Progress and Problems in Discrepancy

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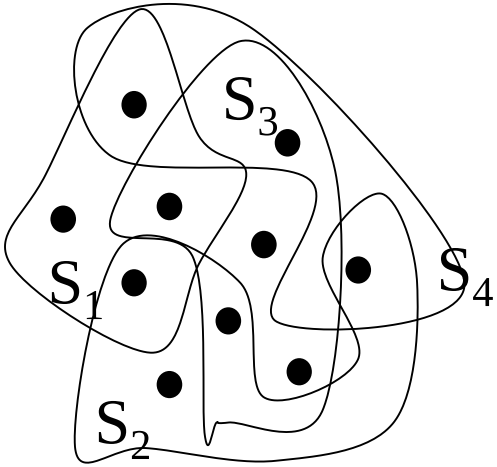
(CWI and Eindhoven)

Discrepancy



Universe: U= [1,…,n]
Subsets: $S_1, S_2, \ldots, S_m$

Color elements red/blue so each set is colored as evenly as possible.

Given χ: $[n] \rightarrow \{ -1,+1\}$
Disc $(\chi) = \max_S |\sum_{i \in S} \chi(i)| = \max_S |\chi(S)|$

Disc (set system) $= \min_\chi \max_S |\chi(S)|$

Capture various properties of the set system.
Lots of questions/applications in various areas.

Discrepancy

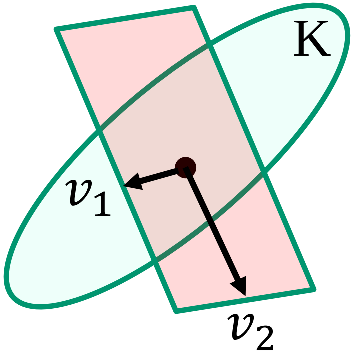
Given an $m \times n$ matrix A,
find $x \in \{-1,1\}^n$, to minimize
disc(A) = $|Ax|_\infty$

$$\text{Incidence matrix } A = \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

Rows: sets
Columns: elements

Vector balancing view: Given vectors $v_1, \ldots, v_n \in R^m$
find $x \in \{-1,1\}^n$ to minimize $\left|\sum_i x_i v_i\right|_\infty$

Can also consider more general norms
K: symmetric convex body
Find $x \in \{-1,1\}^n$ to minimize $\left|\sum_i x_i v_i\right|_K$



Discrepancy: All about beyond the probabilistic method

Two problems: Spencer's setting, Komlos' problem
Open questions (could geometry of polynomials help?)

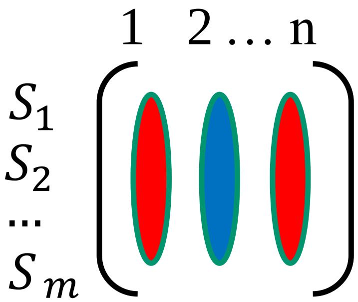
Classical methods from discrepancy:
Partial Coloring Method, Banaszczyk's method
(Non-constructive, argue about all colorings simultaneously)

Recent algorithmic approaches.
New algorithmic ways to go beyond the probabilistic method

Can they help for Kadison Singer, applications of interlacing poly?

Two examples

Spencer Setting: Discrepancy of any set system on n elements and m sets?



[Spencer'85]: (independently by Gluskin'87)

For m = n discrepancy $\leq 6n^{1/2}$

Tight: Cannot beat $0.5\ n^{1/2}$ (Hadamard Matrix).

Random coloring gives $O(n \log n)^{1/2}$

Proof: For set S, Pr [disc(S) $\approx c|S|^{1/2}$] $\approx \exp(-c^2)$

Set c = $O(\log n)^{1/2}$ and apply union bound.

Tight. Random gives $\Omega(n \log n)^{1/2}$ with very high prob.

Spencer setting

More generally: $\mathrm{O}\left(n^{1/2} \log^{1/2}\left(\frac{m}{n}\right)\right)$ for m sets, n elements

Random: $(n \log m)^{1/2}$

Nothing special about 0/1. e.g. $|a_{ij}| \leq 1$ also fine.

Invented the Partial Coloring Method (a key tool in discrepancy)

Open Problem: Matrix Spencer

Given (symmetric) $A_1, \ldots, A_n$ with spectral norm ≤ 1

Is there a signing s.t. $\left|\sum_i x_i A_i\right| = O\left(n^{1/2}\right)$

For simultaneously diagonalizable matrices, follows from Spencer.

Komlos Problem

Given vectors $v_1, \dots, v_n \ \in R^m$ with $|v_i|_2 \leq 1$

find a signing to minimize $\left|\sum_i x_i v_i\right|_\infty$

Random coloring gives $\Omega\left(n^{1/2}\right)$

E.g. If m=1, have $v_i \in [-1,1]$

Partial Coloring: O(log n)

Banaszczyk: $O((\log n)^{1/2})$

Conjectured bound: O(1)

Beck Fiala Conjecture

Discrepancy of low degree set systems, where each element lies in at most t sets? (i.e. 0-1 matrix where each column has $\leq t$ 1's).

