AN UNDETERMINED MATRIX MOMENT PROBLEM AND ITS APPLICATION TO COMPUTING ZEROS OF L-FUNCTIONS

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COMPLEXITY OF COMPUTING ZEROS

π AN AUTOMORPHIC CUSP FORM ON $GL_m/\mathbb{Q}$ $(m=2)$.

$L(s,\pi)$ ITS STANDARD L-FUNCTION

$L(s,\pi_p)$ = LOCAL FACTOR AT p

$$L(s,\pi) = \prod_{p<\infty} L(s,\pi_p)$$

$$\Lambda(s,\pi) = L(s,\pi_\infty)L(s,\pi)$$

ANALYTIC CONTINUATION AND FUNCTIONAL EQ^n:

$$\Lambda(1-s,\pi) = W(\pi)\, N_\pi^{s-1/2}\, \Lambda(s,\pi)$$

$\pi = \tilde{\pi}$ (ASSUMPTION) SELF DUAL

$W(\pi) = \pm 1$: ROOT NUMBER

N_π IS THE CONDUCTOR, IT IS A PRODUCT OVER PRIMES AT WHICH π IS RAMIFIED.

N_π MEASURES THE COMPLEXITY OF π;

WANT TO COMPUTE ZEROS OF $L(s,\pi)$ NEAR ½.

EXPLICATE WITH ELLIPTIC CURVES

WEISTRASS FORM:

$$E: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \quad (1)$$

$a_j \in \mathbb{Z}$. CORRESPONDING INVARIANTS ARE

$$b_2 = a_1^2 + 4a_2,\ b_4 = a_1 a_3 + 2a_4,\ b_6 = a_3^2 + 4a_6,\ b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2$$

$$c_4 = b_2^2 - 24 b_4 \qquad c_6 = -b_2^3 + 36 b_2 b_4 - 216 b_6$$

DISCRIMINAT $\quad \Delta = -b_2^2 b_8 - 8 b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6 \quad$ — (2)

WE ASSUME THAT $(c_4, c_6) = 1$ AND $(c_4, 6) = 1$ SO THAT E HAS ONLY MULTIPLICATIVE BAD REDUCTION $\Rightarrow$ N_E THE CONDUCTOR OF E IS SQUARE FREE PART OF $|\Delta|$, ASSUME LATTER IS SQ. FREE.

$$N_E = |\Delta|$$

$$W(E) = -\left(\frac{c_6}{N_E}\right) \mu(N_E) \ ;$$

μ THE MOBIUS FUNCTION THE PARITY OF THE NUMBER OF PRIME FACTORS.

THE ORDER OF VANISHING OF $L(s,E)$ AT $s = 1/2$ IS THE RANK OF $E/\mathbb{Q}$ (BSD).

· ONE CAN COMPUTE $L(s, E_p)$ THE LOCAL FACTOR AT p IN POLY LOG(p) STEPS (SCHOOF)

$$\Rightarrow \quad L(s,E) = \sum_{n=1}^{\infty} \lambda_E(n) n^{-s}$$

THEN FOR ANY x WE CAN COMPUTE THE $\lambda_E(n)$'s $n \leq x$, IN $x^{1+o(1)}$ STEPS.

RIEMANN'S GOLD STANDARD :

GIVEN $W(E)$, USING THE "APPROXIMATE FUNCTIONAL EQUATION" (RIEMANN–SIEGEL FORMULA) ONE CAN COMPUTE $L(s,E)$ FOR s NEAR $1/2$ IN

$$N_E^{1/2 + o(1)} \quad \text{STEPS.}$$

$W(E)$ IS A PRODUCT OVER p's DIVIDING N_E OF $W_p(E)$, SO $W(E)$ CAN BE COMPUTED IN $N_E^{1/2}$ STEPS TRIVIALLY.

