Location of zeros of the partition function of the Ising model

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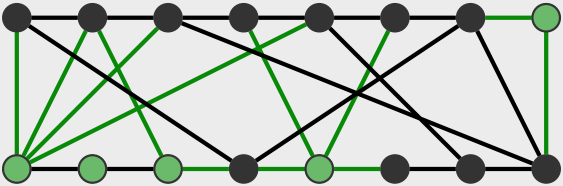
Partition function of the Ising model

For a graph $G = (V, E)$, and $\lambda, \beta \in \mathbb{C}$, the partition function of the Ising model is defined as

$$Z_G(\lambda, \beta) = \sum_{U \subseteq V} \lambda^{|U|} \cdot \beta^{|\delta(U)|}.$$

Here $|\delta(U)|$ denotes the number of edges between U and $V \setminus U$.



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- Invented to study ferromagnetism in statistical physics.
- $Z_G(1, \beta)$ is generating functions of edge cuts in G.
- $Z_G(1, \beta)$ is the partition function of the 2-state Potts model.
- $Z_G(\lambda, \beta)$ for non-real β, λ relates to output probabilities for certain quantum circuits (Mann, Brenner 2018+)

Theorem (Lee and Yang, 1952)

Fix $\beta \in [-1, 1]$. Then for any graph G, the zeros of the univariate polynomial, $Z_G(\lambda, \beta)$, lie on the unit circle in the complex plane.

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- Today: where on the circle are these zeros?
- If $\beta = 1$, $Z_G = (1 + \lambda)^{|V|}$, which has only one zero: -1.
- For any other β, the roots of all graphs are in fact dense on the circle.
- We will consider the class of bounded degree graphs.

Overview of the rest of the talk

- Results for all bounded degree graphs
- Algorithmic consequences
- Ideas of proof (use of complex dynamics)
- Open problems and questions

Zeros for bounded degree graphs: Ferromagnetic case

$\mathcal{G}_{d+1}$ is collection of all graphs of maximum degree at most $d+1$.
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Let $d \in \mathbb{N}_{\geq 2}$ and let $\beta \in (\frac{d-1}{d+1}, 1)$. Then there exists $\theta = \theta_\beta \in (-\pi, \pi)$ such that the following holds:

(i) *for any $\lambda = e^{i\vartheta}$,$|\vartheta| < \theta$ and any graph $G \in \mathcal{G}_{d+1}$ we have $Z_G(\lambda, \beta) \neq 0$;*

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(i) *for any $\lambda = e^{i\vartheta}$,$|\vartheta| < \theta$ and any graph $G \in \mathcal{G}_{d+1}$ we have $Z_G(\lambda,\beta) \neq 0$;*

(ii) *the set $\{\lambda \in \mathbb{C} \mid Z_G(\lambda,\beta) = 0$ for some $G \in \mathcal{G}_{d+1}\}$ is dense in $\partial\mathbb{D} \setminus (-\theta,\theta)$.*

- Part (ii) independently proved by Chio, He, Ji, and Roeder (2018+).
- Extends some results of Barata and Marchetti and Barata and Goldbaum for $d=2$ on Cayley trees.

Zeros for bounded degree graphs: Anti-Ferromagnetic case

$\mathcal{G}_{d+1}$ is collection of all graphs of maximum degree at most $d+1$.

Theorem (Peters, R. 18+)

Let $d \in \mathbb{N}_{\geq 2}$ and let $\beta \in (1, \frac{d+1}{d-1})$. Then there exists $\alpha = \alpha_\beta \in (-\pi, \pi)$ such that the following holds:

(i) *for any $\lambda = e^{i\vartheta}$, $|\vartheta| < \alpha$, any $r \geq 0$ and any graph $G \in \mathcal{G}_{d+1}$ we have $Z_G(r \cdot \lambda, \beta) \neq 0$;*

(ii) *the set $\{\lambda \in \mathbb{C} \mid Z_G(\lambda, \beta) = 0 \text{ for some } G \in \mathcal{G}_{d+1}\}$ accumulates on $e^{i\alpha}$ and $e^{-i\alpha}$.*

Algorithmic consequences

Corollary

There exists an FPTAS for computing $Z_G(\lambda, \beta)$ *for each fixed* β *and* λ *as above and* $G \in \mathcal{G}_{d+1}$.