Scaling by $\frac{1}{t^{1/2}}$ gives unit columns

Random: $\Omega(n^{1/2})$ (a row could have n 1's)

Beck-Fiala'81: $2t-1$, 2t – log^* t [Bukh'16]

Banaszczyk'97: $t^{1/2}(\log n)^{1/2}$

Conjecture : $O(t^{1/2})$

Non-constructive methods

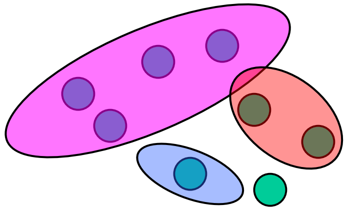
1) Partial Coloring Method:

Beck/Spencer early 80's: Probabilistic Method + Pigeonhole

Gluskin'87: Convex Geometric Approach

Very versatile

Loss adds over O(log n) iterations



2) Banaszczyk'98: Based on a deep convex geometric result

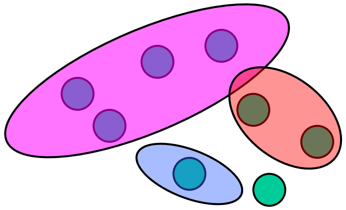
Produces full coloring directly

Spencer's $O(n^{1/2})$ result

Partial Coloring suffices: For any set system with m sets, there exists a coloring on $\geq$ n/2 elements with discrepancy $O(n^{1/2} \log^{1/2} (2m/n))$ [For m=n, disc = $O(n^{1/2})$]

Algorithm for total coloring:

Repeatedly apply partial coloring lemma



Total discrepancy

$O(n^{1/2} \log^{1/2} 2)$ [Phase 1]

$+ O((n/2)^{1/2} \log^{1/2} 4)$ [Phase 2]

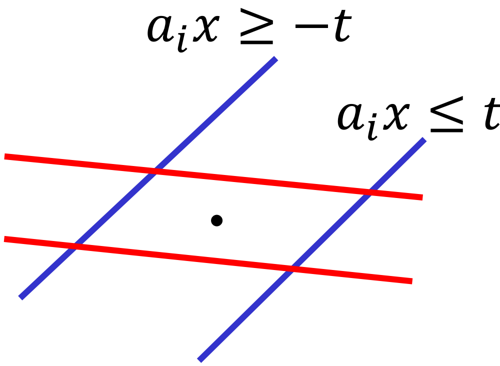
$+ O((n/4)^{1/2} \log^{1/2} 8)$ [Phase 3]

$+ \ldots$ $= O(n^{1/2})$

A geometric view

Spencer’85: Any 0-1 matrix (n x n) has disc ≤ $6\sqrt{n}$

Gluskin’87: Convex geometric approach



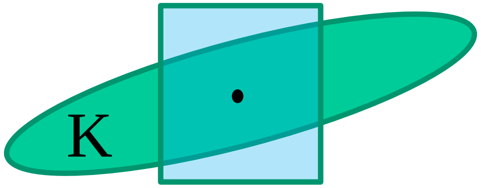
Consider polytope P(t) = $-t\,\mathbf{1} \leq Ax \leq t\,\mathbf{1}$

P(t) contains a point in $\{-1,1\}^n$ for t = $6\sqrt{n}$

Gluskin’87: If K symmetric, convex with large (Gaussian) volume ($> 2^{-n/100}$) then K contains a point with many coordinates {-1,+1}

d-dim Gaussian Measure: $\gamma_d(x) = \exp\left(-|x|^2/2\right)(2\pi)^{-d/2}$

$\gamma_d(K)$: Pr[$(y_1, \dots, y_m) \in K$] each y_i iid N(0,1)

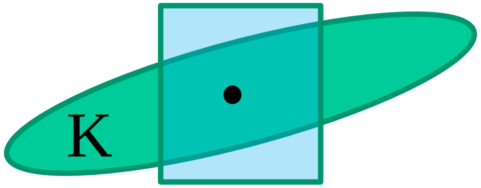


$[-1,1]^n$ cube

What is the Gaussian volume of $[-1,1]^n$ cube

A geometric view

Gluskin’87: If K symmetric, convex with large (Gaussian) volume ($> 2^{-n/100}$) then K contains a point with many coordinates {-1,+1}



Similar to Minkowski’s theorem:
K symmetric has a non-zero point in Z^n, if Vol(K) > 2^n

Proof: Look at K+x for all $x \in \{-1,1\}^n$
Total volume of shifts = $2^{\Omega(n)}$

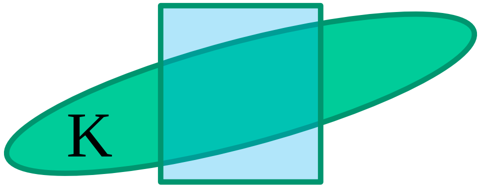
$$\gamma_n(K+x) \geq \gamma_n(K) \exp\left(-|x|^2/2\right)$$

Some point z lies in $2^{\Omega(n)}$ copies

$z = k + x$ and $z = k' + x'$ where x, x' have large hamming distance
Gives $(x - x')/2 = (k - k')/2 \in K$.

