_\kappa[x]$, then

$$\mathcal{C}_w[P \boxplus_\kappa Q] \supseteq \mathcal{C}_w[P] + \mathcal{C}_w[Q] - \kappa w$$

$$= \{x + y - \kappa w : x \in \mathcal{C}_w[P] \text{ and } y \in \mathcal{C}_w[Q]\},$$

where $\kappa w = (\kappa_1 w_1, \ldots, \kappa_n w_n)$.

If $T : \mathbb{R}_\kappa[x] \to \mathbb{R}_\gamma[x]$ is a linear operator with symbol $G_T$, then

$$T(f(x + y)) = (x^\gamma f(y)) \boxplus_{\gamma \oplus \kappa} G_T(x, y),$$

where $\gamma \oplus \kappa = (\gamma_1, \ldots, \gamma_m, \kappa_1, \ldots, \kappa_n)$.

Unifies the proofs for the different convolutions considered by Marcus, Spielman and Srivastava (2015).

Conjecture (Leake, Ryder, 2018).

$$\mathcal{C}[P \boxplus_\kappa Q \boxplus_\kappa R] + \mathcal{C}[R] \supseteq \mathcal{C}[P \boxplus_\kappa R] + \mathcal{C}[Q \boxplus_\kappa R].$$