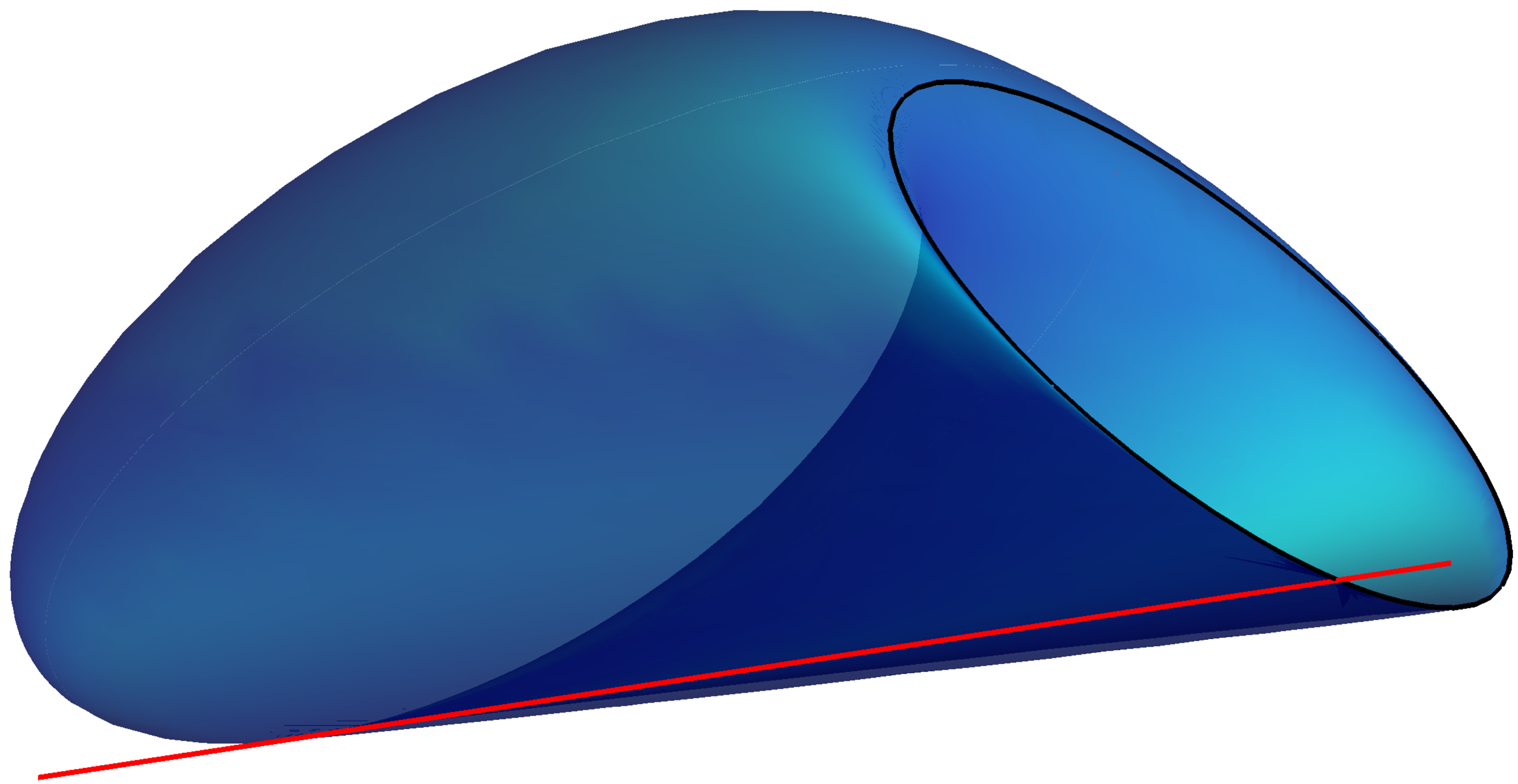
KIPPENHAHN'S THEOREM IN HIGHER DIMENSIONS

Daniel Plaumann
(TU Dortmund)

Rainer Sinn
(FU Berlin)

Stephan Weis
(University of Coimbra)



Simons Workshop on

Hyperbolic polynomials and hyperbolic programming

Berkeley, 3 May, 2019

The numerical range

Let A be a complex $d \times d$-matrix.
The **numerical range** of A is the set

$$W(A) = \left\{ \overline{x^T} A x \;\middle|\; x \in \mathbb{C}^d \text{ with } \|x\| = 1 \right\} \subset \mathbb{C}$$

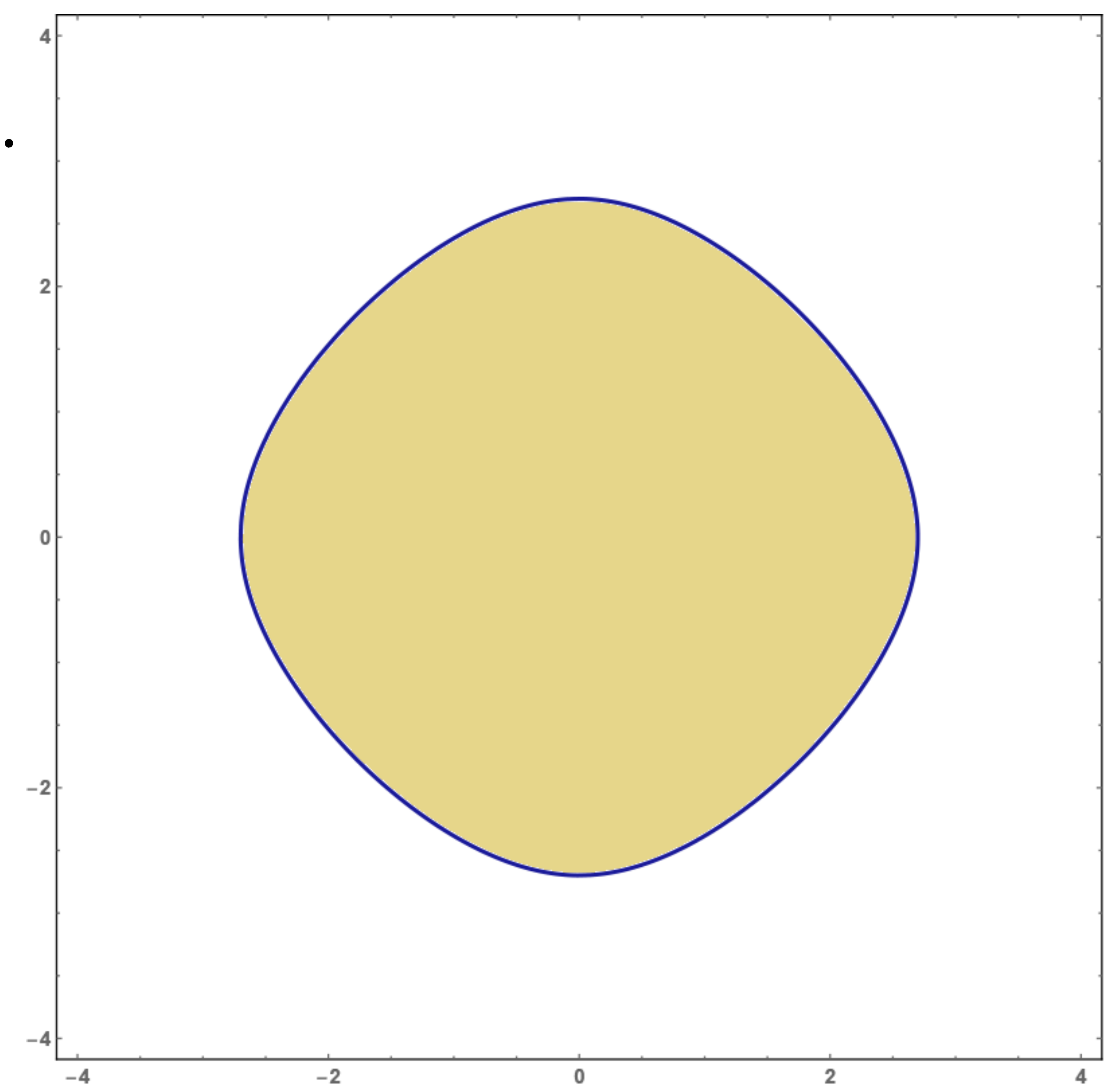
Toeplitz-Hausdorff Theorem (1919).
The set $W(A)$ is a convex subset of $\mathbb{C} = \mathbb{R}^2$.

Examples. (1) A is Hermitian if and only if $W(A)$ is a real line segment.

(2) If A is normal, then $W(A)$ is the convex hull of the eigenvalues of A.

(3) Running example:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$



Hyperbolic curves

Let A be a complex $d \times d$-matrix.

Define Hermitian matrices

$$\mathrm{Re}(A) = \frac{1}{2}\left(A + \overline{A}^T\right) \qquad \text{and} \qquad \mathrm{Im}(A) = \frac{1}{2i}\left(A - \overline{A}^T\right)$$

$$\Longrightarrow A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Re}(A) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 2 & 0 & \frac{3}{2} & 0 \end{pmatrix} \qquad \mathrm{Im}(A) = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 2i \\ \frac{i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{3i}{2} \\ -2i & 0 & \frac{3i}{2} & 0 \end{pmatrix}$$

Hyperbolic curves

Let A be a complex $d \times d$-matrix.