Gluskin for Polytopes

Gluskin'87: If K symmetric, convex with large (Gaussian) volume ($> 2^{-n/100}$) then K contains a point with many coordinates {-1,+1}



Spencer's result proof:

Consider polytope P(t) $= \; -t\,\mathbf{1} \; \leq Ax \; \leq t\,\mathbf{1}$

Show Gaussian volume large enough for t $= c\sqrt{n}$

Sidak's Thm: $\gamma_n(K) \geq \Pi_i\, \gamma_n(Slab_i)$ $\qquad Slab_i = \; -t \; \leq \; a_i x \; \leq t$

Thm: Given an m x n matrix A, there is a partial coloring satisfying

$|a_i x| \leq \; \lambda_i \; |a_i|_2$ for each row i, provided $\sum_i e^{-\lambda_i^2} \; \leq \frac{n}{5}$

Comparison w/ random coloring

Given an m x n matrix A, there is a partial coloring satisfying $|a_i x| \leq \lambda_i \, |a_i|_2$ for each row i, provided $\sum_i e^{-\lambda_i^2} \leq \frac{n}{5}$

Can view as extending Chernoff bounds

1) n/5 vs 1 (Chernoff)

2) Partial Coloring vs Full (Chernoff)

E.g. Can get 0 discrepancy on n/10 rows (very powerful)

Key tool in most discrepancy problems

Application: Komlos

Claim: Get partial coloring with O(1) discrepancy.

Assume $n \leq m$ (linear algebraic argument)

For each column j, $\sum_i a_{ij}^2 \leq 1$

Sum of a_{ij}^2 over all matrix entries $\leq n$

Average sum per row $\leq n/m \leq 1$.

Call a row i big if $\sum_j a_{ij}^2$ > 10. At most n/10 of these.

Set λ_i=0 for big rows. Else λ_i= O(1).

Gluskin: $|a_i x| \leq \lambda_i |a_i|_2$ for each row i, provided $\sum_i e^{-\lambda_i^2} \leq \frac{n}{5}$

Annoying loss of O(log n)
to get full coloring

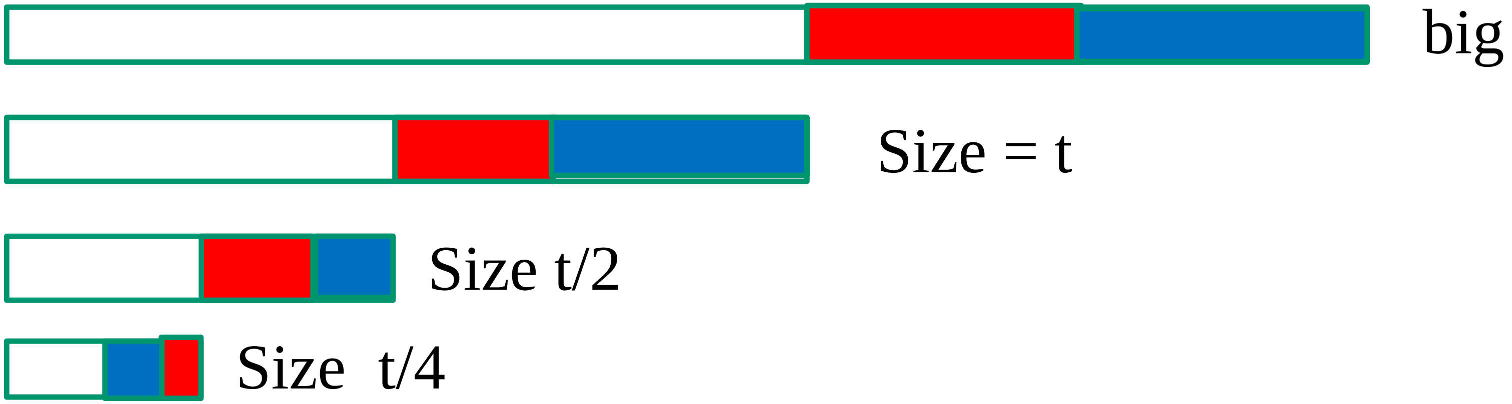
Ideal case

Beck-Fiala Setting: At most n/10 big (>10t) sets

Partial Coloring: 0 for big sets.

