On volume of sublevel sets of polynomials

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☞ to appear in SIAM J. Appl. Alg. Geom.

Consider the compact set $\Omega_g \subset \mathbf{B} = (-1,1)^n$ defined by :

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$$\Omega_g \;:=\; \{\, \mathbf{x} \in \mathbb{R}^n : \quad g(\mathbf{x}) \leq \; 1 \,\}$$

for some nonnegative homogeneous polynomial $g \in \mathbb{R}[\mathbf{x}]$.

Compute

$$\rho \;=\; \mathrm{vol}(\Omega_g) \;=\; \int_{\Omega_g} d\mathbf{x}$$

... and possibly the moments

$$\rho_\alpha \;=\; \int_{\Omega_g} \mathbf{x}^\alpha \, d\mathbf{x}, \quad \alpha \in \mathbb{N}^n,$$

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Motivation

Let g be a nonnegative homogeneous polynomial of degree d and let $\Omega_g = \{\mathbf{x} : g(\mathbf{x}) \leq 1\}$.

Then :

$$\mathrm{vol}(\Omega_g) = \int_{\Omega_g} d\mathbf{x} = \frac{1}{\Gamma(1+n/d)} \int_{\Omega_g} \exp(-g(\mathbf{x}))\, d\mathbf{x}.$$

see e.g. Morozov & Shakirov, *Introduction to integral discriminants*, J. High Energy physics

I. ☞ $\int_{\Omega_g} \exp(-g(\mathbf{x}))\, d\mathbf{x}$, called an integral discriminant, is ubiquitous in statistical and quantum Physics.

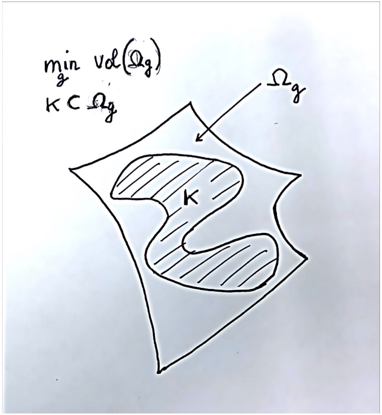
II. From the above formula it follows that

☞ vol(Ω_g) is a strictly CONVEX function of the coefficients of the polynomial g.

☞ very useful for solving Problem P :

P : Compute g nonnegative homogeneous of degree $2d$ such that $\mathbf{K} \subset \Omega_g$ and Ω_g has minimum volume.

where $\mathbf{K} \subset \mathbb{R}^n$ is a given compact (not necessarily convex) set.



Theorem

Problem **P** *is a CONVEX problem with a unique optimal solution* g^*

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Hence if one obtains good approximations of moments ρ_α, $|\alpha| = 2d$, then one may evaluate the function $g \mapsto \text{vol}(\Omega_g)$ and its gradient and Hessian because :

$$\frac{\partial\, \text{vol}(\Omega_g)}{\partial g_\alpha} = \frac{-\Gamma(1 + (n + |\alpha|)/2d)}{\Gamma(1 + n/2d)} \underbrace{\int_{\Omega_g} \mathbf{x}^\alpha \, d\mathbf{x}}_{\rho_\alpha}.$$

$$\frac{\partial^2\, \text{vol}(\Omega_g)}{\partial g_\alpha \partial g_\beta} = \frac{-\Gamma(1 + (n + |\alpha + \beta|)/2d)}{\Gamma(1 + n/2d)} \underbrace{\int_{\Omega_g} \mathbf{x}^{\alpha+\beta} \, d\mathbf{x}}_{\rho_{\alpha+\beta}}.$$

☞ one may approximate the optimal g^* by a standard gradient descent algorithm (or even Newton's method if desired).

Computing vol(Ω)

Let λ be the Lebesgue probability measure on a box $\mathbf{B} \supset \Omega_g$.

General approach

(i) Either approximate vol(Ω_g) by Monte-Carlo : λ-sample on $\mathbf{B}$ and COUNT points that fall into Ω_g. This provides a (random) estimate of vol(Ω_g).

(ii) Or SOLVE† (or approximate)

$$\mathrm{vol}(\Omega_g) \,=\, \max_{\phi}\,\{\,\phi(\Omega_g) : \phi \,\leq\, \lambda\,\}$$

where the “max" is over measures ϕ supported on Ω_g.

☞ † Henrion D., Lasserre J.B., Savorgnan C. (2009) Approximate volume and integration for basic semi-algebraic sets. SIAM Review 51, pp. 722–743

(i) ☞ simple method that can handle potentially relatively large dimensions. On the other hand, it only provides a (random) estimate of $\mathrm{vol}(\Omega_g)$.