Define Hermitian matrices

$$\mathrm{Re}(A) = \frac{1}{2}\left(A + \overline{A}^T\right) \qquad \text{and} \qquad \mathrm{Im}(A) = \frac{1}{2i}\left(A - \overline{A}^T\right)$$

$$\Longrightarrow A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$$

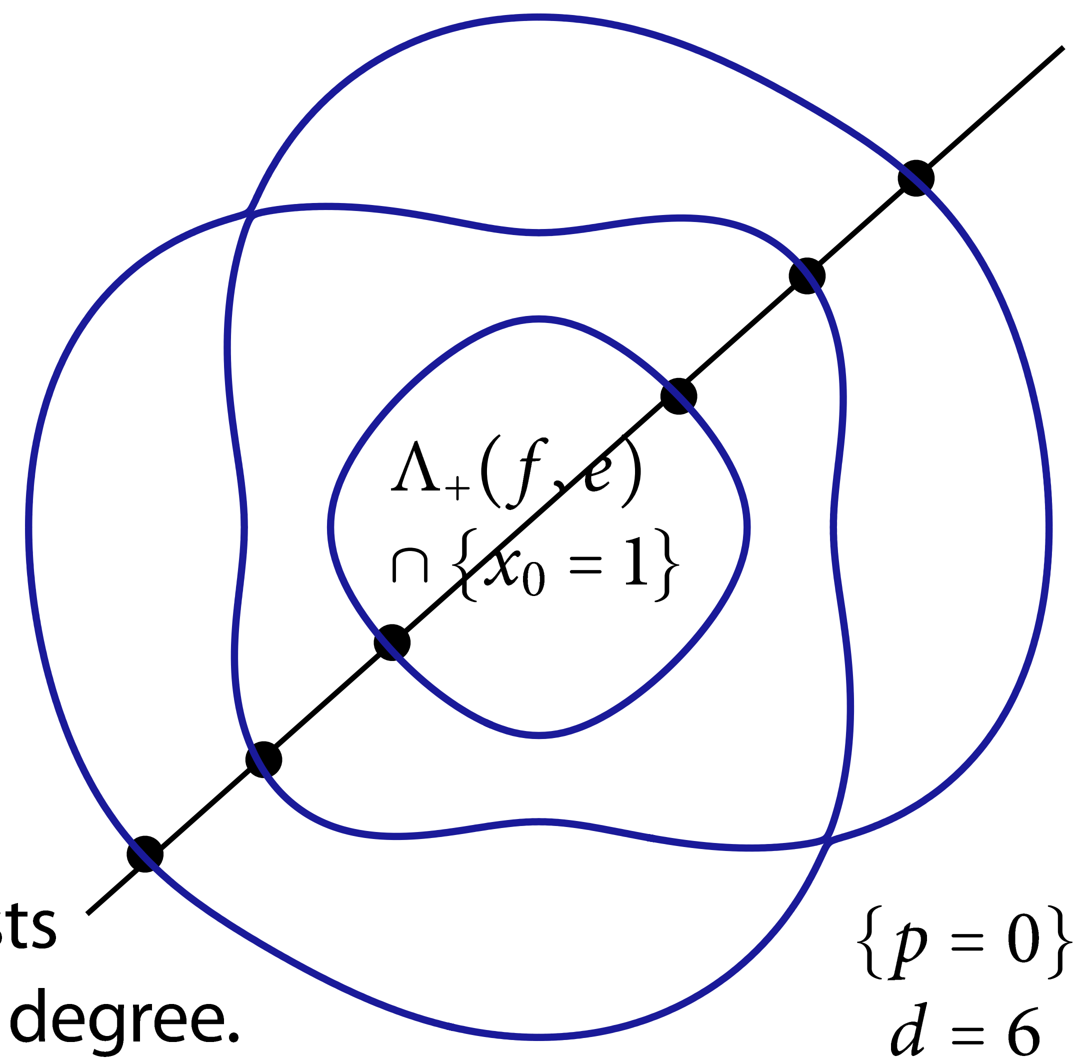
The polynomial

$$p = \det\left(x_0 I_d \; + \; x_1 \mathrm{Re}(A) + \; x_2 \mathrm{Im}(A)\right).$$

is **hyperbolic** with respect to the point $e = (1, 0, 0)$, i.e. all roots of

$$p(t, a_1, a_2)$$

are real for all $(a_1, a_2) \in \mathbb{R}^2$.



In the real projective plane, $\{p = 0\}$ consists of **nested ovals**, if p is irreducible of even degree.

Kippenhahn's Theorem

Let A be a complex $d \times d$ matrix and let

$$p = \det\left(x_0 I_d + x_1 \mathrm{Re}(A) + x_2 \mathrm{Im}(A)\right)$$

with spectrahedron

$$S(A) = \left\{(a_1, a_2) \in \mathbb{R}^2 \,\middle|\, I_d + a_1 \mathrm{Re}(A) + a_2 \mathrm{Im}(A) \succeq 0\right\}$$

Theorem. *(Kippenhahn 1951)*
The numerical range $W(A)$ is the convex dual

$$S(A)^\circ = \left\{(u_1, u_2) \in \mathbb{R}^2 \,\middle|\, \langle u, a\rangle \geqslant -1 \text{ for all } a \in S(A)\right\}$$

of $S(A)$. It is the convex hull of the points (u_1, u_2) for which $[1, u_1, u_2]$ lies on the ***dual curve*** *of $V = \{p = 0\}$.*

The dual curve V^* is the closure of the set of points $(1, u_1, u_2)$ for which the line

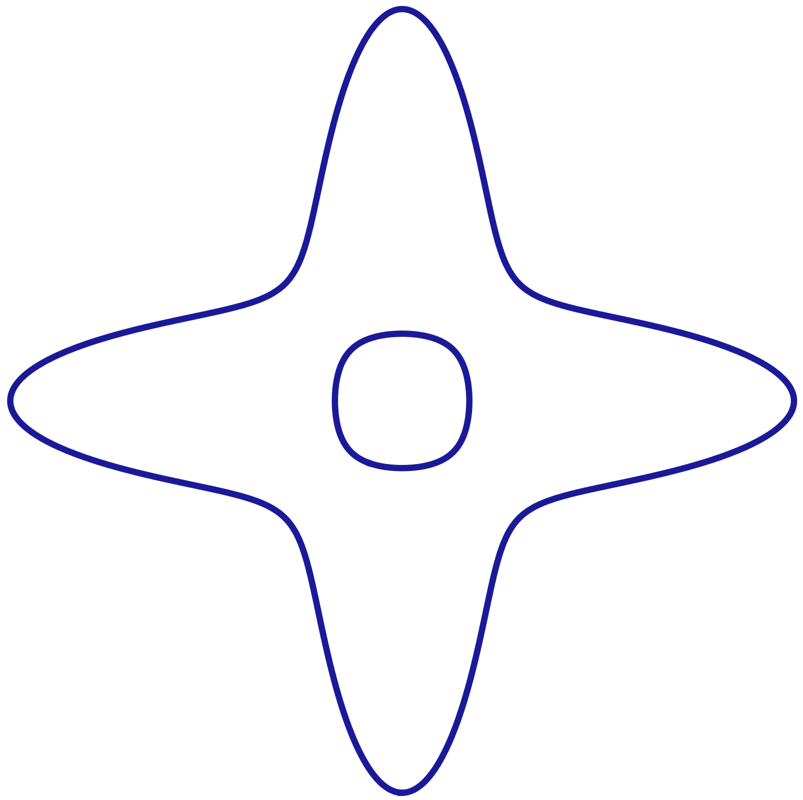
$$x_0 + u_1 x_1 + u_2 x_2 = 0$$

is tangent to V (at some regular point).

Hyperbolic curves

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Re}(A) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 2 & 0 & \frac{3}{2} & 0 \end{pmatrix} \qquad \mathrm{Im}(A) = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 2i \\ \frac{i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{3i}{2} \\ -2i & 0 & \frac{3i}{2} & 0 \end{pmatrix}$$

$$\begin{aligned} p &= \det\left(x_0 I_4 + x_1 \mathrm{Re}(A) + x_2 \mathrm{Im}(A)\right) \\ &= \frac{1}{16}\left(25x_1^4 + 25x_2^4 + 434x_1^2x_2^2 - 120x_0^2x_1^2 - 120x_0^2x_2^2 + 16x_0^4\right) \end{aligned}$$