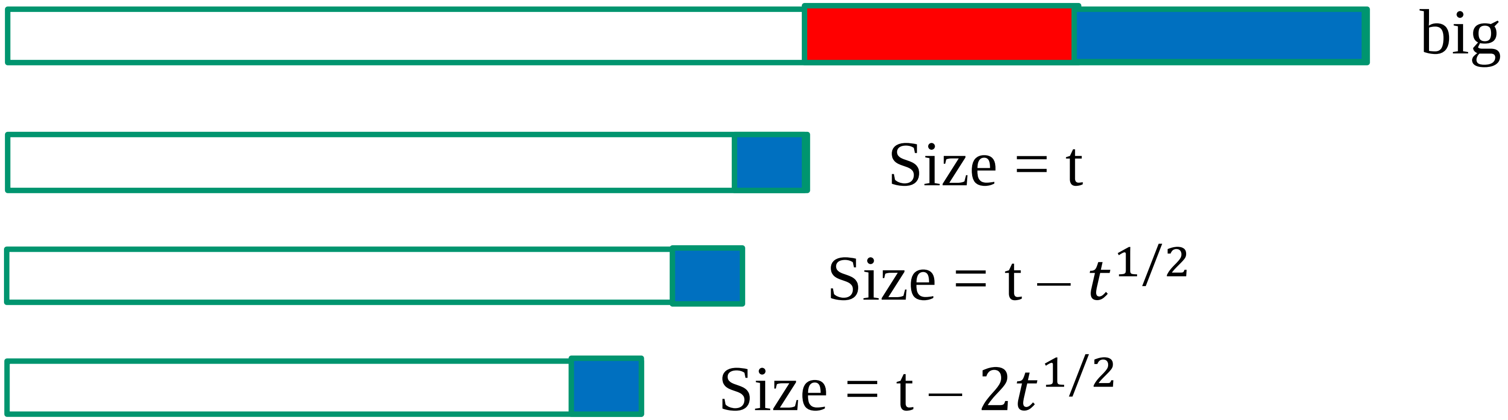
About $s^{1/2}$ for small sets of size s.

“Ideal” life cycle of a set



Ideal case: Discrepancy $= t^{1/2} + (t/2)^{1/2} + (t/4)^{1/2} + \ldots$

What can go wrong



Trouble: A set can get $t^{1/2}$discrepancy, but very few elements colored.

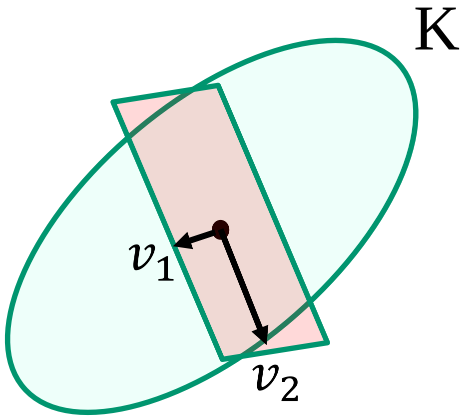
Banaszczyk’s method $O(\log^{1/2} n)$ for Komlos

Banaszczyk's Theorem

Thm: Let A have columns $v_1, \ldots, v_n \in R^m$, $|v_i|_2 \leq 1/5$

K = symmetric convex body with $\gamma_m(K) \geq \frac{1}{2}$

$\exists\ x \in \{-1,1\}^n$ s.t. $Ax \in K$

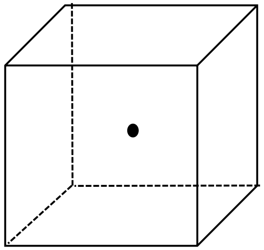


Constants somewhat arbitrary

For non-symmetric K, need

$\gamma_m(K) >$ ½ to ensure $0 \in K$ (e.g. if halfspace)

Banaszczyk's Theorem



Cube: K = $O(\log m)^{1/2}\ [-1,1]^m$ $\gamma_m(K) \geq 1/2$

Komlos: Given unit vectors in R^m,

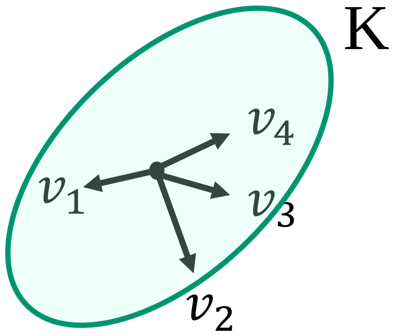
∃ signed sum w/ ℓ_∞-norm $O(\log m)^{1/2}$

Surprising results for various bodies K.

Proof idea

Given $v_1, \dots, v_n$, each $|v_i| < 1/5$. $\gamma_m(K) \geq \frac{1}{2}$

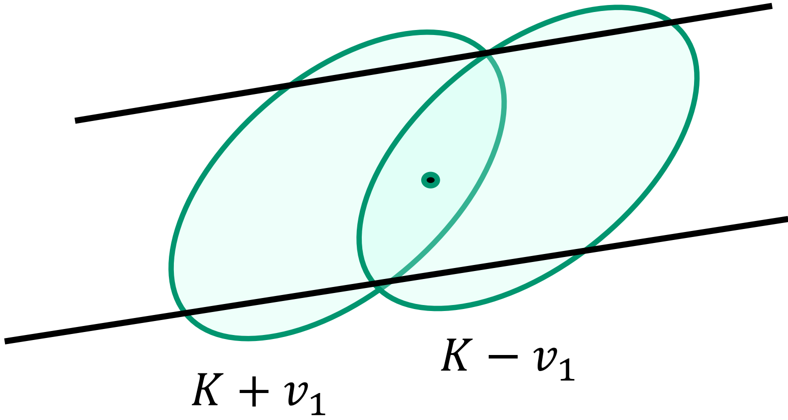
Goal: Find signing $\sum_i x_i v_i \in K$



Key observation: Signing exists iff

Some signing of $v_2, \dots, v_n$ with sum in

$(K + v_1) \cup (K - v_1)$.



