On volume of sublevel sets of polynomials

Jean B. Lasserre*

LAAS-CNRS and Institute of Mathematics, Toulouse, France

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Consider the compact set $\Omega_g \subset \mathcal{B} = (-1, 1)^n$ defined by:

$$
\Omega_g := \{ \mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 1 \}
$$

for some nonnegative homogeneous polynomial $g \in \mathbb{R}[\mathbf{x}]$.

Compute

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\rho = \text{vol}(\Omega_g) = \int_{\Omega_g} d\mathbf{x}
$$

... and possibly the moments

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\rho_\alpha = \int_{\Omega_g} \mathbf{x}^\alpha d\mathbf{x}, \quad \alpha \in \mathbb{N}^n,
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Motivation

Let $g$ be a nonnegative homogeneous polynomial of degree $d$ and let $\Omega_g = \{ x : g(x) \leq 1 \}$.

Then:

$$\text{vol}(\Omega_g) = \int_{\Omega_g} dx = \frac{1}{\Gamma(1+n/d)} \int_{\Omega_g} \exp(-g(x)) \, dx.$$

See e.g. Morozov & Shakirov, *Introduction to integral discriminants*, J. High Energy Physics

I. $\int_{\Omega_g} \exp(-g(x)) \, dx$, called an integral discriminant, is ubiquitous in statistical and quantum Physics.
II. From the above formula it follows that

\[ \text{vol}(\Omega_g) \] is a \textbf{strictly CONVEX} function of the \textbf{coefficients} of the polynomial \( g \).

very useful for solving Problem \( P \) :

\( P \): Compute \( g \) \textbf{nonnegative homogeneous} of degree \( 2d \) such that \( K \subset \Omega_g \) and \( \Omega_g \) has \textbf{minimum volume}.

where \( K \subset \mathbb{R}^n \) is a given compact (not necessarily convex) set.
Introduction

Theorem

Problem $P$ is a CONVEX problem with a unique optimal solution $g^*$
\( d = 2 \) (quadratic case): \( \Omega_g^* \) is the celebrated Löwner-John ellipsoid

However, given \( g \), computing \( \text{vol}(\Omega_g) \) is difficult!
$d = 2$ (quadratic case) : $\Omega_g^*$ is the celebrated Löwner-John ellipsoid

However, given $g$, computing $\text{vol}(\Omega_g)$ is difficult!
Hence if one obtains good approximations of moments $\rho_\alpha$, $|\alpha| = 2d$, then one may evaluate the function $g \mapsto \text{vol}(\Omega_g)$ and its gradient and Hessian because:

$$\frac{\partial \text{vol}(\Omega_g)}{\partial g_\alpha} = -\frac{\Gamma(1 + (n + |\alpha|)/2d)}{\Gamma(1 + n/2d)} \int_{\Omega_g} x^\alpha d\mathbf{x} \cdot \rho_\alpha.$$ 

$$\frac{\partial^2 \text{vol}(\Omega_g)}{\partial g_\alpha \partial g_\beta} = -\frac{\Gamma(1 + (n + |\alpha + \beta|)/2d)}{\Gamma(1 + n/2d)} \int_{\Omega_g} x^{\alpha+\beta} d\mathbf{x} \cdot \rho_{\alpha+\beta}.$$ 

Thus one may approximate the optimal $g^*$ by a standard gradient descent algorithm (or even Newton’s method if desired).
Let \( \lambda \) be the Lebesgue probability measure on a box \( B \supset \Omega_g \).

**General approach**

(i) Either approximate \( \text{vol}(\Omega_g) \) by Monte-Carlo: \( \lambda \)-sample on \( B \) and COUNT points that fall into \( \Omega_g \). This provides a (random) estimate of \( \text{vol}(\Omega_g) \).

(ii) Or SOLVE\(^\dagger\) (or approximate)

\[
\text{vol}(\Omega_g) = \max_{\phi} \{ \phi(\Omega_g) : \phi \leq \lambda \}
\]

where the “max" is over measures \( \phi \) supported on \( \Omega_g \).

(i) A simple method that can handle potentially relatively large dimensions. On the other hand, it only provides a (random) estimate of $\text{vol}(\Omega_g)$.

(ii) $\phi^* := \lambda_{\Omega_g}$ is the unique optimal solution and applying the Moment-SOS hierarchy provides a monotone sequence of upper bounds $(\rho_d)_{d \in \mathbb{N}} \downarrow \text{vol}(\Omega_g)$ as $d \to \infty$.