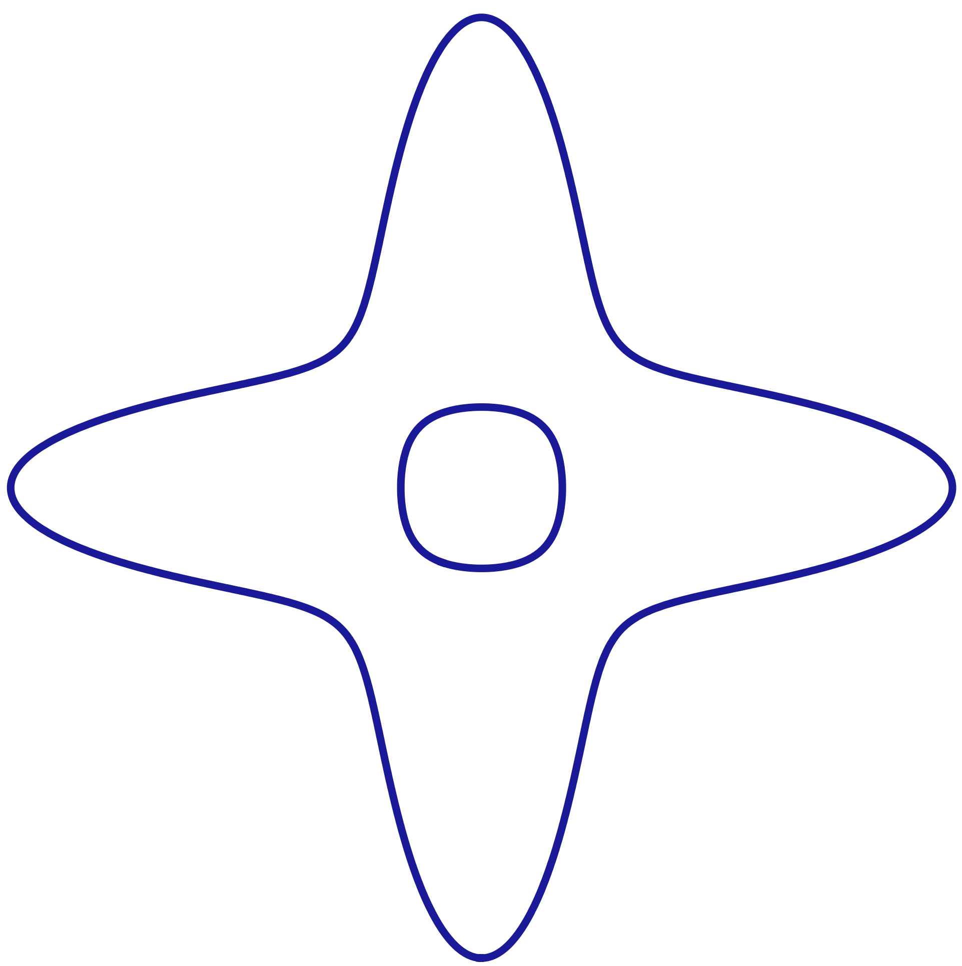


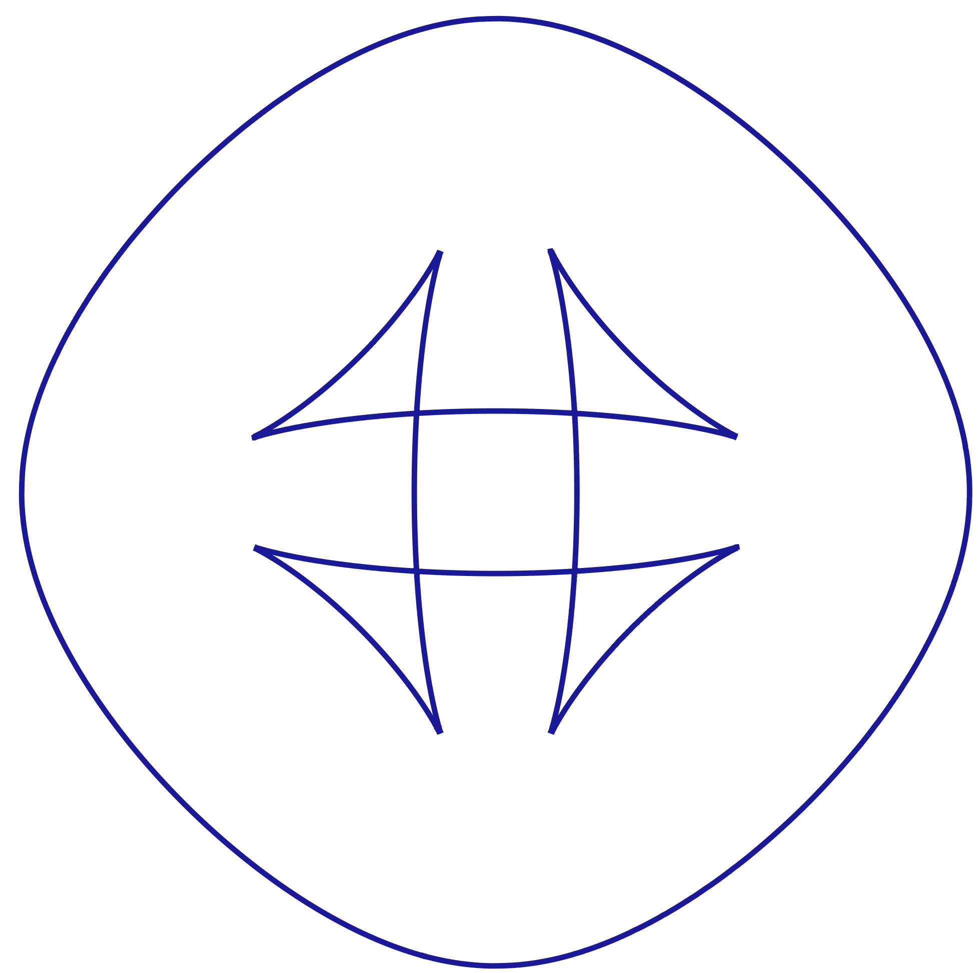
Hyperbolic curves

$$p = \frac{1}{16}\left(25x_1^4 + 25x_2^4 + 434x_1^2x_2^2 - 120x_0^2x_1^2 - 120x_0^2x_2^2 + 16x_0^4\right)$$

Dual curve is given by

$$\begin{aligned}
&250000u_1^{12} + 4380000u_1^{10}u_2^2 - 5475000u_0^2u_1^{10} + 1446000u_1^8u_2^4 - 68559000u_0^2u_1^8u_2^2 + 47610625u_0^4u_1^8 + 8787776u_1^6u_2^6 \\
&+ 179739600u_0^2u_1^6u_2^4 + 429249700u_0^4u_1^6u_2^2 - 209547000u_0^6u_1^6 + 1446000u_1^4u_2^8 + 179739600u_0^2u_1^4u_2^6 - 1058169786u_0^4u_1^4u_2^4 \\
&- 1493997480u_0^6u_1^4u_2^2 + 476341350u_0^8u_1^4 + 4380000u_1^2u_2^{10} - 68559000u_0^2u_1^2u_2^8 + 429249700u_0^4u_1^2u_2^6 - 1493997480u_0^6u_1^2u_2^4 \\
&+ 2442311100u_0^8u_1^2u_2^2 - 476982000u_0^{10}u_1^2 + 250000u_2^{12} - 5475000u_0^2u_2^{10} + 47610625u_0^4u_2^8 - 209547000u_0^6u_2^6 + 476341350u_0^8u_2^4 \\
&- 476982000u_0^{10}u_2^2 + 82355625u_0^{12} \quad = \quad 0
\end{aligned}$$