Convexify:

Remove regions of K width $< 2|v_1|$ along v_1

Lose and gain volume.

(non-trivial) computation to show volume stays ≥ ½

Algorithmic history

Partial Coloring now constructive

Bansal’10: SDP + Random walk

Lovett Meka’12: Random walk + linear algebra

Rothvoss’14: Convex geometric

Many others by now [Harvey, Schwartz, Singh], [Eldan, Singh], [Lee], …

Banaszczyk based approaches:

[B., Dadush, Garg’16]: $O(\log n)^{1/2}$ algorithm for Komlos problem

[B., Dadush, Garg, Lovett 18]: algorithm for general Banaszczyk.

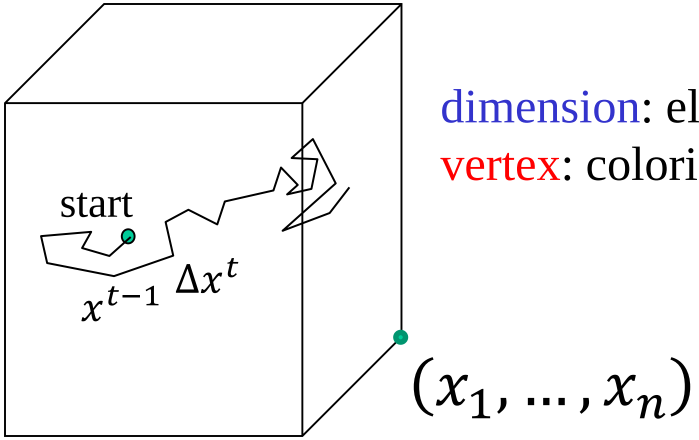
Useful View

Independent rounding.

A (complicated) view

Brownian motion in cube.

Cube: $\{-1,+1\}^n$



dimension: element

vertex: coloring

Same as randomized rounding

Each coordinate rounded independently

(martingale property of the walk)

Useful View

If additional constraints.

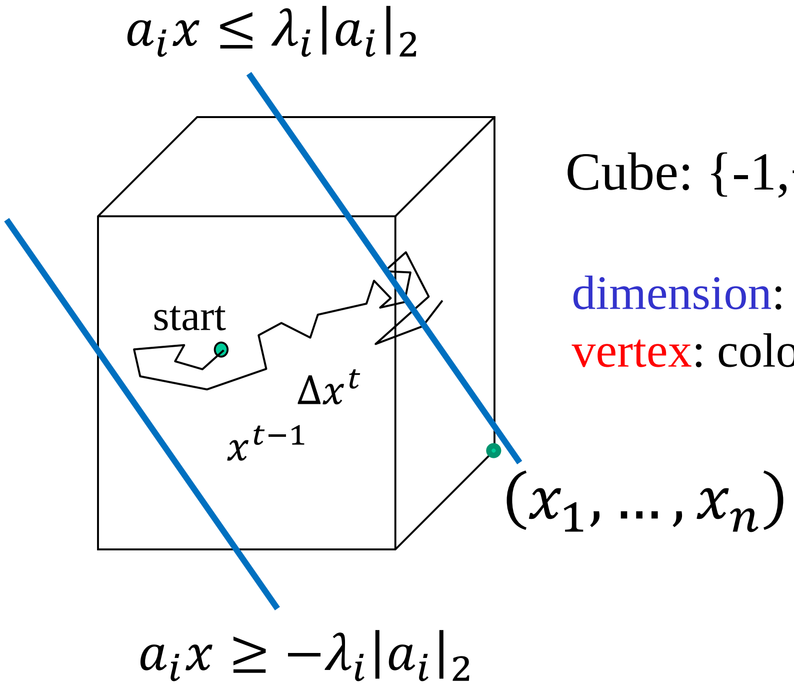
Can tailor walk accordingly.