(ii) ☞ $\phi^* := \lambda_{\Omega_g}$ is the unique optimal solution and applying the Moment-SOS hierarchy provides a monotone sequence of upper bounds $(\rho_d)_{d\in\mathbb{N}} \downarrow \mathrm{vol}(\Omega_g)$ as $d \to \infty$.

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Stokes' theorem

With vector field $X = x$ and $\alpha \in \mathbb{N}^n$ arbitrary :

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\begin{aligned}
0 &= \int_{\Omega_g} \mathrm{Div}(X \cdot \mathbf{x}^\alpha (1-g))\, dx = \int \mathrm{Div}(X \cdot \mathbf{x}^\alpha (1-g))\, d\phi^* \\
&= \int \underbrace{\mathbf{x}^\alpha \, [(n+|\alpha|)\,(1-g) - \langle \mathbf{x}, \nabla g \rangle]}_{p_\alpha(\mathbf{x})} \, d\phi^* \\
&= \int p_\alpha(\mathbf{x})\, d\phi^* \quad \text{a moment constraint on } \phi^*
\end{aligned}
$$

Hence one may equivalently solve :

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\mathrm{vol}(\Omega_g) = \max_{\phi \in \mathscr{M}(\Omega_g)} \{ \phi(\Omega_g) : \phi \leq \lambda; \quad \int p_\alpha \, d\phi = 0, \quad \alpha \in \mathbb{N}^n \}
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A simple transformation

Let the measure $\#\lambda$ on $\mathbb{R}$ be the pushforward of λ by the mapping $g : \mathbf{B} \to \mathbb{R}$. That is :

$$\#\lambda(B) \,=\, \lambda(g^{-1}(B)), \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

Let $I := g(\mathbf{B}) \subset \mathbb{R}$. Notice that :

All moments γ_k of $\#\lambda$ are obtained in closed form. That is :

$$\gamma_k \,:=\, \int_I z^k \, d\#\lambda(z) \,=\, \int_{\mathbf{B}} g(\mathbf{x})^k \, \lambda(d\mathbf{x}), \quad k = 0, 1, \ldots$$

Next, observe that

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Then :

$$\#\lambda([0,1]) \,=\, \int_{0\leq z\leq 1} \#\lambda(dz) \,=\, \lambda(g^{-1}([0,1])) \,=\, \lambda(\Omega_g)$$

That is, computing the n-dimensional volume ρ is computing the one-dimensional measure of the interval $[0,1]$ for the measure $\#\lambda$ on $\mathbb{R}$ ...

☞ Therefore Jasour et al.[†] et al. suggest to solve :

$$\rho \,=\, \max_{\phi} \{\phi([0,1]) : \, \phi \,\leq\, \#\lambda; \; \mathrm{supp}(\phi) = [0,1]\,\}$$

Indeed $\phi^* = 1_{[0,1]}(z)\, d\#\lambda(z)$ is the unique optimal solution.

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Hence

One has replaced computation of the n-dimensional Lebesgue-volume of Ω_g by computation of the 1-dimensional $\#\lambda$-volume of the interval $[0, 1]$

The value ρ can be approximated as closely as desired by solving appropriate SDP relaxations associated with the Moment-SOS hierarchy.

Problem :

☞ Convergence $(\rho_d)_{d\in\mathbb{N}} \downarrow \rho$ is typically VERY SLOW !

☞ One cannot use Stokes constraints because one does not know the density of $\#\lambda$.

The homogeneous case

Take home message

When g is homogeneous then one can do much better !

$$\text{Let} \quad \phi_j^* = \int_{[0,1]} z^j \, d\#\lambda(z), \quad j = 0, 1, \ldots$$

so that $\rho = \lambda(\Omega_g) = \phi_0^*$.