- Additional linear constraints coming from Stokes’ theorem applied to $\phi^*$ significantly accelerate the (otherwise slow) convergence.
- However, in view of the present status of SDP-solvers, this method is limited to problems of modest size.
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- However, in view of the present status of SDP-solvers, this method is limited to problems of modest size.
Stokes’ theorem

With vector field \( X = x \) and \( \alpha \in \mathbb{N}^n \) arbitrary:

\[
0 = \int_{\Omega_g} \text{Div}(X \cdot x^\alpha (1 - g)) \, dx = \int_{\Omega_g} \text{Div}(X \cdot x^\alpha (1 - g)) \, d\phi^*
\]

\[
= \int x^\alpha \left[ (n + |\alpha|) (1 - g) - \left< x, \nabla g \right> \right] d\phi^*
\]

\[
= \int p_\alpha(x) \, d\phi^* \quad \text{a moment constraint on } \phi^*
\]

Hence one may equivalently solve:

\[
\text{vol}(\Omega_g) = \max_{\phi \in \mathcal{M}(\Omega_g)} \left\{ \phi(\Omega_g) : \phi \leq \lambda; \quad \int p_\alpha \, d\phi = 0, \quad \alpha \in \mathbb{N}^n \right\}
\]

\(\Rightarrow\) The associated relaxations of the Moment-SOS hierarchy converge much faster!
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$$= \int \underbrace{\left( n + |\alpha| \right) (1 - g) - \langle x, \nabla g \rangle}_{p_\alpha(x)} \, d\phi^*$$

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The associated relaxations of the Moment-SOS hierarchy converge much faster!
Let the measure \( \#\lambda \) on \( \mathbb{R} \) be the pushforward of \( \lambda \) by the mapping \( g : \mathcal{B} \rightarrow \mathbb{R} \). That is:

\[
\#\lambda(B) = \lambda(g^{-1}(B)), \quad \forall B \in \mathcal{B}(\mathbb{R}).
\]

Let \( I := g(B) \subset \mathbb{R} \). Notice that:

All moments \( \gamma_k \) of \( \#\lambda \) are obtained in closed form. That is:

\[
\gamma_k := \int_I z^k d\#\lambda(z) = \int_B g(x)^k \lambda(dx), \quad k = 0, 1, \ldots
\]

Next, observe that

\[
g(\Omega_g) = \{ z \in I : 0 \leq z \leq 1 \}.
\]
A simple transformation

Let the measure \( \#\lambda \) on \( \mathbb{R} \) be the pushforward of \( \lambda \) by the mapping \( g : B \to \mathbb{R} \). That is:

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Then:

\[
\#\lambda([0, 1]) = \int_{0 \leq z \leq 1} \#\lambda(dz) = \lambda(g^{-1}([0, 1])) = \lambda(\Omega_g)
\]

That is, computing the \(n\)-dimensional volume \(\rho\) is computing the one-dimensional measure of the interval \([0, 1]\) for the measure \(\#\lambda\) on \(\mathbb{R}\) . . .

Therefore Jasour et al.\(^\dagger\) et al. suggest to solve:

\[
\rho = \max_{\phi} \{ \phi([0, 1]) : \phi \leq \#\lambda; \text{ supp}(\phi) = [0, 1] \}
\]

Indeed \(\phi^\ast = 1_{[0,1]}(z) d\#\lambda(z)\) is the unique optimal solution.

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Therefore Jasour et al.‡ et al. suggest to solve:

\[ \rho = \max_{\phi} \{ \phi([0, 1]) : \phi \leq \#\lambda; \text{supp}(\phi) = [0, 1] \} \]

Indeed \(\phi^* = 1_{[0,1]}(z) \ d\#\lambda(z)\) is the unique optimal solution.

Hence

One has replaced computation of the \( n \)-dimensional Lebesgue-volume of \( \Omega_g \) by computation of the 1-dimensional \#\( \lambda \)-volume of the interval \([0, 1]\).

The value \( \rho \) can be approximated as closely as desired by solving appropriate SDP relaxations associated with the Moment-SOS hierarchy.

Problem:

\( \downarrow \) Convergence \((\rho_d)_{d \in \mathbb{N}} \downarrow \rho \) is typically VERY SLOW!

\( \downarrow \) One cannot use Stokes constraints because one does not know the density of \#\( \lambda \).
Take home message

When $g$ is homogeneous then one can do much better!