Pick covariance matrix for Δx^t

(slow down towards bad regions)

Design barrier functions

…



Cube: {-1,+1}^{n}

dimension: element

vertex: coloring

Lovett Meka Algorithm

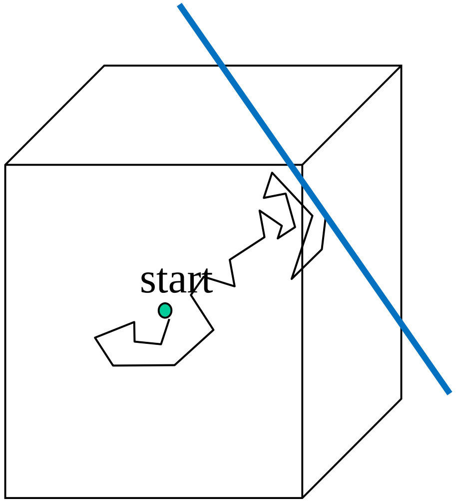
Random walk, γ N(0,1) in each dimension

a) Fix j if $x_j = \pm 1$

b) If row a_i gets tight (disc(a_i) = $\lambda_i |a_i|_2$)

Move in subspace $a_i\mathrm{x} = \lambda_i |a_i|_2$

(not violate discrepancy)



Thm: Given an m x n matrix A, finds a partial coloring satisfying

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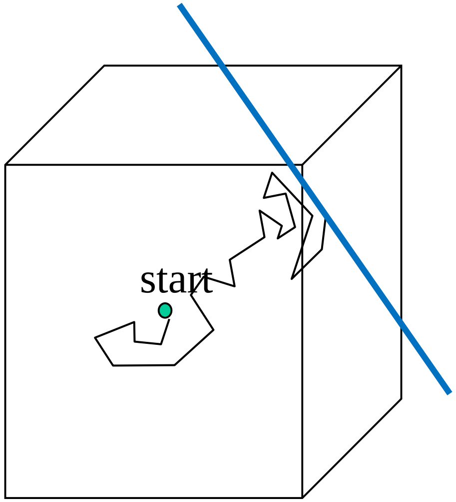
Lovett Meka Algorithm



Random walk, γ N(0,1) in each dimension

a) Fix j if $x_j = \pm 1$

b) If row a_i gets tight (disc(a_i) $= \lambda_i |a_i|_2$)

Move in subspace $a_i x = \lambda_i |a_i|_2$

(not violate discrepancy)

Idea: Walk makes progress as long as dimension = $\Omega(n)$

($E\left[\sum_i x_i^2\right]$ rises by $\Omega(n)\gamma^2$ per step)

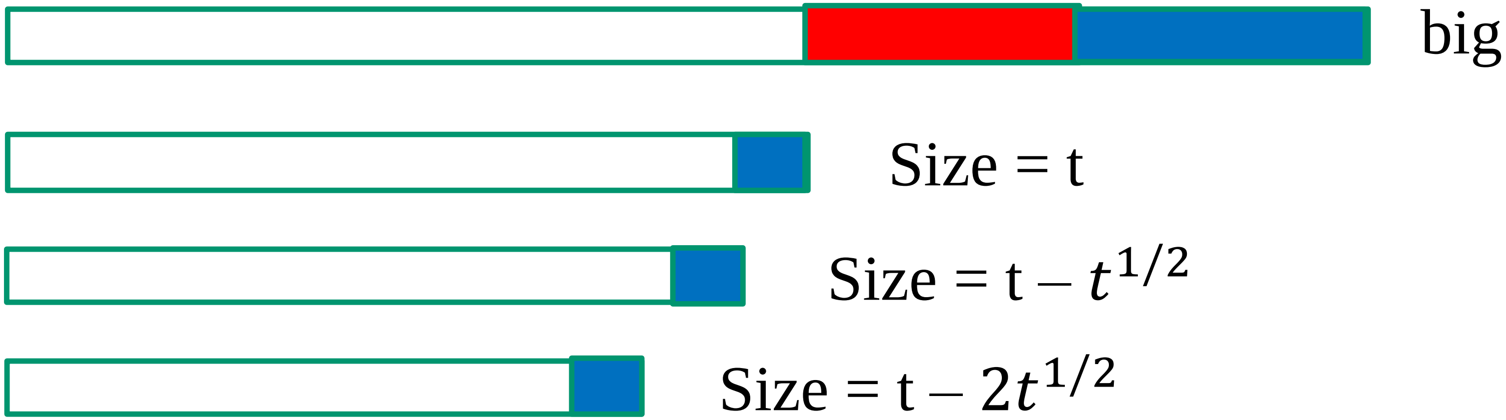
After $\frac{10}{\gamma^2}$ steps: Pr[Row a_i tight] $\approx \exp(-\lambda_i^2)$

As $\sum_i exp\left(-\lambda_i^2\right) \leq \frac{n}{5}$ so n/5 tight rows in expectation

As stays in cube, $\Omega(n)$ variables must have hit ± 1,

Recall trouble with Partial Coloring

Beck Fiala Setting



big

Size = t

Size = t – $t^{1/2}$

Size = t – $2t^{1/2}$

Trouble: A set can get $t^{1/2}$discrepancy, but very few elements colored.

Correlations in Lovett-Meka

Consider set S = {1,2,…,t}

Ideal case: If randomly color each element

Progress = t   discrepancy ≈ $t^{1/2}$

Suppose move in subspace $x_1 = x_2 = \cdots = x_t$

E.g. if have constraints x_1 - x_2 = 0, x_2 - x_3= 0, …

Can only color all +1 or all -1.