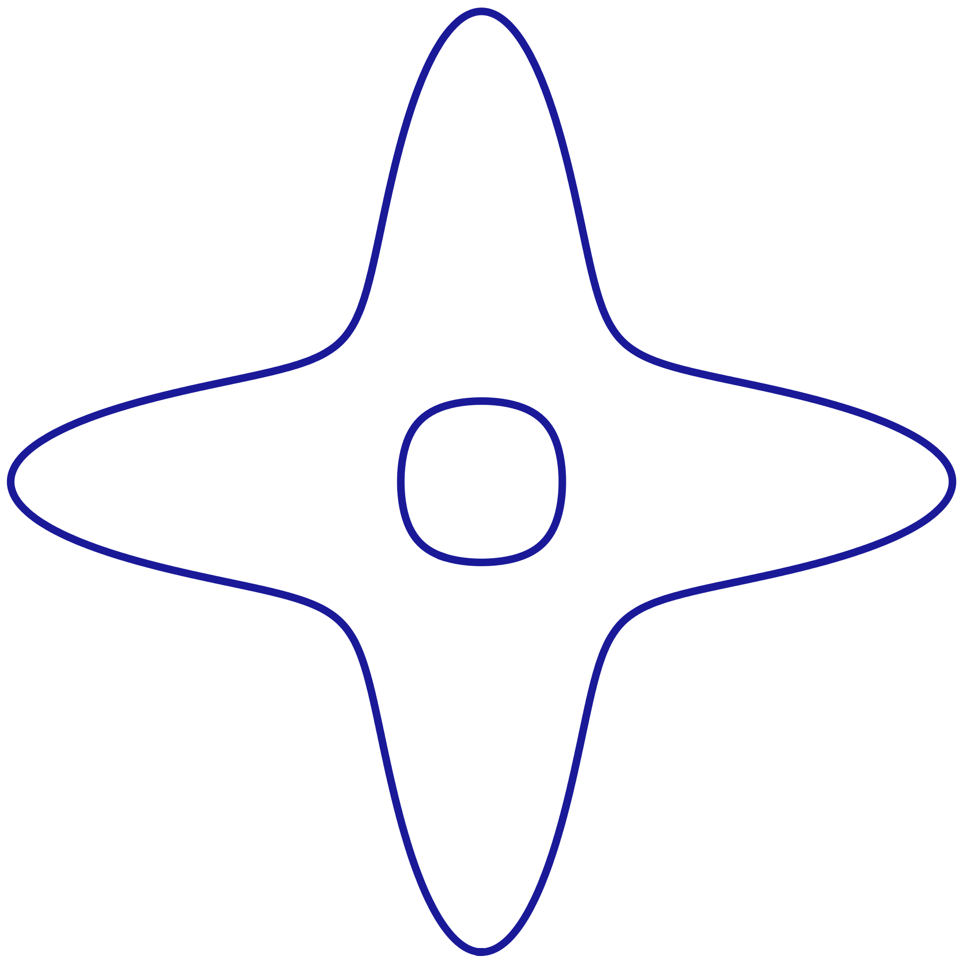
Duality for plane curves

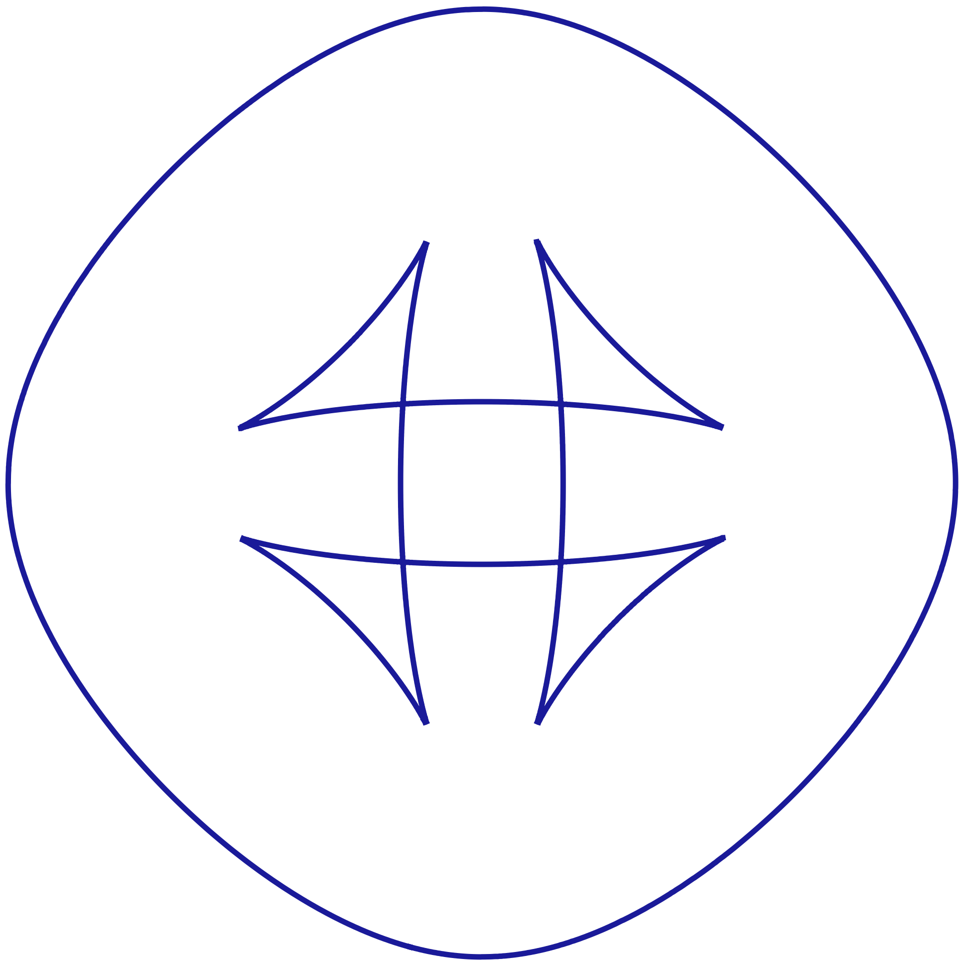
Let $V = \{p = 0\}$ be a plane curve of degree d.

If V is smooth, the dual curve V^* is irreducible of degree $d(d-1)$.

If V is generic (and smooth), then V^* has two types of singularities:

- The **bitangent lines** of V correspond to **nodes** of V^*.
- The **inflection lines** of V correspond to **cusps** of V^*.





Duality for plane curves

Let $V = \{p = 0\}$ be a plane curve of degree d.

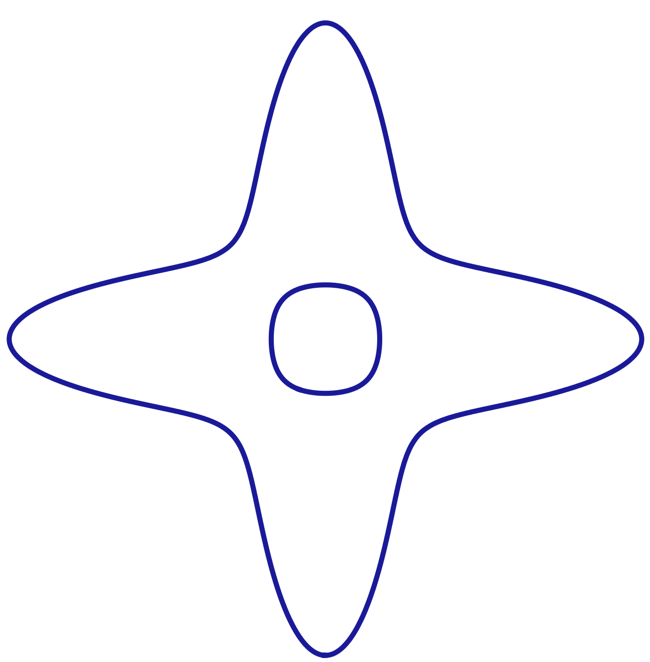
If V is smooth, the dual curve V^* is irreducible of degree $d(d-1)$.

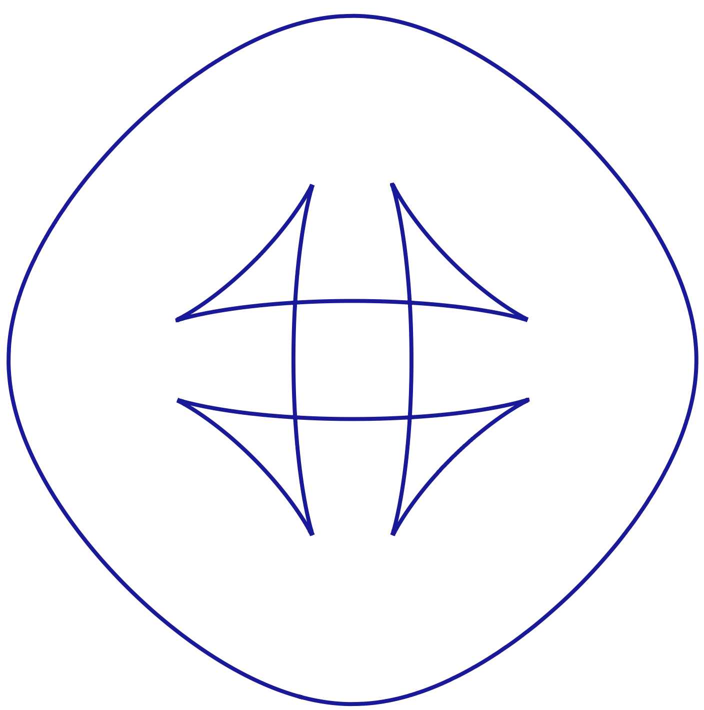
If V is generic (and smooth), then V^* has two types of singularities:

- The **bitangent lines** of V correspond to **nodes** of V^*.
- The **inflection lines** of V correspond to **cusps** of V^*.

For generic V, numbers over $\mathbb{C}$ are given by the **Plücker formulas**:

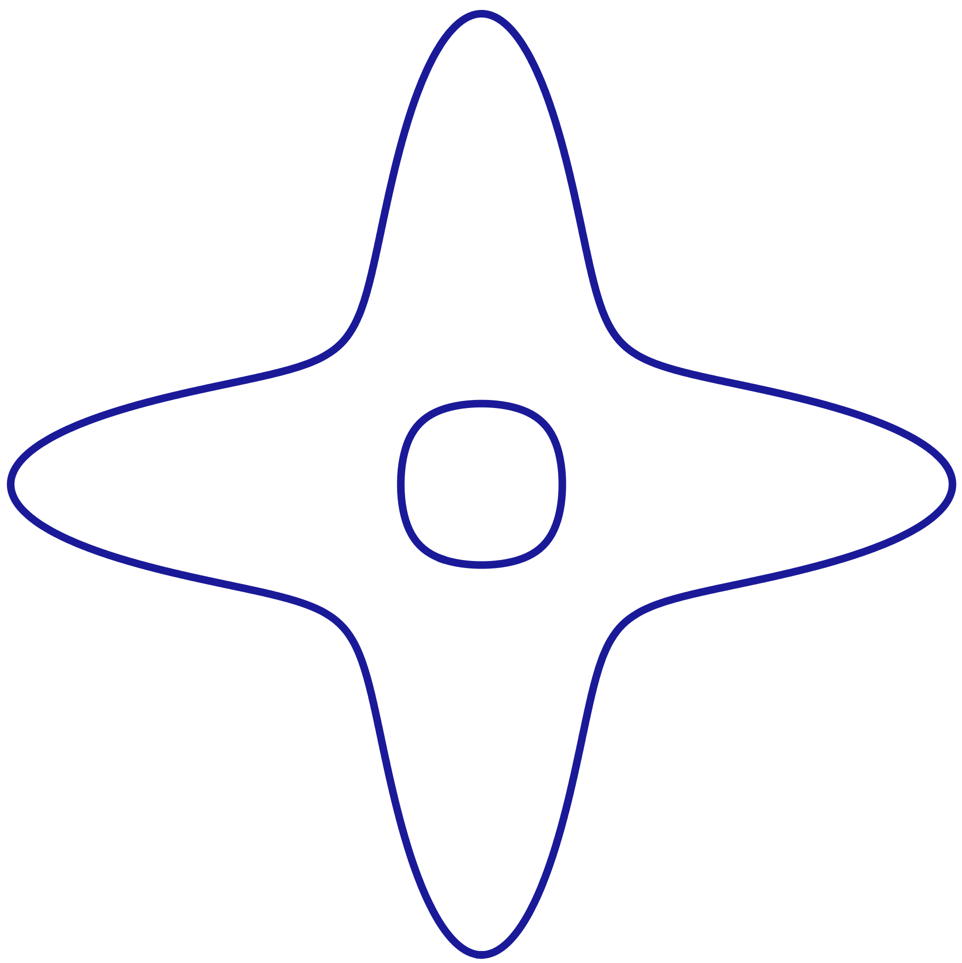
Nodes of V^*/bitangents of V: $\frac{1}{2}d(d-2)(d-3)(d+3)$
Cusps of V^*/inflection lines of V: $3d(d-2)$

