Random Walk on Simplicial Complexes

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Simplicial Complexes

Random walks on Simplicial Complexes
How well can the RW mix
High dimensional local spectral expanders
Decomposition Theorems for Random Walks on HD expanders
Simplicial Complexes - Abstract Definition

A $d$-dimensional simplicial complex $X$ is defined as follows:

1. $V$ is a set of vertices
2. For every $-1 \leq k \leq d$, the set of $k$-simplices of $X$, denoted $X(k)$, is a subset of $\binom{V}{k+1}$ and we denote $X = \bigcup_k X(k)$
3. If $\sigma \in X$, then for every $\tau \subseteq \sigma$, $\tau \in X$
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Below $X$ is always assumed to be finite ($|V| < \infty$) and pure $d$-dimensional (every $k$-simplex is contained in a $d$-dimensional simplex).
Geometric interpretation
Define $C^k(X) = \{\phi : X(k) \to \mathbb{R}\}$, e.g., $C^0(X)$ are functions from vertices of $X$ to $\mathbb{R}$.
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Define the following inner-product on $C^k(X)$:

$$\langle \phi, \psi \rangle = \sum_{\eta \in X(k)} w(\eta)\phi(\eta)\psi(\eta),$$

where $w$ is a weight function which "takes into account" the higher dimensional structure (explicitly, $w(\tau) = (d - k)! \sum_{\sigma \in X(d), \tau \subseteq \sigma} w(\sigma), \forall \tau \in X(k)$).
Random Walks on Simplicial Complexes
The $k$-random walk is a random walk on $X(k)$ defined as follows: for $\tau \in X(k)$

1. **Up step**: Choose $\eta \in X(k + 1)$ such that $\tau \subseteq \eta$ at random (according to the weight function $w$)

2. **Down step**: Choose at random $\tau' \in X(k)$ such that $\tau' \subseteq \eta$

We denote by $M_k^+ : C^k(X) \to C^k(X)$ the operator corresponding to this random walk.
Up and Down operators

Define the $Up$ operator $U_k : C^k(X) \rightarrow C^{k+1}(X)$: for $\phi \in C^k(X), \eta \in X(k + 1)$,

$$(U_k \phi)(\eta) = \sum_{\tau \in X(k), \tau \subseteq \eta} \phi(\tau).$$
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Define the $Down$ operator $D_{k+1} : C^{k+1}(X) \to C^k(X)$: for $\psi \in C^{k+1}(X), \tau \in X(k)$,

$$(D_{k+1} \psi)(\tau) = \sum_{\eta \in X(k+1), \tau \subseteq \eta} \frac{w(\eta)}{w(\tau)} \psi(\eta).$$
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$U_k^* = D_{k+1}$, $M_k^+ = \frac{1}{k+2} D_{k+1} U_k$
The 0-random walk in graphs

Assume that $X$ is a regular graph. What is $M_0^+$?
The 0-random walk in graphs

Assume that $X$ is a regular graph. What is $M_0^+$?

1. Up step

2. Down step

Note: This is not the usual random walk, but a lazy RW (has probability 0.5 to stay at the vertex).
Motivating questions

Note:

- $M_k^+ 1 = 1$.
- $M_k^+$ is self-adjoint and all its eigenvalues are in $[0, 1]$. 
- Under mild connectivity conditions on $X$, every eigenfunction $\phi \perp 1$ has eigenvalue $< 1$. 

Questions:

1. Can we bound the second largest eigenvalue of $M_k^+ + k$, in other words, can we find $\mu$ such that for all $\phi \perp 1$, $\langle M_k^+ \phi, \phi \rangle \leq \mu \| \phi \|_2$?

2. What can we say about $\langle M_k^+ \phi, \phi \rangle$ for a specific $\phi$ beyond the bound on the second eigenvalue?
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- \( M_k^+ \mathbb{1} = \mathbb{1} \).
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- Under mild connectivity conditions on \( X \), every eigenfunction \( \phi \perp \mathbb{1} \) has eigenvalue \(< 1\).

Questions:

1. Can we bound the second largest eigenvalue of \( M_k^+ \), in other words, can we find \( \mu \) s.t. for all \( \phi \perp \mathbb{1} \),
\[
\langle M_k^+ \phi, \phi \rangle \leq \mu \| \phi \|^2?
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How well can the RW mix
How well can the RW mix? (1)
How well can the RW mix? (2)

\[ M_1^+ \phi = \]

\begin{array}{cccccc}
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{3}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{3}{3} & \frac{2}{3} & \frac{1}{3} \\
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\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\
\end{array}
How well can the RW mix? (3)

\[
\phi = \\
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

\[
M_1^+ \phi = \\
\begin{bmatrix}
-1 & -1 & 0 \\
0 & -1 & -1 \\
0 & 0 & -1 \\
\end{bmatrix}
\]

\[
\langle M_1^+ \phi, \phi \rangle = \frac{2}{3} \| \phi \|^2
\]
Observe that the obstruction to $\frac{1}{3}$-mixing came from “below”: $\phi = U_0 \psi$ where $\psi$ is 1 on one vertex and $-1$ on the other (0 everywhere else)
Observe that the obstruction to $\frac{1}{3}$-mixing came from “below”: $\phi = U_0 \psi$ where $\psi$ is 1 on one vertex and $-1$ on the other (0 everywhere else).

\[ \psi = \quad \vdots \quad \cdots\]

This is a general phenomenon: in general, for $k > 0$, in the $k$-walk we should expect to see $\frac{2}{k+2}, \ldots, \frac{k+1}{k+2}$ “obstructions” coming from dimensions $k - 1, \ldots, 0$. 
High dimensional local spectral expanders
Given a simplex $\tau \in X$, the \textit{link} of $\tau$ is the subcomplex of $X$, denoted $X_\tau$ and defined as

$$X_\tau = \{ \sigma \in X : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in X \}$$
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1-Skeleton

The 1-Skeleton of a complex is the graph $(X(0), X(1))$: 
High dimensional local spectral expanders - definition

For a constant $0 < \lambda < 1$, $X$ is called a one-sided (two sided) $\lambda$-local spectral expander if:

1. The 1-skeleton of $X$ is connected and normalized spectrum of the 1-skeleton of $X$ is contained in $[-1, \lambda] \cup \{1\}$ (two-sided: $[-\lambda, \lambda] \cup \{1\}$).