Progress = t   discrepancy = t

In Lovett-Meka, such sets hit subspace at $t^{1/2}$ discrepancy, but progress is only $t^{1/2}$

Suggests a solution

Used to get an algorithmic $O\left(\log^{1/2} n\right)$ bound for Komlos [B., Dadush, Garg'16]

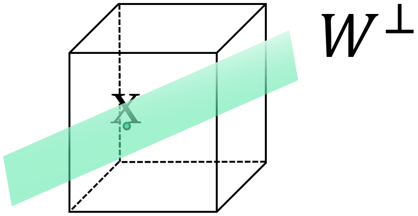
Can we design a walk that moves in some subspace, but still looks "random" enough?

E.g. If constrained to move in subspace $x_1 = x_2 = \cdots = x_t$

Just set $\Delta x_i = 0$ for i=1,2,..,t
Can still do a random walk for i = t+1,..,n.

Better covariance matrices

W: arbitrary subspace dim(W) $\leq (1-\delta)n$
Need to walk in $W^\perp$



-1/+1 cube

Property 1: $w^T(\Delta x) = 0 \quad \forall w \in W$

$$E\left[\, w^T \Delta x \, \Delta x^T w\right] = 0 \quad \text{or} \quad w^T Y w \; = \; 0$$

Covariance matrix
$Y(i,j) = E\left[\Delta x_i, \Delta x_j\right]$

Property 2: Still looks almost independent.

For any direction $c = (c_1, \ldots, c_n)$

$$E[\left(\sum_i c_i \Delta x_i\right)^2] \leq \frac{1}{\delta} \sum_i c_i^2 \, E\left[\, \Delta x_i^2\right]$$

$$c^T Y\, c \leq \left(\frac{1}{\delta}\right) c^T diag(Y)\, c \qquad \forall c \in R^n.$$

$$Y \preccurlyeq \left(\frac{1}{\delta}\right) diag(Y)$$

Can find such a good walk

Key Thm: If $\dim(W) \leq (1 - \delta)n$

There is a non-zero solution Y to the SDP

$$w^T Y w \; = \; 0 \qquad \forall w \; \in W$$

$$Y \preccurlyeq \left(\frac{1}{\delta}\right) diag(Y)$$

$$Y \succcurlyeq 0$$

Proof: Using SDP duality

Algorithm for Komlos

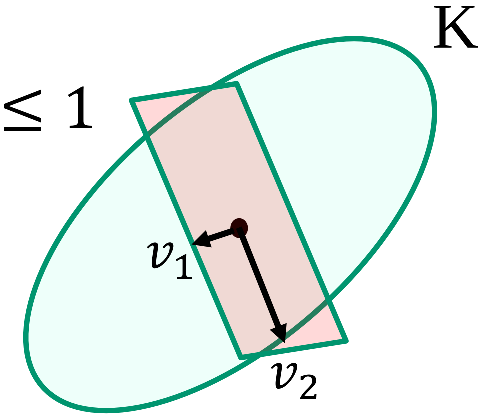
Time t: If n_t variables alive, at most $n_t/10$ big rows
Pick W = span of these constraints.

Run the SDP walk.
No phases, continue till all variables -1/+1 (i.e. $n_t = 0$).

If row big = discrepancy 0
When becomes small, just like a random walk.

"Freedman type" martingale analysis (avoid dependence on time steps), gives the result.

Making Banaszczyk Algorithmic



Thm [Banaszczyk 97]: Input $v_1, \ldots, v_n \in R^d$, $|v_i|_2 \leq 1$

$\forall$ convex body K, with $\gamma_d(K) \geq \frac{1}{2}$

$\exists$ coloring $x \in \{-1,1\}^n$ s.t. $\sum_i x(i) v_i \in 5K$

Coloring depends on the convex body K.

How is K specified? (input size could be exponential)

Idea [Dadush, Garg, Lovett, Nikolov'16]: Minimax Thm. (2-player game)

Universal distribution on colorings that works for all convex bodies

Equivalent formulation

Alternate formulation [Dadush, Garg, Lovett, Nikolov’16]:

∃ distribution on colorings $x \in \{-1,1\}^n$,

s.t. Y = $\sum_i x(i) v_i$ is $\underbrace{\approx N(0,1) \text{ in every direction}}_{O(1)\text{ subgaussian}}$

No body K anymore

$Y \in R^d$ is σ-subgaussian if in all directions $\theta \in R^d, |\theta|_2 = 1$,

$\langle \theta, Y \rangle$ has same tails as $N(0, \sigma^2)$ i.e. $\Pr[\, |\langle \theta, Y \rangle| \geq \lambda] \leq 2 \exp(-\lambda^2/2\sigma^2)$

Lemma: $Y \in K$ (for K convex, $\gamma_d(K) \geq \frac{1}{2}$) with constant prob.

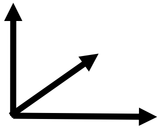
Suffices to sample x implicitly from such a distribution.