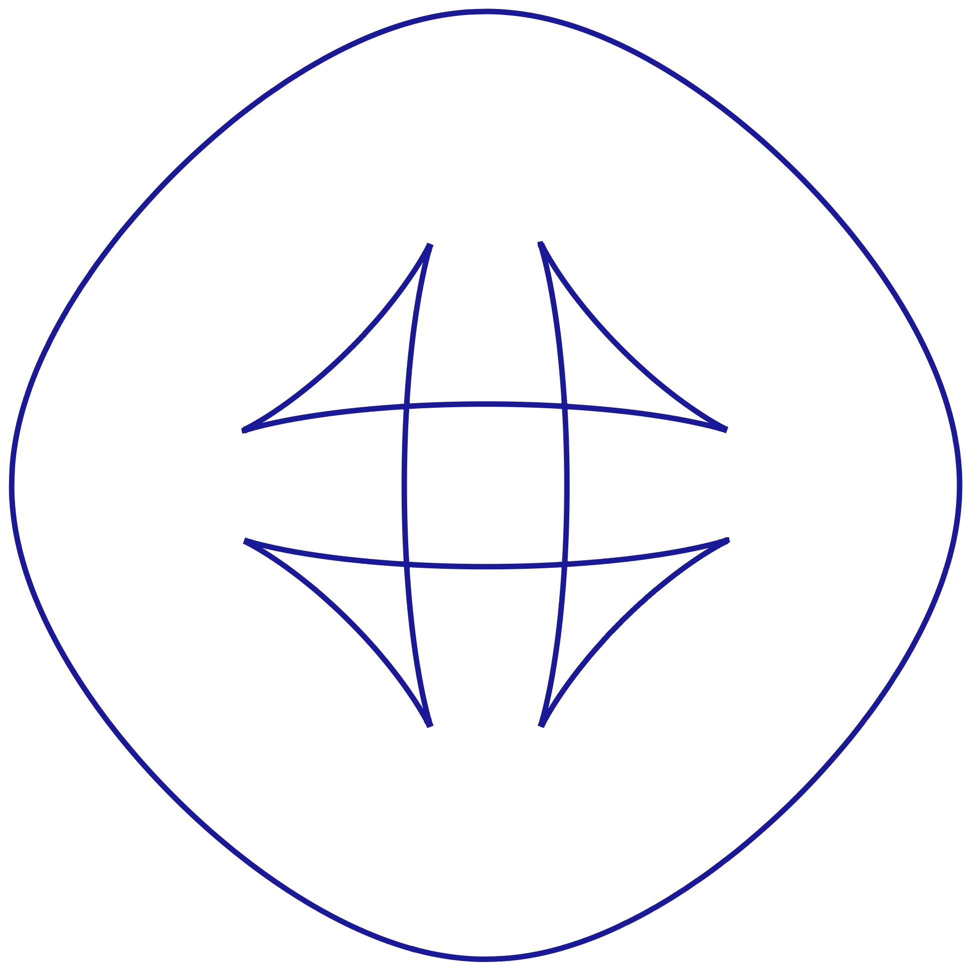




Duality for hyperbolic curves

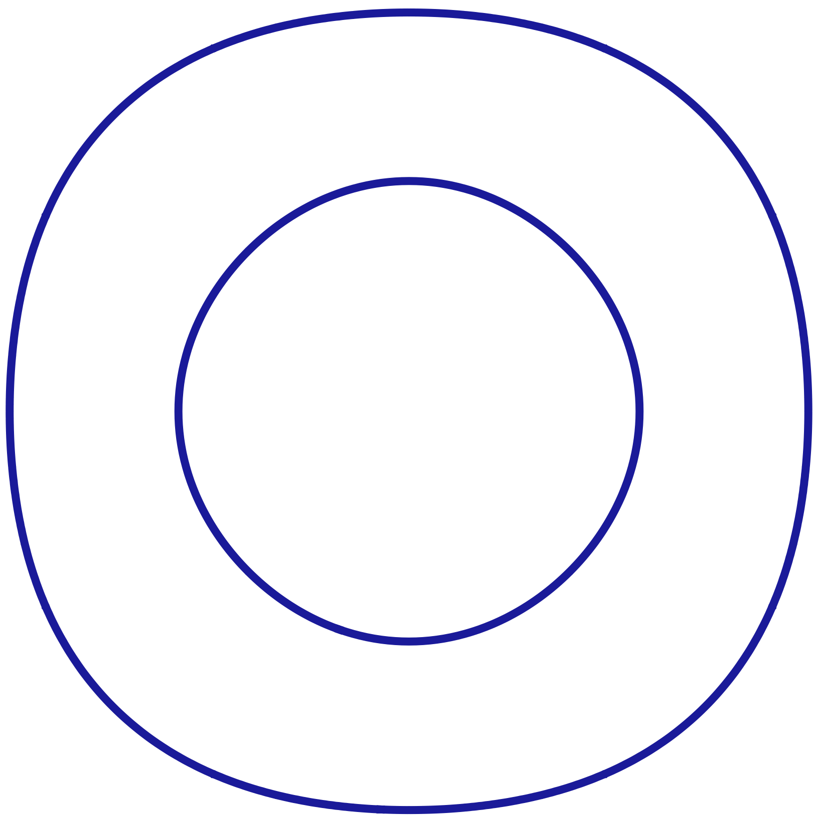
Theorem. *(Kippenhahn for hyperbolic curves)*
Let $p \in \mathbb{R}[x_0, x_1, x_2]$ *be hyperbolic with respect to* $e = (1, 0, 0)$.
The convex dual of the hyperbolicity region $\Lambda_+(f, e) \cap \{x_0 = 1\}$ *is the convex hull of the dual curve of* $\{f = 0\}$ *in the dual plane* $\{u_0 = 1\}$.

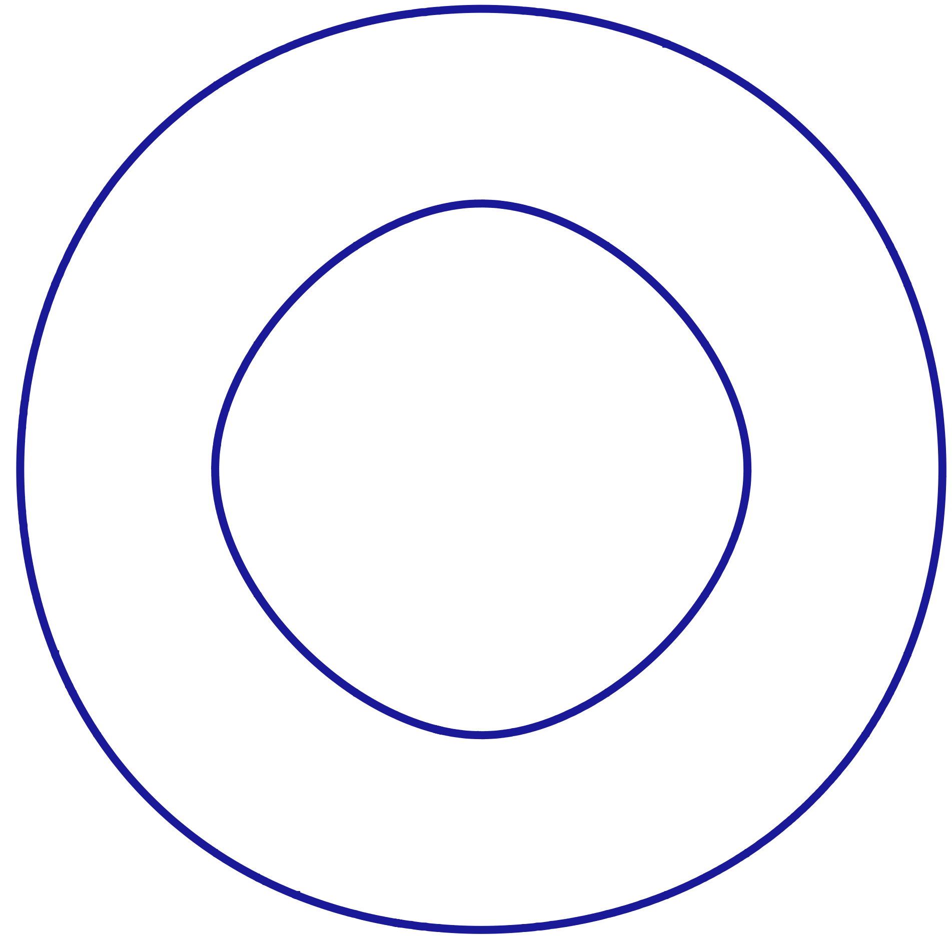




Problem: What about isolated real points (nodes) of the dual curve?

