Remarks on the Riemann Hypothesis

Charles M. Newman

Courant Institute of Mathematical Sciences & NYU Shanghai
For a function $f \geq 0$ with $\int_{-\infty}^{\infty} f(u)du < \infty$; let

$$L_{f,\lambda}(w) := \int_{-\infty}^{\infty} e^{wu} e^{\lambda u^2} f(u)du$$

for $w \in \mathbb{C}$, $\lambda \in \mathbb{R}$ where possible (e.g., $\lambda < 0$); and

$$L_{\rho,\lambda}(w) := \int_{\mathbb{R}} e^{wu + \lambda u^2} d\rho(u).$$

We take $f$ (or $\rho$) to be even (and $\rho$ is usually a probability measure).
For a specific function $\Phi$, 

$$\text{RH} \iff \text{zeros in } \mathbb{C} \text{ of } L_{\Phi,0} \text{ all pure imaginary;}$$

we’ll say $L_{\Phi,0}$ is PIZ. $\Phi$ is defined so that 

$$L_{\Phi,0} = Cs(s - 1)\pi^{-s/2}\Gamma(s/2) \sum_{1}^{\infty} n^{-s} \bigg|_{s=1/2+w/2}$$

and its explicit formula is 

$$\Phi = \sum_{1}^{\infty} \left( n^4 \pi^2 e^{9u} - \frac{3}{2} n^2 \pi e^{5u} \right) e^{-n^2 \pi e^{4u}}$$
Graph of $\Phi$
Some History

- Polya ‘20s: hoped that $L_{\Phi,\lambda}$ is PIZ $\forall \lambda \in \mathbb{R}$; he proved that PIZ for $\lambda_1 \implies$ PIZ for $\lambda \geq \lambda_1$. (I.e., increasing/decreasing $\lambda$ helps/hurts PIZ.)

- de Bruijn ‘50: $L_{\Phi,\lambda}$ is PIZ for $\lambda \geq 1/2$. (Based on zeros of $L_{\Phi,0}$ being in critical strip.)

- N. ‘76: $\exists \lambda$ s.t. $L_{\Phi,\lambda}$ is not PIZ and thus $\exists \Lambda \in (-\infty, 1/2]$ such that PIZ for $\lambda \geq \Lambda$ but not for $\lambda < \Lambda$. $\Lambda$ is now called the de Bruijn-Newman constant.

\[ \text{RH} \iff \Lambda \leq 0 \]
Some History

- de B. ‘50: $\Lambda \leq 1/2$,
- N. ‘76: $\Lambda > -\infty$.

There is also

N. ‘76 Conjecture: $\Lambda \geq 0$;

i.e., the RH, if true, is only barely so.

\exists series of bounds on $\Lambda$ better than $\Lambda > -\infty$ and $\Lambda \leq 1/2$:

- $\Lambda > -50$ (Csordas-Norfolk-Varga ‘88), ... ,
- $\Lambda > -4.3 \times 10^{-6}$ (Csordas-Smith-Varga ‘94), ... ,
- $\Lambda > -1.1 \times 10^{-11}$ (Saouter-Gourdon-Demichel ‘11);
- $\Lambda < 1/2$ (Ki-Kim-Lee ‘09).

**Proof of N. Conjecture: \( \Lambda \geq 0 \)**

Methods — extend Csordas-Smith-Varga work to study motion in \( t \) of zeros of \( L_{\Phi,t} \).

- New Project (see terrytao.wordpress.com) to improve upper bound \( \Lambda < 1/2 \) of Ki-Kim-Lee: this is Polymath 15 project; as of October 2018: \( \Lambda < 0.22 \) (with possibility of \( \Lambda < 0.11 \); uses Ki-Kim-Lee result that \( \lambda > 0 \Rightarrow \) number of \( L_{\Phi,\lambda} \) zeros off imaginary axis is finite.
Math. Phys. interest starts from the ‘52 Ising model Thm. of Lee and Yang that generates $\rho$’s s.t. $L_{\rho,\lambda}$ is PIZ for $\lambda \geq 0$.

For Euclidean Field Theory, would like $f = e^{-V}$ s.t. $L_{f,\lambda}$ is PIZ also for all $\lambda < 0$; call such an $f$ “perfect”.

Example, Polya ‘20s, Simon-Griffiths ‘73

$e^{-au^4-bu^2}$ for $a > 0$, $b \in \mathbb{R}$ is perfect.

Motivated by $e^{-a \cosh(u)}$, N ‘76 determined all perfect $f$’s; they did not include $e^{-a \cosh(u)}$ or $\Phi$ of RH (which proved $\Lambda > -\infty$).
Theorem A (N., Wei WU '17)

If \( \int e^{\lambda u^2} d\rho = \infty \; \forall \; \lambda > 0 \); then for every \( \lambda < 0 \), \( L_{\rho,\lambda} \) is not PIZ.

Proof is based on a surprising weak convergence result (Thm. B below). (Also a connection to Gaussian Multiplicative Chaos.)
Some related results

Definition

A random variable $X$ is in $\mathcal{L}$ if:

(i) $X \overset{d}{=} -X$, and (ii) $E[e^{bX^2}] < \infty$ for some $b > 0$, and
(iii) $E(e^{\lambda X})$ has only PIZ.

Theorem B (N., WU ‘17)

If each $X_n \in \mathcal{L}$ (with $b = b(X_n)$) and $X_n \overset{d}{\to} X$, then $X \in \mathcal{L}$.

How Th. B $\implies$ Th. A: If conclusion of Th. A not valid, then $\rho_{\lambda_0} \equiv C_{\lambda_0} e^{\lambda_0 u^2} d\rho \in \mathcal{L}$ for some $\lambda_0 < 0$; then by Polya would be in $\mathcal{L}$ $\forall \lambda \in (\lambda_0, 0)$, but $\rho_{\lambda} \to \rho$ as $\lambda \uparrow 0$. So by Th. B, $\rho \in \mathcal{L}$. But $\rho \notin \mathcal{L}$ since by assumptions of Th. A, it doesn’t satisfy (ii).
Proof of Theorem B

Key to the proof of Th. B is a Hadamard factorization:

\[ X \in \mathcal{L} \Rightarrow E(e^{zX}) = e^{Bz^2} \prod_{k}(1 + \frac{z^2}{y_k^2}) \]

with \( B \geq 0, y_k \in \mathbb{R}, \sum 1/y_k^2 < \infty \) and \( E(X^2) = 2(B + \sum 1/y_k^2) \).

Remark about N. ‘76:

A perfect \( f(u) \) must be of form

\[ Ku^{2m}e^{-au^4 - bu^2} \prod (1 + \frac{u^2}{y_k^2})e^{-u^2/y_k^2} \]

with \( \sum 1/y_k^4 < \infty, a > 0, b \in \mathbb{R} \) (or \( a = 0, b + \sum 1/y_k^2 > 0 \)).
Thanks!