Let $\phi_j^* = \int_{[0,1]} z^j d\#\lambda(z), \quad j = 0, 1, \ldots$

so that $\rho = \lambda(\Omega_g) = \phi_0^*$. 
Suppose that \( g \) is **NONNEGATIVE** and **HOMOGENEOUS** of degree \( t \). Then by Stokes’ Theorem with vector field \( X = x \):

\[
0 = \int_{\Omega} \left[ n \left( 1 - g(x)^j \right) + \langle x, \nabla (1 - g(x)^j) \rangle \right] d\lambda(x)
\]

\[
= n \lambda(\Omega) - (n + jt) \int_{\Omega} g(x)^j d\lambda(x)
\]

\[
= n \lambda(\Omega) - (n + jt) \int_{g(\Omega)} z^j d\#\lambda(z)
\]

\[
= n \phi_0^* - (n + jt) \phi_j^*, \quad j = 1, 2, \ldots
\]
**Theorem**

Let \((\phi_j^*)_{j \in \mathbb{N}}\) be the moments of \(\phi^*\). Then:

\[
\phi_j^* = \frac{n}{n + j \cdot t} \phi_0^*, \quad j = 1, 2, \ldots
\]

As a consequence the moment matrix \(H_d(\phi^*)\) of \(\phi^*\), is just \(\phi_0^* H_d^*\) with:

\[
H_d^* = \begin{bmatrix}
1 & \frac{n}{n+t} & \cdots & \frac{n}{n+d \cdot t} \\
\frac{n}{n+t} & \cdots & \cdots & \frac{n}{n+(d+1) \cdot t} \\
\cdots & \cdots & \cdots & \cdots \\
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\end{bmatrix}
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which is the moment matrix of the probability measure

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d\gamma(x) = \frac{n}{t} x^{n-1} \, dx \quad \text{on } [0, 1]
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But then:

$$\rho = \max_{\phi} \{ \phi(\mathbb{R}) : \phi \leq \#\lambda; \text{ supp}(\phi) = [0, 1] \}$$

can be approximated as closely as desired by

$$\tau_d = \max_{\theta} \{ \theta : \theta H^*_d \preceq H_d(\#\lambda) \}$$

$$= \lambda_{\min}(H_d(\#\lambda), H^*_d)$$

a **GENERALIZED EIGENVALUE PROBLEM** associated with two **HANKEL** moment matrices.

**Theorem**

$$\tau_d \downarrow \rho \text{ as } d \to \infty.$$
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Theorem

$$\tau_d \downarrow \rho \text{ as } d \to \infty.$$
To visualize & appreciate the simplicity of the approach, let $n = 2$ and $g = \|x\|^2 = x_1^2 + x_2^2$, and $B = [-1, 1]^2$, so that $\text{vol}(\Omega_g) = \pi$. Then:

$$H_1^* = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} , \quad H_1(#\lambda) = \begin{bmatrix} 1 & 2/3 \\ 2/3 & 28/45 \end{bmatrix}$$

This yields $4 \cdot \tau_1 \approx 3.20$ which is already a good upper bound on $\pi$ whereas $4 \cdot \rho_1 = 4$.

$$H_2^* = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} , \quad H_2(#\lambda) = \begin{bmatrix} 1 & 2/3 & 2 \\ 2/3 & 28/45 & 2 \\ 28/45 & 24/35 & 2/9 + 8/21 \end{bmatrix}$$

This yields $4 \cdot \tau_2 \approx 3.1440$ while $4 \cdot \rho_2 = 3.8928$. Hence $4 \cdot \tau_2$ already provides a very good upper bound on $\pi$ with only moments of order 4.
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<table>
<thead>
<tr>
<th>$d$</th>
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<th>$d = 3$</th>
<th>$d = 4$</th>
<th>$d = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_d$</td>
<td>12.19</td>
<td>11.075</td>
<td>9.163</td>
<td>8.878</td>
<td>8.499</td>
</tr>
<tr>
<td>$\tau_d$</td>
<td>6.839</td>
<td>5.309</td>
<td>5.001</td>
<td>4.945</td>
<td>4.936</td>
</tr>
</tbody>
</table>