Duality for hyperbolic curves

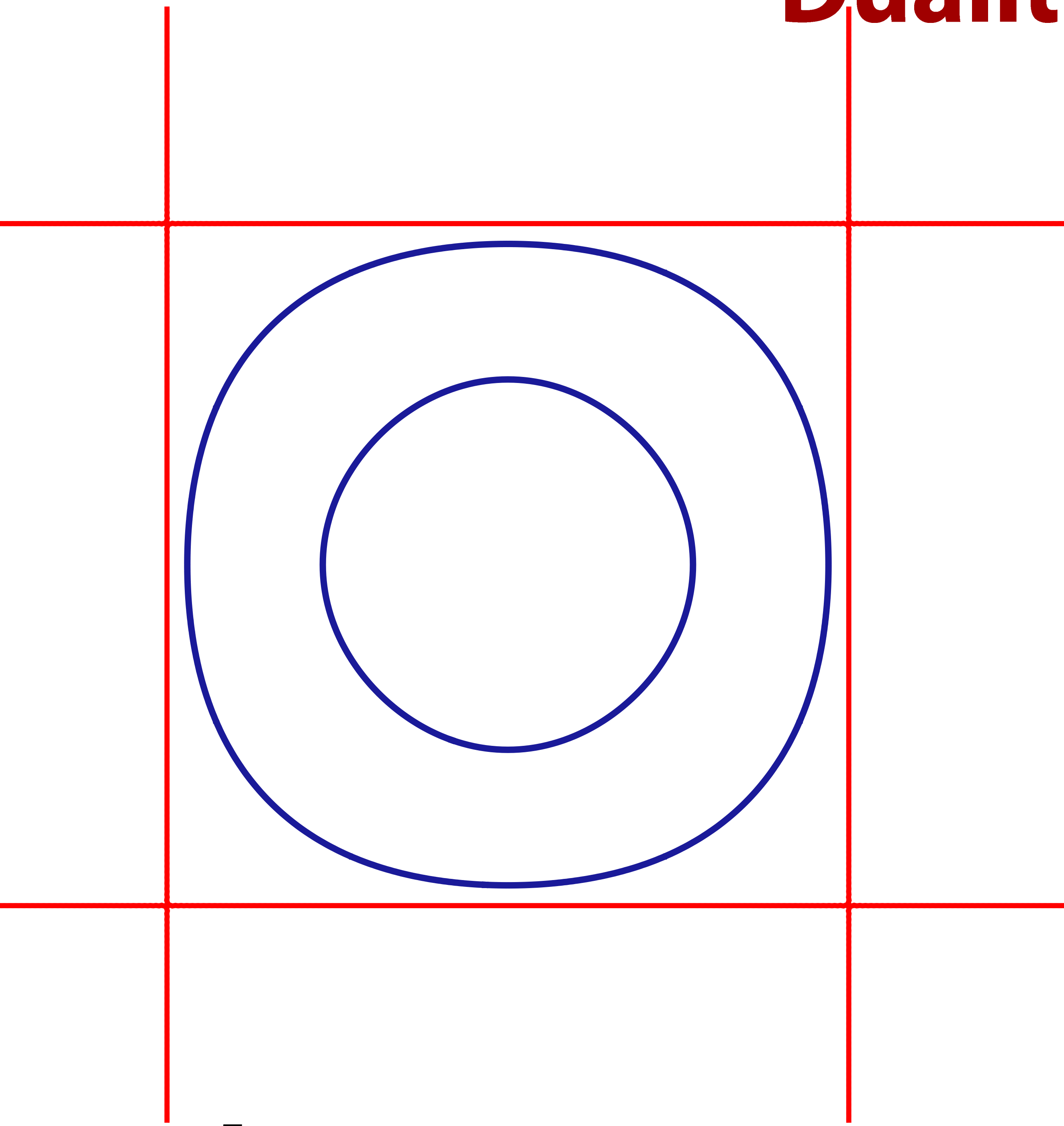


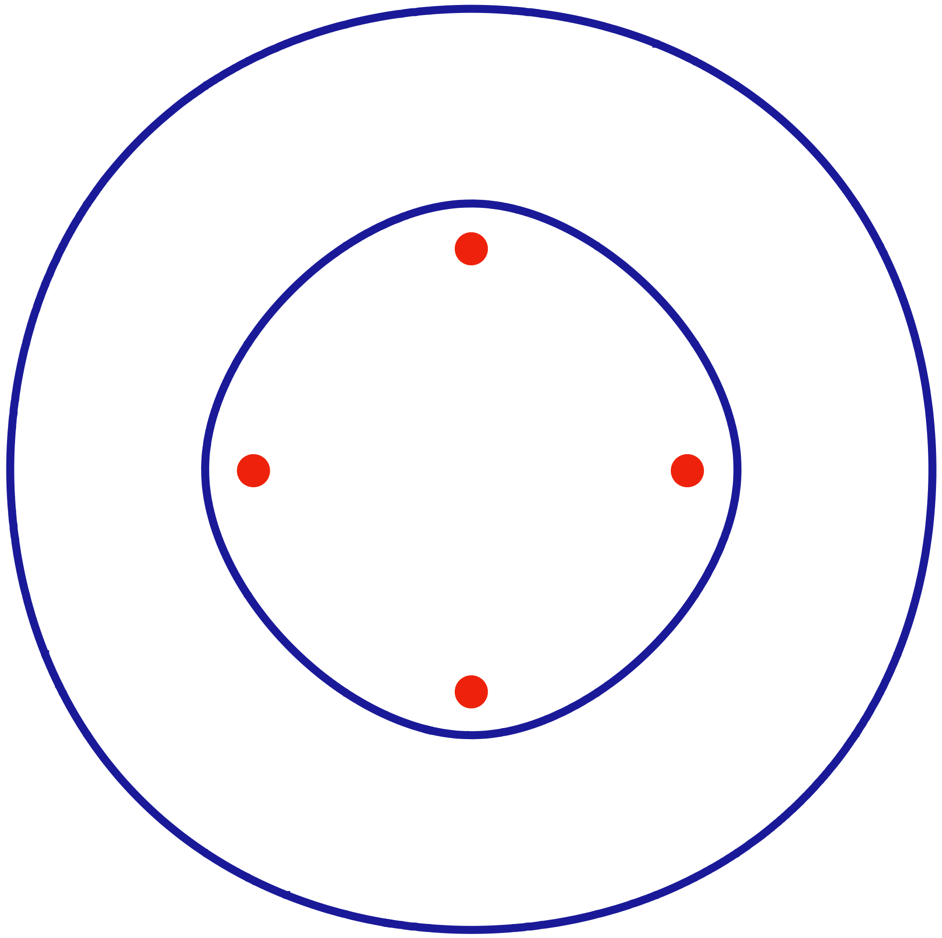


$$x_1^4 + x_2^4 + \frac{7}{4}x_1^2x_2^2 - 4x_0^2x_1^2 - 4x_0^2x_2^2 + 3x_0^4 = 0$$

$$\begin{aligned}
&12288u_1^{12} + 89088u_1^{10}u_2^2 - 4096u_1^{10}u_0^2 + 248064u_1^8u_2^4 - 150784u_1^8u_2^2u_0^2 - 14976u_1^8u_0^4 + 340800u_1^6u_2^6 - 410560u_1^6u_2^4u_0^2 \\
&+ 137328u_1^6u_2^2u_0^4 + 4800u_1^6u_0^6 + 248064u_1^4u_2^8 - 410560u_1^4u_2^6u_0^2 + 283881u_1^4u_2^4u_0^4 - 85260u_1^4u_2^2u_0^6 + 3619u_1^4u_0^8 \\
&+ 89088u_1^2u_2^{10} - 150784u_1^2u_2^8u_0^2 + 137328u_1^2u_2^6u_0^4 - 85260u_1^2u_2^4u_0^6 + 23152u_1^2u_2^2u_0^8 - 1860u_1^2u_0^{10} + 12288u_2^{12} \\
&- 4096u_2^{10}u_0^2 - 14976u_2^8u_0^4 + 4800u_2^6u_0^6 + 3619u_2^4u_0^8 - 1860u_2^2u_0^{10} + 225u_0^{12} \quad = \quad 0
\end{aligned}$$

Duality for hyperbolic curves





$$x_1^4 + x_2^4 + \frac{7}{4}x_1^2x_2^2 - 4x_0^2x_1^2 - 4x_0^2x_2^2 + 3x_0^4 = 0$$

$$\begin{aligned}
&12288u_1^{12} + 89088u_1^{10}u_2^2 - 4096u_1^{10}u_0^2 + 248064u_1^8u_2^4 - 150784u_1^8u_2^2u_0^2 - 14976u_1^8u_0^4 + 340800u_1^6u_2^6 - 410560u_1^6u_2^4u_0^2 \\
&+ 137328u_1^6u_2^2u_0^4 + 4800u_1^6u_0^6 + 248064u_1^4u_2^8 - 410560u_1^4u_2^6u_0^2 + 283881u_1^4u_2^4u_0^4 - 85260u_1^4u_2^2u_0^6 + 3619u_1^4u_0^8 \\
&+ 89088u_1^2u_2^{10} - 150784u_1^2u_2^8u_0^2 + 137328u_1^2u_2^6u_0^4 - 85260u_1^2u_2^4u_0^6 + 23152u_1^2u_2^2u_0^8 - 1860u_1^2u_0^{10} + 12288u_2^{12} \\
&- 4096u_2^{10}u_0^2 - 14976u_2^8u_0^4 + 4800u_2^6u_0^6 + 3619u_2^4u_0^8 - 1860u_2^2u_0^{10} + 225u_0^{12} = 0
\end{aligned}$$

The joint numerical range

Let $A_1, \dots, A_n$ be Hermitian $d \times d$-matrices.
The **joint numerical range** of $A_1, \dots, A_n$ is the set

$$W(A_1, \dots, A_n) = \left\{ \left(\overline{x}^T A_1 x, \dots, \overline{x}^T A_n x \right) \;\middle|\; x \in \mathbb{C}^n \text{ with } \|x| = 1 \right\} \subset \mathbb{R}^n$$

The joint numerical range is **not convex** in general (studied by Li&Poon 2000).

The convex hull can be described as

$$\operatorname{conv} W(A_1, \dots, A_n) = \left\{ \left(\langle A_1, X \rangle, \dots, \langle A_n, X \rangle \right) \;\middle|\; X \succeq 0,\ \operatorname{trace}(X) = 1 \right\}$$

where $\langle A, B \rangle = \operatorname{trace}(A\overline{B^T})$ and $X \succeq 0$ means that X is Hermitian and positive semidefinite.