Suppose that g is NONNEGATIVE and HOMOGENEOUS of degree t. Then by Stokes' Theorem with vector field $X = \mathbf{x}$:

$$\begin{aligned} 0 &= \int_{\Omega_g} \left[n\,(1 - g(\mathbf{x})^j) + \langle \mathbf{x}, \nabla (1 - g(\mathbf{x})^j) \rangle \right] \, d\lambda(\mathbf{x}) \\ &= n\,\lambda(\Omega) - (n + jt) \int_{\Omega} g(\mathbf{x})^j \, d\lambda(\mathbf{x}) \\ &= n\,\lambda(\Omega) - (n + jt) \int_{g(\Omega)} z^j \, d\#\lambda(z) \\ &= n\,\phi_0^* - (n + jt)\,\phi_j^*, \quad j = 1, 2, \ldots \end{aligned}$$

Theorem

Let $(\phi^*_j)_{j\in\mathbb{N}}$ be the moments of ϕ^*. Then :

$$\phi^*_j \;=\; \frac{n}{n+j\,t}\,\phi^*_0,\quad j=1,2,\ldots$$

As a consequence the moment matrix $H_d(\phi^*)$ of ϕ^*, is just $\phi^*_0\,H^*_d$ with :

$$H^*_d \;=\; \left[\begin{array}{cccc} 1 & \frac{n}{n+t} & \cdots & \frac{n}{n+d\,t} \\ \frac{n}{n+t} & \cdots & \cdots & \frac{n}{n+(d+1)\,t} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{n}{n+d\,t} & \cdots & \cdots & \frac{n}{n+2d\,t} \end{array}\right]$$

☞ which is the moment matrix of the probability measure

$$d\gamma(x) \;=\; \frac{n}{t}\,x^{\frac{n}{t}-1}\,dx \quad \text{on } [0,1]$$

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But then :

$$\rho = \max_{\phi} \{ \phi(\mathbb{R}) : \phi \leq \#\lambda; \ \mathrm{supp}(\phi) = [0,1] \}$$

can be approximated as closely as desired by

$$\begin{aligned} \tau_d &= \max_{\theta} \{ \theta : \theta H_d^* \preceq H_d(\#\lambda) \} \\ &= \lambda_{\min}(H_d(\#\lambda), H_d^*) \end{aligned}$$

a GENERALIZED EIGENVALUE PROBLEM associated with two HANKEL moment matrices.

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Theorem

$\tau_d \downarrow \rho$ *as* $d \to \infty$.

To visualize & appreciate the simplicity of the approach, let $n = 2$ and $g = \|\mathbf{x}\|^2 = x_1^2 + x_2^2$, and $\mathbf{B} = [-1, 1]^2$, so that $\mathrm{vol}(\Omega_g) = \pi$. Then :

$$H_1^* = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}; \quad H_1(\#\lambda) = \begin{bmatrix} 1 & 2/3 \\ 2/3 & 28/45 \end{bmatrix}$$

This yields $4 \cdot \tau_1 \approx 3.20$ which is already a good upper bound on π whereas $4 \cdot \rho_1 = 4$.

$$H_2^* = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}; H_2(\#\lambda) = \begin{bmatrix} 1 & 2/3 & 2 \\ 2/3 & 28/45 & 2 \\ 28/45 & 24/35 & 2/9 + 8 \end{bmatrix}$$

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d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
ρ_d	12.19	11.075	9.163	8.878	8.499
τ_d	6.839	5.309	5.001	4.945	4.936

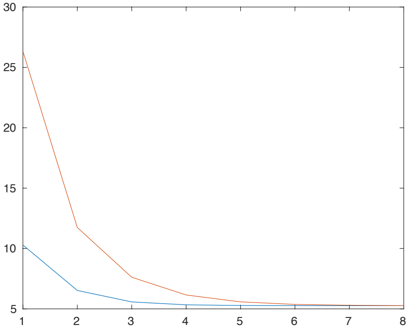
TABLE – $n = 4$, $\rho = 4.9348$; ρ_d versus τ_d

d	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	d
$2^n\tau_d$	15.04	7.97	5.569	4.639	4.272	4.133	4.
$\frac{(2^n\tau_d-\rho^*)}{\rho^*}$	270%	96%	37%	14%	5.26%	1.83%	0.

TABLE – $n = 8$, $\rho = 4.0587$; τ_d and relative error

Influence of the size of $\mathbf{B} = [-r, r]^n$

$n = 8$ and τ_d with $r = 1.3$ (red above) and $r = 1$ (blue below)



Other moments

Suppose that one has already computed (or approximated) ϕ_0^* (and so all moments ϕ_j^*) of $\#\lambda$.