$\Rightarrow$ ONE CAN COMPUTE THE COUNTING FN.

$$S(E,t) = \#\{\rho = \tfrac{1}{2} + i\gamma : 0 \le \gamma \le t,\ L(\rho, E) = 0\} \quad (4)$$

AND IN PARTICULAR THE RANK IN

$N_E^{1/2 + o(1)}$ STEPS.

• CAN ONE DO BETTER; BREAK THE 1/2 BARRIER, OR EVEN $N_E^{o(1)}$ STEPS I.E. SUBEXPONENTIAL.

THE ELUSIVE PARITY:

THE PARITY OF THE NUMBER OF PRIME FACTORS OF A NUMBER IS ONE OF ITS BEST KEPT SECRETS. THEORETICALLY IN SIEVE THEORY THERE IS THE WELL KNOWN SIEVE LIMIT ON RECOGNISING PARITY. COMPUTATIONALLY AS FAR AS WE KNOW THE FASTEST WAY TO COMPUTE $\mu(N)$ IS TO FACTOR N - AND THESE ARE PROBABLY OF THE SAME COMPLEXITY. THE BEST FACTORING ALGORITHMS ARE SUBEXPONENTIAL IN N, AND GIVE BENCH MARKS FOR OUR PROBLEM.

THEOREM (RUBINSTEIN/S):

ONE CAN COMPUTE $W(E)$ (WITHOUT FACTORING N_E) AND $S(E,t)$ FOR MANY t's, IN SUBEXPONENTIAL TIME.

REMARKS:

(A) WE ARE ASSUMING GRH THROUGHOUT.

(B) THE ALGORITHM WHEN IT TERMINATES GIVES CORRECT ANSWERS. THE FACT THAT IT DOES TERMINATE QUICKLY DEPENDS ON CONJECTURES (KATZ/S) RELATING THE DISTRIBUTION OF THE ZEROS TO RANDOM MATRIX ENSEMBLES.

THE BASIS OF THE COMPUTATION IS THE EXPLICIT FORMULA OF RIEMANN, GUINAND, WEIL :

$\phi \in \mathcal{S}(\mathbb{R})$, $\hat{\phi}$ F.T $\hat{\phi}$ COMPACT SUPPORT

ϕ EVEN

$\frac{1}{2}+i\gamma=\rho$ THE ZEROS OF $\Lambda(s,\pi)$

$$\sum \phi(\gamma) = \frac{\hat{\phi}(0)}{\pi} \log N_\pi +$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi(t) \, Re\left(\frac{L_\infty'(\frac{1}{2}+it,\pi)}{L_\infty(\frac{1}{2}+it,\pi)} \right) dt$$

$$- \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{c(n)}{\sqrt{n}} \hat{\phi}\left(\frac{\log n}{2\pi} \right)$$

WHERE $L'/L\,(s,\pi) = -\sum_{n=1}^{\infty} \frac{c(n)}{n^s}$

IF WE COMPUTE $c(n)$ FOR $n \leq x$, BY LIMITING SUPPORT $\hat{\phi}$, THE ABOVE IS A SYSTEM OF EQUATIONS FOR THE γ'S.

THIS SYSTEM REDUCES TO THE FOLLOWING BASIC PROBLEM WITH

$$\log N_\pi \approx n$$

UNDERDETERMINED MOMENT PROBLEM

$O(2n+1)$ ORTHOGONAL GROUP SIZE $2n+1$

FOR $A \in O(2n+1)$

$$P_A(\lambda) = \det(\lambda I - A) = \lambda^{2n+1} + a_1\lambda^{2n} + \cdots + a_{2n}\lambda + a_{2n+1} \quad \text{—(4)}$$

$$a_{2n+1} = -\det A, \quad a_j = -(\det A)\, a_{2n+1-j}$$

SINCE $\lambda^{2n+1} P_A(\lambda^{-1}) = (-\det A) P_A(\lambda)$ SELF RECIPROCAL

WE ARE GIVEN THE FIRST k-MOMENTS

$$S_j(A) = \operatorname{trace}(A^j), \quad 0 \le j \le k$$

$$= (\det A)^j + \sum_{\nu=1}^{n} 2\cos(j\theta_\nu) \quad \text{—(5)}$$

HERE THE $2n+1$ EIGENVALUES OF A ARE $\det A, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}$

- IF $k=n$ AND $\det A = \pm 1$ IS KNOWN THEN (4) AND (5) AND NEWTON GIVE THE a_j's FOR ALL j AND HENCE θ_j's.

