The complexity of ground states

Zeph Landau
Ground State

condensed matter

structure gap algorithm

many body

Area Law

AGSP entanglement viable set

Local Hamiltonian
The difficulty of understanding many-body physics

Each particle in a d-dimensional space—Cd n particles = tensor the individual spaces together = space of dimension d^n. H = (Cd) ⊗ n.

System described by a state: a unit vector |v⟩ ∈ H.

The same property that leads to the power of quantum computation is the major barrier for understanding many-body physics: Exponential Dimensional Space. So even describing a state requires exponential amount of information.
The difficulty of understanding many-body physics

Each particle a $d$ dimensional space—$\mathbb{C}^d$

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A Basic Question

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- Do they have a special structure?
- Does that structure allow for meaningful short descriptions?
- Does that structure allow us to compute various properties of them?
A Basic Question

All states

Physically relevant states

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Physically Relevant States: Ground States of Local Hamiltonians

Local Term: $H_i$ is a linear operator. (self-adjoint). Acts "locally": non-trivial on only a few particles.

Local Hamiltonian: $H = \sum_i H_i$ is an operator formed from the sum of local terms.

Ground State: The ground state $|\Gamma\rangle$ is the smallest eigenvector of $H$.

Gap: The distance between the lowest two eigenvalues.

Focus on unique ground state and constant gap.

Ground states model the state of the system at low temperatures.
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[Diagram of a grid with connected dots, possibly representing a lattice]
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Fundamental Connection: Local Hamiltonians and Constraint Satisfaction Problems.

Classical Constraint Satisfaction Problems (CSP’s).
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Solving, classifying, and understanding the structure of the solutions of CSP’s at the heart of complexity theory.
Local Hamiltonians = non-commutative CSP’s

Complexity Theory

Constraint Satisfaction Problems

non-commutative generalization

Condensed Matter Physics

Local Hamiltonians

Number of colors $\leftrightarrow$ Dimension of single particle

Local constraint diagonal only $\leftrightarrow$ Local term $H_i$

Assignment that violates fewest constraints $\leftrightarrow$ Ground state: lowest eigenvalue

Least number of constraints violated $\leftrightarrow$ Lowest eigenvalue

CSP constraints correspond to $H_i$ that are diagonal in the standard basis. In particular they all commute.
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For (gapped) 1D systems: yes
For higher dimensions: ?
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Understanding ground states of local Hamiltonians: A journey

"In theory, there is no difference between theory and practice. In practice, there is."

How do you do Physics?

[92, White] Density Matrix Renormalization Group (DMRG):

1D – remarkably successful in practice.
2D – open.
However not a great understanding of what is going on.
Sure you can do it in practice . . .
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Quantum Complexity Theory viewpoint

[ ’97, Kitaev]:
- Introduction of QMA—quantum analogue of NP.
- Finding ground states of general quantum systems is QMA complete.

[ ’05, Oliveira, Terhal, ’06, Kempe, Kitaev, Regev]: Finding solutions to 2D systems is QMA hard.
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Area Law formulation

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[’01, Vidal, Latorre, Rico, Kitaev] Area Law formalized in terms of entanglement entropy.

- Effect on DMRG: speedup, simplification, better understanding of the heuristics used.
Area Law in 1D systems

1D Area law proved [Hastings ’07].

- Established that many 1D solutions (constant gap) satisfy an area law and are in NP rather than QMA-complete.
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[’08, Cirac, Schuch, Verstraete] Example of finding a solution that satisfies the area law that is NP-hard.
The birth of Approximate Ground State Projections

"If there is a problem you can’t solve, then there is an easier problem you can’t solve: find it." - George Polya
The birth of Approximate Ground State Projections

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A special case: frustration-free commuting case.

- Can assume $H_i$ are projections.
- $P = \prod_i (1 - H_i)$ projects onto the ground space.
- $P$’s complexity across a cut proportional to number of terms acting across the cut.
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**Approximate Ground State Projection (AGSP)**

Properties:
- It "approximately" projects onto one vector you want (ground state).
- It isn’t too complex.
New blood: Approximate Ground State Projections (AGSPs)

Two new results:

[11',12', Arad, Kitaev, Landau, Vazirani] Exponential improvement in parameters of the 1D area law which $\to$ a sub-exponential time algorithm for finding solutions.

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Current view of 1D local Hamiltonians

From AGSP’s:
- exponential improvement on the constants for the 1D Area Law algorithm for 1D,
- gives insight as to what is going on,
- tools for attacking the 2D questions.

Zeph Landau ()
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Role of AGSP in proof of Area Law I

Two main steps:

1. Find a not very complex state that has constant overlap with the ground state.

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Ground State
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2. Repeatedly apply an AGSP to that state to rapidly get a good approximation to the ground state.
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Both steps use AGSPs— the first is much more delicate.
A state on $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the form $\sum_{i=1}^{C} a_i \otimes b_i$ will be said to have entanglement rank $C$. 