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(What is known about approximating Z_G when $G \in \mathcal{G}_{d+1}$)

- FPRAS on all graphs when $0 < \beta < 1$ and $\lambda > 0$ (Jerrum and Sinclair 1993)
- FPTAS when $\lambda = 1$ and $\beta \in (1, \frac{d+1}{d-1})$ (Sinclair, P. Srivastava, and Thurley, 2014)
- FPTAS when $\lambda = 1$ and $|\beta - 1| \leq O(1/d)$, (Barvinok and Soberón 2017 combined with Patel, R. 2017)
- FPTAS when $\beta \in [-1, 1]$ and $|\lambda| < 1$ (Liu, Sinclair, P. Srivastava, 2017)

High level idea of the proof

- Transform the problem to ratios of partition functions.
- Express the ratio as an iteration of a rational map and apply techniques/ideas from complex dynamics.
- Same structure/idea was used by Peters and R. to solve a conjecture of Sokal concerning the location of zeros for the independence polynomial.

Ratios of partition functions

$$Z_G(\lambda, \beta) = \sum_{U \subseteq V} \lambda^{|U|} \cdot \beta^{|\delta(U)|}.$$

$$Z_G = Z_{G,v,\text{in}} + Z_{G,v,\text{out}}$$

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$$Z_G = Z_{G,v,\text{in}} + Z_{G,v,\text{out}}$$

$$R_{G,v} := \frac{Z_{G,v,\text{in}}}{Z_{G,v,\text{out}}}$$

Then ('ignoring' the situation that $Z_{G,v,\text{in}} = Z_{G,v,\text{out}} = 0$),

$$Z_G \neq 0 \iff R_{G,v} \neq -1.$$

High level idea of proof II

- Step 1: Analyse the ratio on Cayley trees using complex dynamics. (This allows to prove parts (ii))
- Step 2: Extend results to all trees with boundary conditions.
- Step 3: Use Weitz' self avoiding walk tree to go from trees to all graphs.

A recurrence for ratios

Let $T_{k,d}$ be the rooted Cayley tree of down degree d with k layers , i.e. $T_{0,d}$ consists of a single vertex and $T_{k,d}$ consists of d copies of $T_{k-1,d}$ connected to the root.

Lemma

$$R_{T_{k,d}} = \lambda \left(\frac{R_{T_{k-1,d}} + \beta}{\beta R_{T_{k-1,d}} + 1} \right)^d .$$

Towards dynamical systems

Define

$$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ by } R \mapsto \lambda \left(\frac{R+\beta}{\beta R+1} \right)^d .$$

Lemma

For Cayley trees $T_k = T_{k,d}$:

$$Z_{T_k}(\beta, \lambda) \neq 0 \textit{ for all } k \iff f^{\circ k}(1) \neq -1 \textit{ for all } k.$$

Basic observations when $\beta \in (0,1)$

Let

$$g(R) = \frac{R+\beta}{\beta R + 1} \qquad \text{then } f(R) = \lambda \cdot g(R)^d$$

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- g is a Möbius transformation and preserves the circle, $\partial\mathbb{D}$, its interior and its exterior. (Implies Lee-Yang Thm. for trees.)
- f is an orientation preserving d-fold covering of $\partial\mathbb{D}$
- $$f'(R) = f(R)\frac{d(1-\beta^2)}{(R+\beta)(\beta R+1)}.$$

  So $|f'(R)|$ is minimal at $R = 1$ and increasing with $|\text{Arg}(R)|$.

Observations from complex dynamics

Definition (Informal)

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- Montel's theorem implies that the Julia set is contained in the unit circle, $\partial\mathbb{D}$.
- Two options for the Julia set J:
  - J is the entire circle (so no attracting fixed points on the circle).
  - J is *not* the entire circle, in which case the Fatou set is a single component and contains a unique *attracting* or *parabolic* fixed point on $\partial\mathbb{D}$.