**Table** – $n = 4$, $\rho = 4.9348$; $\rho_d$ versus $\tau_d$

<table>
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<tr>
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<th>$d = 3$</th>
<th>$d = 4$</th>
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<th>$d = 6$</th>
<th>$d = 7$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^n \tau_d$</td>
<td>15.04</td>
<td>7.97</td>
<td>5.569</td>
<td>4.639</td>
<td>4.272</td>
<td>4.133</td>
<td>4.0587</td>
</tr>
<tr>
<td>$\frac{(2^n \tau_d - \rho^<em>)}{\rho^</em>}$</td>
<td>270%</td>
<td>96%</td>
<td>37%</td>
<td>14%</td>
<td>5.26%</td>
<td>1.83%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

**Table** – $n = 8$, $\rho = 4.0587$; $\tau_d$ and relative error
Influence of the size of $\mathbf{B} = [-r, r]^n$

$n = 8$ and $\tau_d$ with $r = 1.3$ (red above) and $r = 1$ (blue below)
Suppose that one has already computed (or approximated) $\phi_0^*$ (and so all moments $\phi_j^*$) of $\#\lambda$.

To compute the moment $\int_{\Omega} x^\alpha \, dx$:

- Replace the measure $\lambda$ on $B$ with the measure

$$
\lambda^\alpha := (1 - x^\alpha) \, d\lambda \quad \text{on} \ B, \ \text{and}
$$

- Use the pushforward $\#\lambda^\alpha$ of $\lambda^\alpha$ by the mapping $g$.

- Then again:

$$
\lambda^\alpha(\Omega_g) = \#\lambda^\alpha([0,1]) = \int_{\Omega_g} (1 - x^\alpha) \, dx
$$

$$
= \vol(\Omega_g) - \int_{\Omega_g} x^\alpha \, dx
$$
So again the moments of $\#\lambda^\alpha$ are obtained in closed form

$$\#\lambda_j^\alpha = \int z^j \, d\#\lambda^\alpha(z) = \int_B g(x)^j (1 - x^\alpha) \, d\lambda(x).$$

w.l.o.g. $B = [-1, 1]^n$. From Stokes’ theorem for the integral:

$$\int_{\Omega_g} \text{Div}(X \cdot (1 - g_j)(1 - x^\alpha)) \, dx,$$

the moments $(\phi^\alpha_j)_{j \in \mathbb{N}}$ of the restriction of $\#\lambda^\alpha$ to $[0, 1]$ satisfy:

$$\phi^\alpha_j = \frac{n + |\alpha|}{n + |\alpha| + tj} \phi^\alpha_0 + \frac{|\alpha|}{n + |\alpha| + tj} (\phi^*_j - \phi^*_0), \quad \forall j \in \mathbb{N}.$$  

Again $\phi^\alpha_0$ determines all $\phi^\alpha_j$!
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Introduction

Quasi-homogeneous polynomials

Let $u \in \mathbb{Q}^n$:

A polynomial $p \in \mathbb{R}[x]$ is $u$-quasi-homogeneous if

$$p(\lambda^{u_1} x_1, \ldots, \lambda^{u_n} x_n) = \lambda p(x), \quad \forall \lambda > 0, \ x \in \mathbb{R}^n.$$ 

Euler’s identity becomes:

$$\sum_{i=1}^{n} u_i \frac{\partial p(x)}{\partial x_i} x_i = p(x), \quad \forall x \in \mathbb{R}^n$$

Stokes’ constraints also work for nonnegative such $p$:

Just use the vector field $X = (u_1 x_1, \ldots, u_n x_n)$.

And with $u := \sum_i u_i$, one obtains:

$$\phi^*_j = \frac{u}{u+j} \phi^*_0, \quad j = 1, 2, \ldots$$
If one use $\tilde{g} = g^{1/m}$ with $m \in \mathbb{N}$, then $\tilde{g}$ is $t/m$-homogeneous (and not a polynomial anymore). In particular, with $s = \max_x \{ \tilde{g}(x) : x \in B \}$

$$\#\lambda_k = \int_0^s z^k d\#\lambda(z) = \int_B g(x)^{k/m} \, dx, \quad k = 0, 1, \ldots$$

are not provided in closed-form anymore.

However, suppose that one may obtain $(\#\gamma_k)_{k \in \mathbb{N}}$ (e.g. by cubature formula on $B$ or by Monte-Carlo). Then again:

$$\tau_d = \max_{\theta} \{ \theta : \theta H_d^* \preceq H_d(\#\lambda) (= H_d(\gamma)) \}$$

$$= \lambda_{\min}(H_d(\gamma), H_d^*)$$

but the convergence is even faster ... !
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I. Extension to the non-homogeneous case

Again $\Omega_g = \{x : 0 \leq g(x) \leq 1 \} \subset [-1, 1]^n$ but $g \in \mathbb{R}[x]_t$ is not homogeneous.