Entanglement rank behavior

Multiplicative for operators applied to states or product of operators.

Additive for sums of states or operators.
Measure of Complexity: Entanglement rank

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AGSP: almost projection with small entanglement rank

We are looking for an operator $K$ with 2 properties:

1. It approximately projects onto the ground state:
2. It has small entanglement rank:
AGSP: almost projection with small entanglement rank

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![Diagram of ground state and eigenvalues]

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\begin{align*}
\text{Ground state} & \downarrow \\
\Delta & \\
\text{Eigenvalues of AGSP} & \\
\Delta & \\
\text{Eigenvalues of H} & \\
\end{align*}
\]

Critical threshold $D\Delta < 1.$
Theorem (Area Law) [Arad, Landau, Vazirani] The existence of an AGSP $K$ for which $D\Delta < 1/2$ proves that the ground state has entropy $O(1) \log D$.

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- Start with an unentangled state $|v\rangle$. 

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Finding the ground state of 1D systems: solving a large convex program

finding the minimal energy state

\[ \min_{\rho} \text{tr}(\rho H) \]
\[ \text{with the conditions} \]
\[ \rho \succeq 0 \]
\[ \text{tr}(\rho) = 1. \]

Exponential size space is too costly. What we'll need:
- A restriction of the convex program to a polynomial size subspace,
- A succinct description of the elements of that subspace that allows us to perform linear algebra efficiently.
Finding the ground state of 1D systems: solving a large convex program

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Exponential size space is too costly. What we’ll need:

- A restriction of the convex program to a **polynomial size subspace**,
- A **succinct** description of the elements of that subspace that allows us to perform linear algebra efficiently.
The algorithm: a bird’s eye view

A sequence of spaces $S_i$ termed **viable sets**:

- all **polynomial** size
- all with **succinct** descriptions that allow efficient linear algebra,
- each containing a good approximation of the "left" side of the ground state.

$$|\Gamma\rangle \approx \sum_j |a_j\rangle |b_j\rangle \text{ with each } |a_j\rangle \in S_i.$$
The algorithm: a bird’s eye view

Key components:

- **Splitting**: 1D structure allows reduction of convex program to left side by iterating over a net of boundary conditions. **Structural result allows this net to be fixed polynomial size.**
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Arxiv, find me, later workshop.
Where do we go from here?

AGSP

Structure

Simulation

Quantum Many-Body Systems

A 2D area law?
A more local 1D algorithm?
Degenerate ground space?
Different questions?

"The future ain't what it used to be." – Yogi Berra

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The complexity of ground states
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AGSP construction: norm reduction

Looking for low entanglement operators that look like:

\[
f(x) \Delta \varepsilon ||H||
\]

Smaller \( ||H|| \) would be better but we don’t want to lose the local structure around the cut.

**Solution:** Replace \( H = \sum_i H_i \) with \( H' = H_L + H_1 + H_2 + \cdots + H_s + H_R \).
AGSP construction: Chebyshev polynomials

Chebyshev polynomials: small in an interval:

The desired AGSP is a dilation and translation of the Chebyshev polynomial:

\[ K = C_l(H') \]
AGSP complexity: Entanglement rank analysis

\[(H')^\ell = \sum (\text{product of } H_j).\]

For a single term:
AGSP complexity: Entanglement rank analysis

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For a single term:
- Across some cut, an average number of terms are involved → \(d^{2\ell/s}\).
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For a single term:
- Across some cut, an average number of terms are involved $\rightarrow d^{2\ell/s}$.
- Roundtrip cost of going and coming back from center cut: $\rightarrow d^s$. 

Cost $d^s$

Cost $d^{2\ell/s}$

\[\ldots\]
AGSP complexity: Entanglement rank analysis

\[(H')^\ell = \sum (\text{product of } H_j)\].

For a single term:

- Across some cut, an average number of terms are involved $\rightarrow d^{2\ell/s}$.
- Roundtrip cost of going and coming back from center cut: $\rightarrow d^s$.

**Total:** $d^{2\ell/s} + s$
**Problem:** Too many \((s^\ell)\) terms in naive expansion of \((H^\prime)^\ell\).
**Problem:** Too many \((s^\ell)\) terms in naive expansion of \((H')^\ell\).

Need to group terms in a nice way but it all works out with total entanglement increase of the same order as the single term.
Putting things together: Area Law for $H'$

Chebyshev $C_\ell(H')$ has $\Delta \approx e^{-O(\ell/\sqrt{s})}$:

Entanglement analysis yields $D \approx O(d^{\ell/s} + s)$.

Choosing $\ell = s^2$ yields $D\Delta \approx e^{-s^{3/2} + s \log d} < 1$ for appropriate choice of $s \approx \log^2 d$. 