2. For every $\tau \in X(k), k < d - 1$, 1-skeleton of $X_\tau$ is connected and normalized spectrum of the 1-skeleton of $X_\tau$ is contained in $[-1, \lambda] \cup \{1\}$ (two-sided: $[-\lambda, \lambda] \cup \{1\}$).

Normalized spectrum = normalized according to the weight function $w$. 
Local spectral expansion can be deduces “very” locally

Theorem (O.): If $X$ and all the links (of dim. $\geq 1$) are connected and the second e.v. for all the 1-dimensional links is $\leq \frac{\lambda}{1+(d-1)\lambda}$, then $X$ is $\lambda$-local spectral expander.
Local spectral expansion can be deduced “very” locally.

Theorem (O.): If $X$ and all the links (of dim. $\geq 1$) are connected and the second e.v. for all the 1-dimensional links is $\leq \frac{\lambda}{1+(d-1)\lambda}$, then $X$ is $\lambda$-local spectral expander.

If in addition the smallest e.v. all the 1-dimensional links is $\geq \frac{-\lambda}{1+(d-1)\lambda}$, then $X$ is a two-sided $\lambda$-local spectral expander.
Previous work on high order walks

- First introduced by Kaufman and Mass, who studied it for ONE sided local spectral expanders; they got $1 - \frac{1}{(k+2)^2} + f(\lambda, k)$ on second e.v of $M_k^+$.  
- Later improved by Dinur and Kaufman who studied it for TWO sided local spectral expanders; they $1 - \frac{1}{k+2} + O(\lambda(k+1))$ on second e.v of $M_k^+$; This was useful for agreement expansion questions.
Decomposition Theorems for Random Walks on HD expanders.

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Decomposition Theorems for Random Walks on HD expanders
If $X$ is $\lambda$-local spectral expander and $\phi \in C^k(X)$, $\phi \perp 1$, then

1. $\phi$ can be “projected” on $C^i(X)$, $0 \leq i \leq k$
2. These projections control how well the random walk mixes: the more $\phi$ is concentrated at the higher dimensions, the faster the mixing.
Decomposition Theorem - exact formulation

**Main Theorem:** Let $X$ be a $\lambda$-local spectral expander and $0 \leq k \leq d - 1$ constant. For any $\phi \in C^k(X)$, $\phi \perp 1$ there are $\phi^k \in C^k(X), \phi^{k-1} \in C^{k-1}(X), \ldots, \phi^0 \in C^0(X)$ such that

$$\phi^k \perp 1, \ldots, \phi^0 \perp 1,$$

$$\|\phi\|^2 = \|\phi^k\|^2 + \|\phi^{k-1}\|^2 + \ldots + \|\phi^0\|^2,$$

$$\langle M_k^+ \phi, \phi \rangle \leq \sum_{i=0}^{k} \left( \frac{k+1-i}{k+2} + \lambda f(k, i) \right) \|\phi^i\|^2,$$

$$(f(k, i) = \frac{(k+i+2)(k+1-i)}{2(k+2)}).$$
Bound on the second eigenvalue

\[ \langle M_k^+ \phi, \phi \rangle \leq \sum_{i=0}^{k} \left( \frac{k + 1 - i}{k + 2} + O((k + 1)\lambda) \right) \| \phi^i \|^2. \]

When \( \lambda \) is small, we note that the coefficients of the \( \| \phi^i \| \)'s in the sum above become larger as \( i \) becomes smaller. Therefore, the “worst case scenario” is when \( \| \phi \|^2 = \| \phi^0 \|^2. \)
Bound on the second eigenvalue

\[ \langle M_k^+ \phi, \phi \rangle \leq \sum_{i=0}^{k} \left( \frac{k+1-i}{k+2} + O((k+1)\lambda) \right) \| \phi^i \|^2. \]

When \( \lambda \) is small, we note that the coefficients of the \( \| \phi^i \| \)'s in the sum above become larger as \( i \) becomes smaller. Therefore, the “worst case scenario” is when \( \| \phi \|^2 = \| \phi^0 \|^2 \). In that case

\[ \langle M_k^+ \phi, \phi \rangle \leq \left( \frac{k+1}{k+2} + \lambda \frac{k+1}{2} \right) \| \phi \|^2, \]

and therefore the second eigenvalue is bounded by \( \frac{k+1}{k+2} + \lambda \frac{k+1}{2} \).
A more explicit decomposition for 2-sided $\lambda$-local spectral expanders

(Inspired by Dikstein, Dinur, Filmus and Harsha)

Assuming 2-sided $\lambda$-local spectral gap:

- The non-trivial spectrum of $M_k^+$ is contained in $\left[\frac{1}{k+2} - f(k)\lambda, \frac{1}{k+2} + f(k)\lambda\right] \cup \ldots \cup \left[\frac{k+1}{k+2} - f(k)\lambda, \frac{k+1}{k+2} + f(k)\lambda\right]$.

- The eigenspaces are $O(\lambda)$-approximated by the $U$ operators images.
Some words about the proofs (if time permits)
Thank you for listening