Goal: ∃ distribution on colorings $x \in \{-1,1\}^n$,
s.t. random vector $Y = \sum_i x(i) v_i$ is O(1) subgaussian

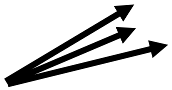
$\forall \theta \in S^{m-1}$, $\langle Y, \theta \rangle = \sum_i x(i) \langle v_i, \theta \rangle$ decays like N(0,1).

Special cases:

1) v_i are Orthogonal: Random ± coloring x_i works
As $\sum_i c_i x_i \approx N\left(0, \sum_i c_i^2\right)$
Var$(\langle Y, \theta \rangle) = \sum_i \langle v_i, \theta \rangle^2 \leq |\theta|^2 \leq 1$

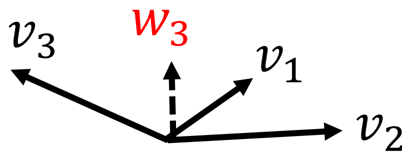


2) All equal vectors



$v_1 = \cdots = v_n = v$ random coloring bad: $\Omega\left(\sqrt{n}\right)$ in direction v
Need dependent coloring: n/2 +1's and n/2 -1's

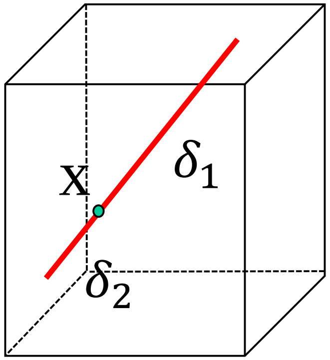
Gram Schmidt Walk



Algorithm: Consider vectors $v_1, \ldots, \boxed{v_n}$

Write $v_n = c_1 v_1 + \ldots c_{n-1} v_{n-1} + w_n$

where $w_n \in span\,(v_1, \ldots, v_{n-1})^{\perp}$



Let direction $c = (c_1, \ldots, c_{n-1}, -1)$

Update coloring x as δc s.t. $E[\delta] = 0$

i.e. $\Delta x = +\delta_1 c$ or $-\delta_2\, c$

Key Point: $\Delta Y = \sum_i \Delta x(i)\, v_i = \delta(\sum_{i=1}^{n-1} c_i v_i - v_n) = -\delta w_n$.

As $\delta \leq 2$ and $E[\delta] = 0$

$\Delta\langle Y, \theta\rangle$ evolves as a martingale with variance $O(\langle \theta, w_n\rangle^2)$

Proof Idea (ideal case)

$v_1, \dots, \enclose{circle}{v_n}$ Suppose pivot is the one to freeze every time

Pivot v_n ΔY: $\delta_n w_n$

Pivot v_{n-1} ΔY: $\delta_{n-1}\, w_{n-1}$

….

$w_1, \dots, w_n$ obtained by Gram Schmidt process.

$w_1 = v_1$ $\hat{w}_1 = w_1 / |w_1|$

$w_2 = v_2 - \langle v_2, \hat{w}_1 \rangle\, \hat{w}_1$ $\hat{w}_2 = w_2 / |w_2|$

$w_3 = v_3 - \langle v_3, \hat{w}_1 \rangle\, \hat{w}_1 - \langle v_3, \hat{w}_2 \rangle\, \hat{w}_2$ $\hat{w}_3 = w_3 / |w_3|$

$$Y = \delta_n w_n + \delta_{n-1} w_{n-1} + \cdots + \delta_1\, w_1$$

$$Var\,(\langle Y, \theta \rangle) = \sum_i \delta_i^2 \langle w_i, \theta \rangle^2 \leq \sum_i \delta_i^2\, \langle\, \hat{w}_i, \theta\, \rangle^2 \leq 4|\theta|^2 = 4$$

Some more details

$v_1, \dots, \cancel{v_5}, \dots, \enclose{circle}{v_n}$ No reason why pivot should get fixed.

Suppose v_5 gets fixed.

w_n becomes w_n' which can be longer.

Proof idea: Can charge increase in $|w_n|^2$ to v_5 disappearing.

Track evolution of $E\left[e^{\lambda\langle\theta,Y\rangle}\right]$ by a suitable potential

and show $E\left[e^{\lambda\langle\theta,Y\rangle}\right] = e^{O\left(\lambda^2\right)}$ for each θ, λ

(Recall Z is σ-subgaussian iff $E\left[e^{\lambda Z}\right] = e^{O\left(\lambda^2\sigma^2\right)}$ for all λ)

Concluding remarks

Besides Matrix Spencer and Komlos conjecture, many problems in discrepancy still open

(Steinitz problem, Tusnady's problem, … $(\log n)^{1/2}$ gap)

Lots of progress on lower bounds (SDP duality, convex geometry)

[Rothvoss'14] Algorithm for Gluskin (general convex bodies)

[Nikolov, Talwar'15] Approximating hereditary discrepancy

Various new uses in algorithm design, beating "union bound"

Bin-packing [Rothvoss'13], iterated + randomized rounding [B.'19]

Questions!