To compute the moment $\int_{\Omega_g} \mathbf{x}^\alpha \, dx$:

- Replace the measure λ on $\mathbf{B}$ with the measure

$$d\lambda^\alpha := (1 - \mathbf{x}^\alpha) d\lambda \quad \text{on } \mathbf{B}, \text{ and}$$

- Use the pushforward $\#\lambda^\alpha$ of λ^α by the mapping g.
- Then again :

☞
$$\lambda^\alpha(\Omega_g) = \#\lambda^\alpha([0,1]) = \int_{\Omega_g} (1 - \mathbf{x}^\alpha) \, d\mathbf{x}$$
$$= \text{vol}(\Omega_g) - \int_{\Omega_g} \mathbf{x}^\alpha \, d\mathbf{x}$$

So again the moments of $\#\lambda^\alpha$ are obtained in closed form

$$\#\lambda^\alpha_j \;=\; \int z^j\, d\#\lambda^\alpha(z) \;=\; \int_{\mathbf{B}} g(\mathbf{x})^j\,(1-\mathbf{x}^\alpha)\,d\lambda(\mathbf{x}).$$

w.l.o.g. $\mathbf{B} = [-1,1]^n$. From Stokes' theorem for the integral :

$$\int_{\Omega_g} \mathrm{Div}(\,\mathbf{X}\cdot(1-g_j)\,(1-\mathbf{x}^\alpha)\,)\,d\mathbf{x},$$

the moments $(\phi^{\alpha *}_j)_{j\in\mathbb{N}}$ of the restriction of $\#\lambda^\alpha$ to $[0,1]$ satisfy :

$$\phi^{\alpha *}_j \;=\; \frac{n+|\alpha|}{n+|\alpha|+tj}\,\phi^{\alpha *}_0 + \frac{|\alpha|}{n+|\alpha|+tj}\,(\phi^*_j-\phi^*_0),\quad \forall j\in\mathbb{N}.$$

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Quasi-homogeneous polynomials

Let $u \in \mathbf{Q}^n$:

A polynomial $p \in \mathbb{R}[\mathbf{x}]$ is u-quasi-homogeneous if

$$p(\lambda^{u_1} x_1, \ldots, \lambda^{u_n} x_n) \;=\; \lambda\, p(\mathbf{x}), \quad \forall \lambda > 0,\ \mathbf{x} \in \mathbb{R}^n.$$

Euler's identity becomes :

$$\sum_{i=1}^{n} u_i \, \frac{\partial p(\mathbf{x})}{\partial x_i} \, x_i \;=\; p(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$$

Stokes' constraints also work for nonnegative such p :

Just use the vector field $X = (u_1\, x_1, \ldots, u_n\, x_n)$.

☞ and with $\mathbf{u} := \sum_i u_i$, one obtains :

$$\phi_j^* \;=\; \frac{\mathbf{u}}{\mathbf{u} + j}\, \phi_0^*, \quad j = 1, 2, \ldots$$

Digression

If one use $\tilde{g} = g^{1/m}$ with $m \in \mathbb{N}$, then $\tilde{g}$ is t/m-homogeneous (and not a polynomial anymore). In particular, with $s = \max_{\mathbf{x}}\{\,\tilde{g}(\mathbf{x}) : \mathbf{x} \in \mathbf{B}\}$

$$\#\lambda_k \,=\, \int_0^s z^k\, d\#\lambda(z) \,=\, \int_{\mathbf{B}} g(\mathbf{x})^{k/m}\, d\mathbf{x}, \quad k = 0, 1, \ldots$$

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However, suppose that one may obtain $(\#\gamma_k)_{k\in\mathbb{N}}$ (e.g. by cubature formula on $\mathbf{B}$ or by Monte-Carlo). Then again :

$$\begin{aligned} \tau_d &= \max_\theta \{\, \theta : \, \theta\, H_d^* \preceq H_d(\#\lambda)\, (= H_d(\gamma))\,\} \\ &= \lambda_{\min}(H_d(\gamma), H_d^*) \end{aligned}$$

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I. Extension to the non-homogeneous case

Again $\Omega_g = \{\mathbf{x} : 0 \leq g(\mathbf{x}) \leq 1\} \subset [-1, 1]^n$ but $g \in \mathbb{R}[\mathbf{x}]_t$ is not homogeneous.

Basic idea

• Write $g = \sum_{k=1}^{t} g_k$ with g_k being homogeneous of degree k.