---

**Basic idea**

- Write $g = \sum_{k=1}^{t} g_k$ with $g_k$ being homogeneous of degree $k$.
- Use the pushforward measure $\#\lambda$ of $\lambda$ w.r.t. the polynomial mapping $G : \mathbb{R}^n \rightarrow \mathbb{R}^t$, $x \mapsto G(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_t(x) \end{bmatrix}$.
\[ I := G(B) \subset \mathbb{R}^t. \]

- The moments of \( \#\lambda \) are given by:

\[
\gamma_\alpha := \int_{I} z_1^{\alpha_1} \cdots z_t^{\alpha_t} \#\lambda(dz), \quad \alpha \in \mathbb{N}^t
\]

\[
= \int_B g_1(x)^{\alpha_1} \cdots g_t(x)^{\alpha_t} \lambda(dx), \quad \alpha \in \mathbb{N}^t
\]

- \( G(\Omega_g) = \{ z \in I : 0 \leq \sum_{k=1}^{t} z_k \leq 1 \} =: \Theta. \)

\[
\rho = \max_{\phi} \{ \phi(\mathbb{R}^t) : \phi \leq \#\lambda; \text{supp}(\phi) = \Theta \}
\]

One has replaced a Lebesgue-volume computation in \( \mathbb{R}^n \)
with a \( \#\lambda \)-volume computation in \( \mathbb{R}^t \) ... Interesting if \( t \ll n. \)
Recall that $g = \sum_{k=1}^{t} g_k$. For instance, with $t = 2$, Stokes’ theorem yields:

$$0 = \int_{\Omega_g} [n g (1 - g) g_1^i g_2^j + \langle x, \nabla (g (1 - g) g_1^i g_2^j) \rangle] \lambda(dx),$$

that is:

$$0 = n \int \left[ (n + i + 2j) z_1^i z_2^j (1 - z_1 - z_2)(z_1 + z_2) \\
+ (z_1 + 2z_2) z_1^i z_2^j (1 - 2z_1 - 2z_2) \right] d\#\lambda \n\n= \int q_{ij}(z) d\#\lambda.$$
Hence with every couple \((i,j) \in \mathbb{N}^2\) one obtains a linear moment constraint \(L_{\phi^*}(q_{ij}) = 0\), where:

\[
q_{ij}(z) = (n + i + 2j) z_1^i z_2^j (1 - z_1 - z_2)(z_1 + z_2) + (z_1 + 2z_2) z_1^i z_2^j (1 - 2z_1 - 2z_2)
\]

Therefore with \(h(z) = (1 - z_1 - z_2)(z_1 + z_2)\), one solves the hierarchy of semidefinite programs:

\[
\tau_d = \max_{\phi} \{ \phi(\mathbb{R}^t) : 0 \preceq M_d(\phi) \preceq M_d(\#\lambda) \\
M_{d-1}(h \phi) \succeq 0 \\
L_{\phi}(q_{ij}) = 0, \quad \deg(q_{ij}) \leq 2d \}
\]
II. Multi-homogeneous constraints

In this case \( \Omega = \{ x : g_j(x) \leq 1, \ j = 1, \ldots, t \} \subset [-1, 1]^n \), where each \( g_j \) is homogeneous of degree \( t_j \). Again use the pushforward measure \( \#\lambda \) of \( \lambda \) w.r.t. the polynomial mapping

\[
G : \mathbb{R}^n \to \mathbb{R}^t \quad x \mapsto G(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_t(x) \end{bmatrix}
\]

\[
I := G(B) \subset \mathbb{R}^t.
\]

- The moments of \( \#\lambda \) are given by:

\[
\gamma_\alpha := \int \prod_{j=1}^t z_j^{\alpha_j} \#\lambda(dz), \quad \alpha \in \mathbb{N}^t
\]

\[
= \int_{B} g_1(x)^{\alpha_1} \cdots g_t(x)^{\alpha_t} \lambda(dx), \quad \alpha \in \mathbb{N}^t
\]

- \( G(\Omega) = \{ z \in I : z_k \leq 1, \ k = 1, \ldots, t \} =: \Theta \).
Again one may use Stokes’ Theorem to provide additional linear constraints on the moments of $\phi^*$
Thank You!