The set $\operatorname{conv} W(A_1, \dots, A_n)$ is again the convex dual of the spectrahedron

$$\left\{ x \in \mathbb{R}^n \;\middle|\; I_d + x_1 A_1 + \cdots + x_n A_n \succeq 0 \right\}$$

Projective duality in higher dimensions

Let $V \subset \mathbb{P}^n$ be a projective variety. The **dual variety** of V (over $\mathbb{C}$) is

$$V^* = \overline{\left\{ u \in (\mathbb{P}^n)^* \mid \exists p \in V_{\text{reg}}\colon T_p(V) \subset \left\{ \sum u_i x_i = 0 \right\} \right\}}.$$

In words: V^* parametrizes all hyperplanes tangent to V at regular points.

Facts:

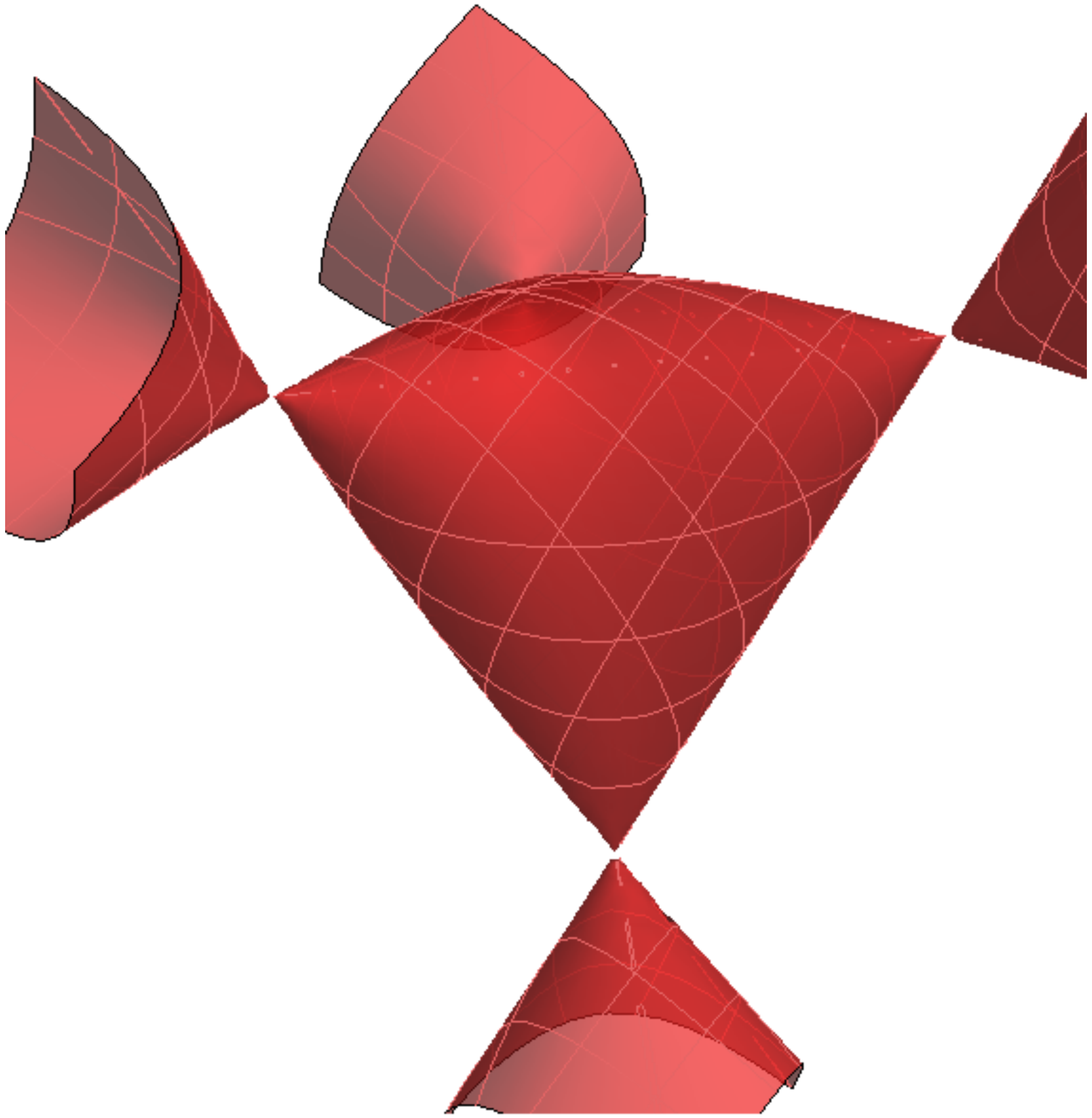
(1) If V is irreducible, then **biduality** holds: $(V^*)^* = V$.

(2) If $V = \{f = 0\}$ is a **generic** hypersurface of degree d, then V^* is a hypersurface of degree $d(d-1)^{n-1}$.

(3) Determinantal hypersurfaces are **not generic** in this sense, for $n \geqslant 3$: The variety V^* is usually **not** a hypersurface.

Examples. For the general determinantal hypersurface $\{\det(X) = 0\}$ in the space $\mathbb{P}^{\binom{d+1}{2}}$ of all symmetric $d \times d$-matrices, the dual variety is the set of all symmetric matrices of rank 1 (the **Veronese variety**).

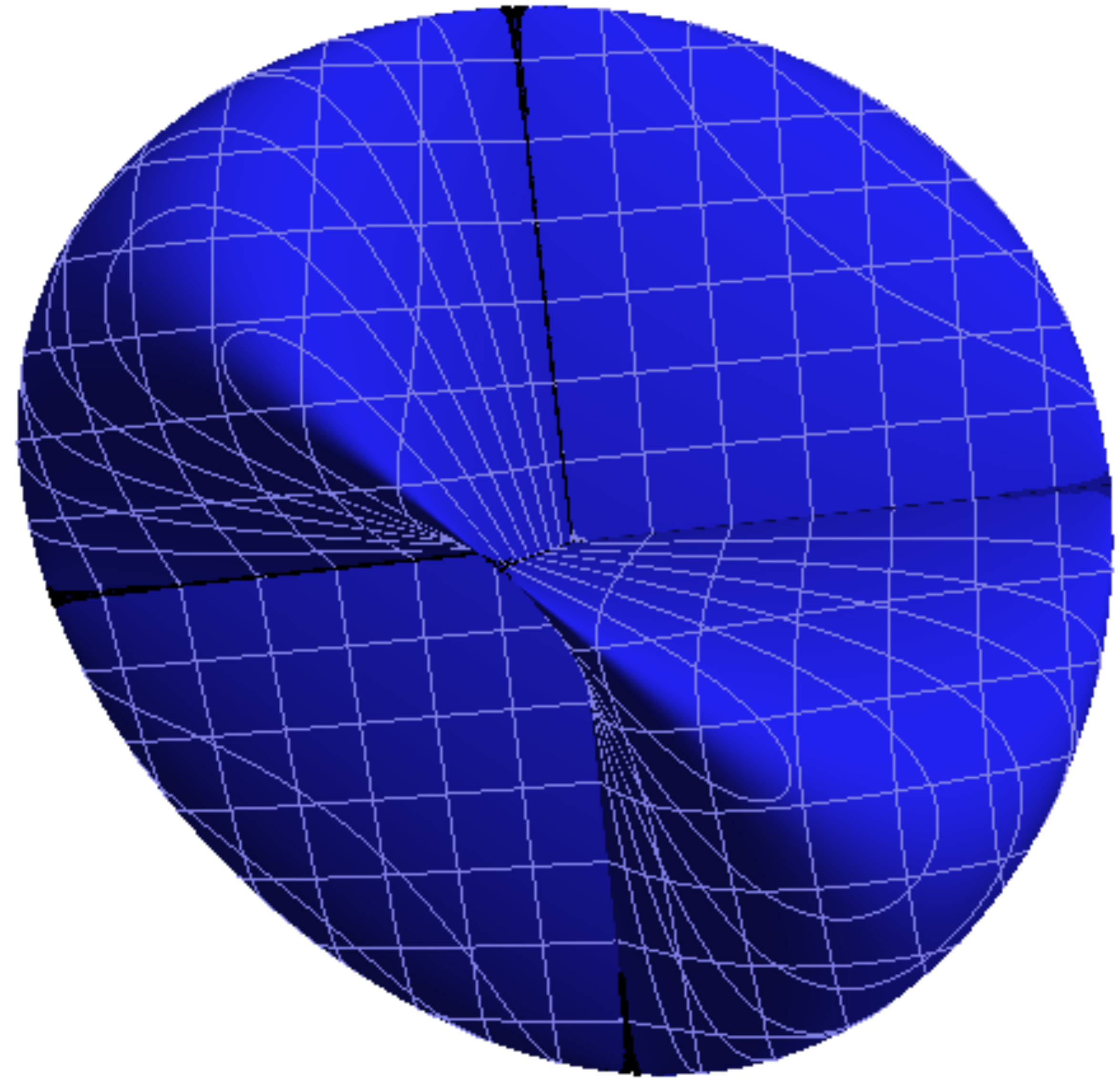
Famous example



Cayley's cubic

$$2x_1x_2x_3 - x_0x_1^2 - x_0x_2^2 - x_0x_3^2 + x_0^3$$

$$= \det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \\ x_2 & x_3 & x_0 \end{pmatrix} = 0$$



Steiner's quartic

$$u_1^2u_2^2 - u_1^2u_3^2 - u_2^2u_3^2 - 2u_0u_1u_2u_3 = 0$$

Example

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Chien and Nakazato 2010)

$$\begin{aligned} p(u_0, u_1, u_2, u_3) &= \det(u_0 \,\mathrm{id} + u_1 A_1 + u_2 A_2 + u_3 A_3) \\ &= u_0^3 + u_0^2 u_3 - 2u_0 u_1^2 - u_0 u_2^2 - u_1^3 - u_1^2 u_3 + u_1 u_2^2 \end{aligned}$$