• Use the pushforward measure $\#\lambda$ of λ w.r.t. the polynomial mapping

$$G : \mathbb{R}^n \to \mathbb{R}^t \quad \mathbf{x} \mapsto G(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ \dots \\ \dots \\ g_t(\mathbf{x}) \end{bmatrix}$$

$$I := G(\mathbf{B}) \subset \mathbb{R}^t.$$

• The moments of $\#\lambda$ are given by :

$$\begin{aligned}\gamma_\alpha &:= \int z_1^{\alpha_1} \cdots z_t^{\alpha_t} \, \#\lambda(d\mathbf{z}), \quad \alpha \in \mathbb{N}^t \\ &= \int_{\mathbf{B}} g_1(\mathbf{x})^{\alpha_1} \cdots g_t(\mathbf{x})^{\alpha_t} \, \lambda(d\mathbf{x}), \quad \alpha \in \mathbb{N}^t\end{aligned}$$

• $G(\Omega_g) = \{\mathbf{z} \in I : \, 0 \leq \sum_{k=1}^t z_k \leq 1\} =: \Theta.$

$$\rho = \max_\phi \{ \phi(\mathbb{R}^t) : \, \phi \leq \#\lambda; \, \mathrm{supp}(\phi) = \Theta \}$$

☞ One has replaced a Lebesgue-volume computation in $\mathbb{R}^n$ with a $\#\lambda$-volume computation in $\mathbb{R}^t$ ... Interesting if $t \ll n$.

Stokes' constraints

Recall that $g = \sum_{k=1}^{t} g_k$. For instance, with $t = 2$, Stokes' theorem yields :

$$0 = \int_{\Omega_g} \left[n\, g(1-g) {g_1}^i\, {g_2}^j + \langle \mathbf{x}, \nabla(g(1-g){g_1}^i {g_2}^j)\rangle \right] \lambda(d\mathbf{x}),$$

that is :

$$\begin{aligned} 0 &= n \int \Big[(n+i+2j)\, z_1^i z_2^j\, (1 - z_1 - z_2)(z_1 + z_2) \\ &\quad + (z_1 + 2z_2)\, z_1^i z_2^j\, (1 - 2z_1 - 2z_2) \Big]\, d\#\lambda \\ &= \int q_{ij}(\mathbf{z})\, d\#\lambda \end{aligned}$$

Hence with every couple $(i,j)\in\mathbb{N}^2$ one obtains a linear moment constraint $L_{\phi^*}(q_{ij})=0$, where :

$$\begin{aligned} q_{ij}(\mathbf{z}) &= (n+i+2j)\, z_1^i z_2^j\,(1-z_1-z_2)(z_1+z_2) \\ &\quad +(z_1+2z_2)\, z_1^i z_2^j\,(1-2z_1-2z_2) \end{aligned}$$

Therefore with $h(\mathbf{z})=(1-z_1-z_2)(z_1+z_2)$, one solves the hierarchy of semidefinite programs :

$$\tau_d = \max_{\phi}\{\,\phi(\mathbb{R}^t) : \quad \begin{array}{l} 0 \preceq \mathbf{M}_d(\phi) \preceq \mathbf{M}_d(\#\lambda) \\ \mathbf{M}_{d-1}(h\,\phi) \succeq 0 \\ L_\phi(q_{ij}) = 0, \quad \deg(q_{ij}) \leq 2d\} \end{array}$$

II. Multi-homogeneous constraints

In this case $\Omega = \{\mathbf{x} : \; g_j(\mathbf{x}) \leq 1, \quad j = 1, \ldots, t\} \subset [-1, 1]^n$, where each g_j is homogeneous of degree t_j. Again use the pushforward measure $\#\lambda$ of λ w.r.t. the polynomial mapping

$$G : \mathbb{R}^n \to \mathbb{R}^t \quad \mathbf{x} \mapsto G(\mathbf{x}) \; = \; \begin{bmatrix} g_1(\mathbf{x}) \\ \cdots \\ \cdots \\ g_t(\mathbf{x}) \end{bmatrix}$$

$$I := G(\mathbf{B}) \; \subset \; \mathbb{R}^t.$$

• The moments of $\#\lambda$ are given by :

$$\begin{aligned} \gamma_\alpha \quad &:= \quad \int z_1^{\alpha_1} \cdots z_t^{\alpha_t} \, \#\lambda(d\mathbf{z}), \quad \alpha \in \mathbb{N}^t \\ &= \quad \int_{\mathbf{B}} g_1(\mathbf{x})^{\alpha_1} \cdots g_t(\mathbf{x})^{\alpha_t} \, \lambda(d\mathbf{x}), \quad \alpha \in \mathbb{N}^t \end{aligned}$$

• $G(\Omega) = \{\mathbf{z} \in I : \; z_k \; \leq 1, \quad k = 1, \ldots, t\} \; =: \; \Theta$.

Again one may use Stokes' Theorem to provide additional linear constraints on the moments of ϕ^*

Thank You !