The derivative at 1

$$f'(R) = f(R)\frac{d(1-\beta^2)}{(R+\beta)(\beta R+1)} \qquad \text{let } \beta_c = \frac{d-1}{d+1}.$$

- if $\beta \in (0, \beta_c)$, $|f'(1)| > 1$ (Julia set is $\partial\mathbb{D}$)
- if $\beta = \beta_c$, $|f'(1)| = 1$.
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Corollary

If $\beta \in (0, \beta_c)$, then the zeros of $Z_{T_{k,d}}(\lambda, \beta)$ are dense in $\partial\mathbb{D}$.

Analysis of parabolic fixed points

Fix $\beta \in (\beta_c, 1)$.

Lemma

There exists a unique $\theta \in (0, \pi)$ such that for the two parameters $\lambda = e^{\pm i\theta}$, f has a unique parabolic fixed point R. It satisfies the equation:

$$R^2 + \frac{d(\beta^2 - 1) + (1 + \beta^2)}{\beta} R + 1 = 0.$$

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Lemma

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This can be used to prove our theorem for Cayley trees.

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  - The recurrence for general trees is given as

$$(R_1, \ldots, R_d) \mapsto F(R_1, \ldots, R_d) := \lambda \prod_{i=1}^{d} \frac{R_i + \beta}{\beta R_i + 1}.$$

  - Let I be the circular interval $[1, \hat{R}]$ ($\hat{R}$ is the attracting fixed point.) Then for any $R \in I$, $f(R) \in I$.
  - Let C be the cone through I. Then for any $R_1, \ldots, R_d \in C$, $F(R_1, \ldots, R_d) \in C$.

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Questions/Open Problems I

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Theorem (Liu, Sinclair, Srivastava, 2018+)

for each $d \geq 2$ there exists a region $B \subset \mathbb{C}$ containing the interval $(\frac{d-1}{d+1}, \frac{d+1}{d-1})$ such that for all $\beta \in B$, and all graphs $G \in \mathcal{G}_{d+1}$, $Z_G(1, \beta) \neq 0$.

Question

What is the maximal domain B containing $(\frac{d-1}{d+1}, \frac{d+1}{d-1})$ such that the above statement still holds?

Questions/Open Problems II

Definition

The partition function of the Potts model is defined for $\beta \in \mathbb{C}$, $k \in \mathbb{N}$ and a graph G by

$$P_G(\beta, k) = \sum_{\phi: V \to [k]} \beta^{\#\text{ monochromatic edges}}.$$

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Note $Z_G(1, \beta) = \beta^{|E|} P_G(1/\beta, 2)$.

Question

Let $k \in \mathbb{N}$. Is it true that there exists a region B containing the interval $(\frac{d+1-k}{d+1}, 1)$ such that for all $\beta \in B$ and graphs $G \in \mathcal{G}_{d+1}$, $P_G(\beta, k) \neq 0$?

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With Bencs, Davies and Patel: can find a region that contains the interval

$$\left[\frac{d+1-(k-1)/e}{d+1}, 1\right).$$

Questions/Open Problems III

More antiferromagnetic ($\beta > 1$) zeros:

Theorem (Bencs, Buys, Guerini, Peters, 19+)

Let $d \in \mathbb{N}_{\geq 2}$ and let $\beta \in (1, \frac{d+1}{d-1})$. Then there exists $\theta = \theta_\beta > \alpha_\beta$ such that the set $\{\lambda \mid Z_G(\lambda, \beta) = 0\}$ for some $G \in \mathcal{G}_{d+1}$ is dense in the circular interval $(-\theta, \theta)$.

Questions/Open Problems III

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Question

What happens in between θ and α?

Preliminary work of Bencs, Buys, Guerini and Peters suggests that there is an interval $I \subset (\alpha, \theta)$ on which the roots accumulate.

Questions/Open Problems IV

Let $d \in \mathbb{N}_{\geq 2}$, let $\beta \in (\frac{d-1}{d+1}, 1)$ and let $\theta = \theta_\beta$.

Corollary

For any $\lambda = e^{i\vartheta}$,$|\vartheta| < \theta$ there is an FPTAS for computing $Z_G(\lambda, \beta)$ for all graphs $G \in \mathcal{G}_{d+1}$.

Question

How hard is it to approximate $Z_G(\lambda, \beta)$ when $\lambda = e^{i\vartheta}$,$|\vartheta| > \theta_\beta$?

Thank you for your attention!