The projective dual is a surface defined by

$$\begin{aligned} q(x_0, x_1, x_2, x_3) = {}& 4x_0^2 x_3^2 + 8x_0 x_1 x_3^2 - 4x_0 x_2^2 x_3 - 24 x_0 x_3^3 + 4x_1^2 x_3^2 \\ & - 4x_1 x_2^2 x_3 - 8x_1 x_3^3 + x_2^4 + 8x_2^2 x_3^2 + 20 x_3^4. \end{aligned}$$

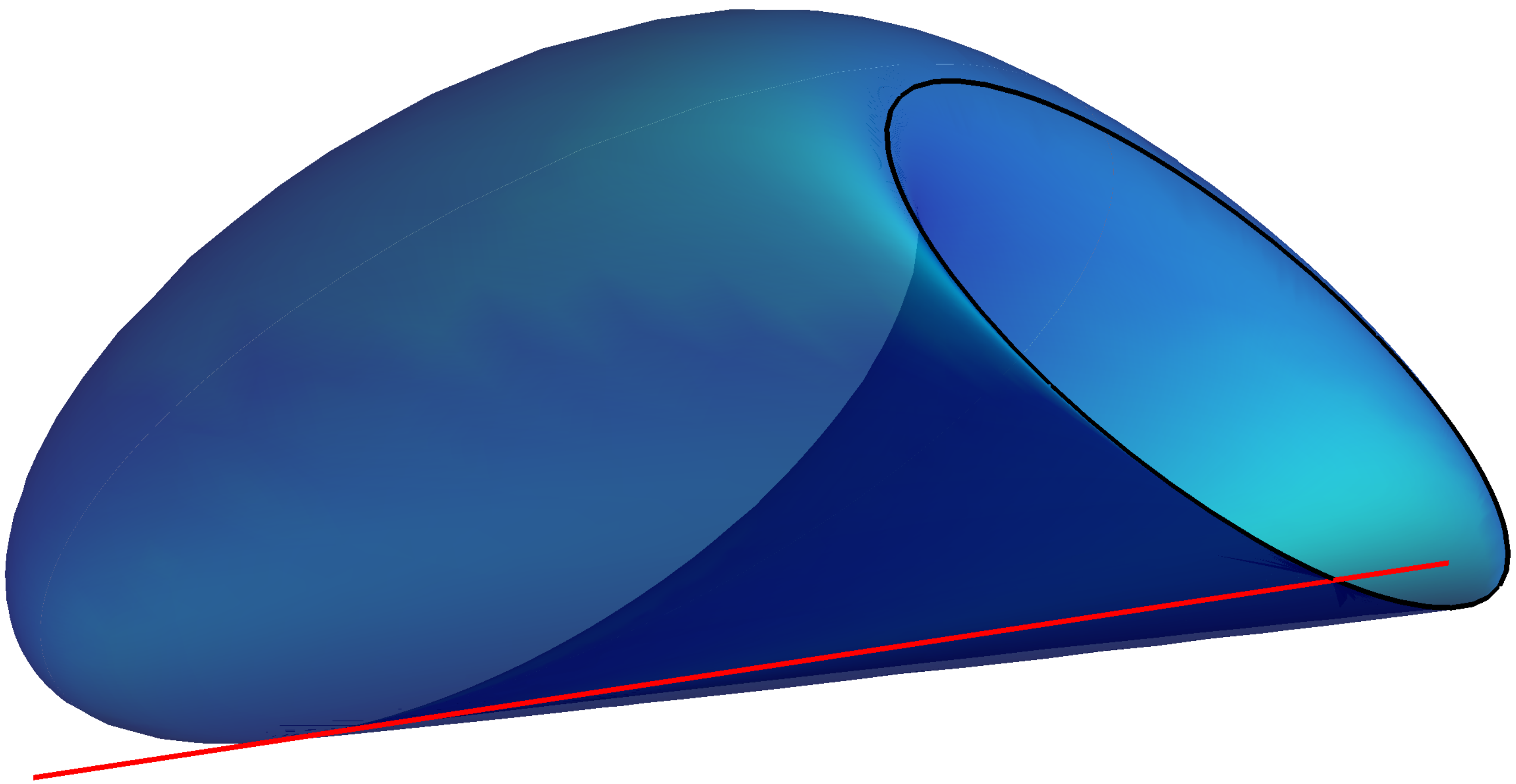
Its singular locus is $\{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 : x_2 = x_3 = 0\}$

Example

The projective dual is a surface defined by

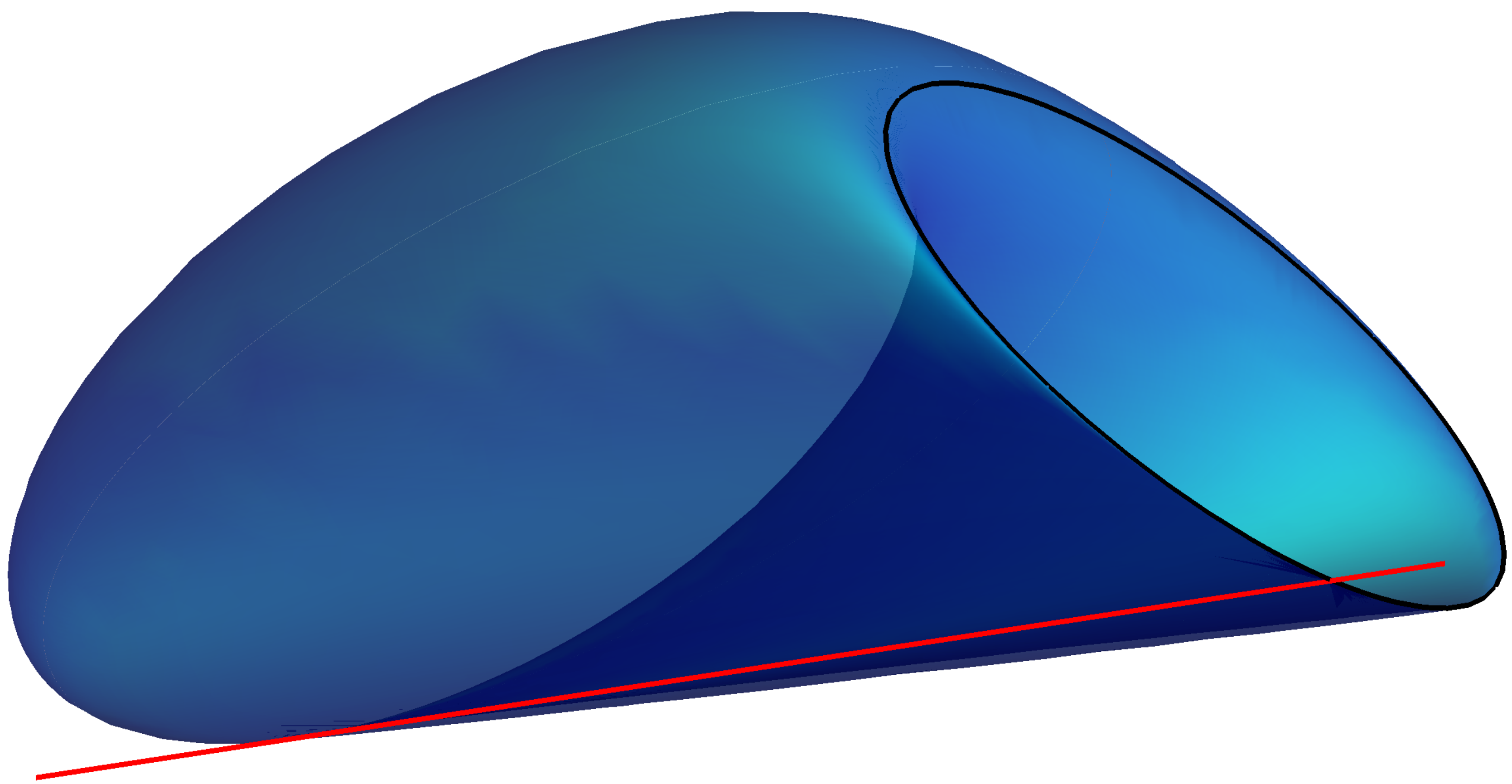
$$q(x_0,x_1,x_2,x_3) = 4x_0^2x_3^2 + 8x_0x_1x_3^2 - 4x_0x_2^2x_3 - 24x_0x_3^3 + 4x_1^2x_3^2 - 4x_1x_2^2x_3 - 8x_1x_3^3 + x_2^4 + 8x_2^2x_3^2 + 20x_3^4.$$

Its singular locus is $\{(x_0,x_1,x_2,x_3) \in \mathbb{P}^3 : x_2 = x_3 = 0\}$



How to fix it

Theorem. *(Sinn 2015/P-Sinn-Weis 2019)*
Let $p \in \mathbb{R}[x_0, \ldots, x_n]$ *be irreducible and hyperbolic with respect to* $e = (1, 0 \ldots, 0)$.
Let $V = \{p = 0\} \subset \mathbb{P}^n$ *and let* V^* *be the dual projective variety.*
The convex dual of the hyperbolicity region $C(p, e) \cap \{x_0 = 1\}$ *is the closure of the convex hull of* $V^*_{\mathrm{reg}}(\mathbb{R}) \cap \{u_0 = 1\}$, *where* $V_{\mathrm{reg}}(\mathbb{R})$ *is the set of regular real points of* V^*.



Corollary. *(PSW 2019) The convex hull of the joint numerical range of Hermitian* $d \times d$ *matrices* $A_1, \ldots, A_n$ *is the closure of the convex hull of the real non-singular part of the dual variety of the hyperbolic hypersurface* $\det(x_0 I_d + x_1 A_1 + \cdots + x_n A_n)$.