- OUR L-FUNCTION PROBLEM ONCE WE HAVE COMPUTED THE $c(n)$'s FOR $n \leq N_\pi^\alpha$, REDUCES TO (5) WITH

$$\frac{k}{n} = 2\alpha \qquad \text{——(6)}$$

SO $\alpha = \frac{1}{2}$ (RIEMANN'S GOLD STANDARD) CORRESPONDS EXACTLY TO $k=n$ WHEN EVERYTHING CAN BE COMPUTED.

COMPLEXITY $\alpha < 1/2$ YIELDS THE CORRESPONDING UNDERDETERMINED PROBLEM:

GIVEN $y \in \mathbb{R}^k$, $y = (s_1, s_2, \dots s_k)$ OUR GIVEN MOMENTS WHAT CAN WE SAY ABOUT $\det A$ AND THE θ_j's ?

FORBIDDEN SET $F(y)$:

LET $F(y) \subset [0, \pi]$ BE THE SET OF t's WHICH ARE NOT THE EIGENVALUES OF ANY A WHOSE FIRST k-MOMENTS ARE y.

- IF $F(y)$ IS LARGE WE LEARN SOMETHING ABOUT THE EIGENVALUES — THEY ARE RESTRICTED TO LIE IN THE UNION OF INTERVALS WHICH FORM THE <u>ADMISSIBLE</u> SET

$$G(y) = [0, \pi] \setminus F(y)$$

- IF $k < n$, $F(y)$ MAY BE EMPTY FOR EXAMPLE IF THE θ'S LIE ON AN ARITHMETIC PROGRESSION, BUT FOR TYPICAL y AND α NOT TOO SMALL THIS WON'T HAPPEN.

EXACT COUNT SET $E(y)$:

$E(y)$ CONSISTS OF ALL t's IN $[0, \pi]$ FOR WHICH THE NUMBER $S_y(t)$ OF EIGENVALUES θ_j, θ_j, ..detA, LIE IN $[0, t]$ IS DETERMINEN INDEPENDENT OF A.

$E(y) \subset F(y)$

HOW TO COMPUTE THESE EFFICIENTLY AND WHAT DO THEY LOOK LIKE AS A FUNCTION OF $\alpha = k/n$?

MOMENT CURVE:

CLOSELY RELATED TO THE ABOVE IS THE MOMENT CURVE; $t = \cos\theta$

$-1 \le t \le 1$

$M : [-1,1] \to \mathbb{R}^k$; $M(t) = (t, t^2, \dots, t^k)$

$C := M([-1,1]) \subset \mathbb{R}^k$

FOR $n \geq 1$ (OUR INTEREST IS $n > k$)

$A(k,n) := C + C + C \dots + C$; n-times

$\subset \mathbb{R}^k$

BASIC COMPLEXITY PROBLEM:

FOR $\xi \in \mathbb{R}^k$ IS $\xi \in A(k,n)$?

DOES THIS HAVE AN EFFICIENT SOLUTION (IE POLYNOMIAL TIME)?

- NOTE THAT FOR $A(k,k)$ IT DOES SINCE IN THIS CASE WE GET THE FULL CHARACTERISTIC POLYNOMIAL AND HENCE RECOVER THE ROOTS EFFICIENTLY.

BUT HOW ABOUT $n = 2k$?

<u>RELEVANCE:</u> TO COMPUTE $F(y)$ WE CONSIDER FOR EACH $t \in [-1,1]$ WHETHER

$$y - M(t) \in A(k, n-1)$$

THIS IS SO IFF $t \in F(y)$.

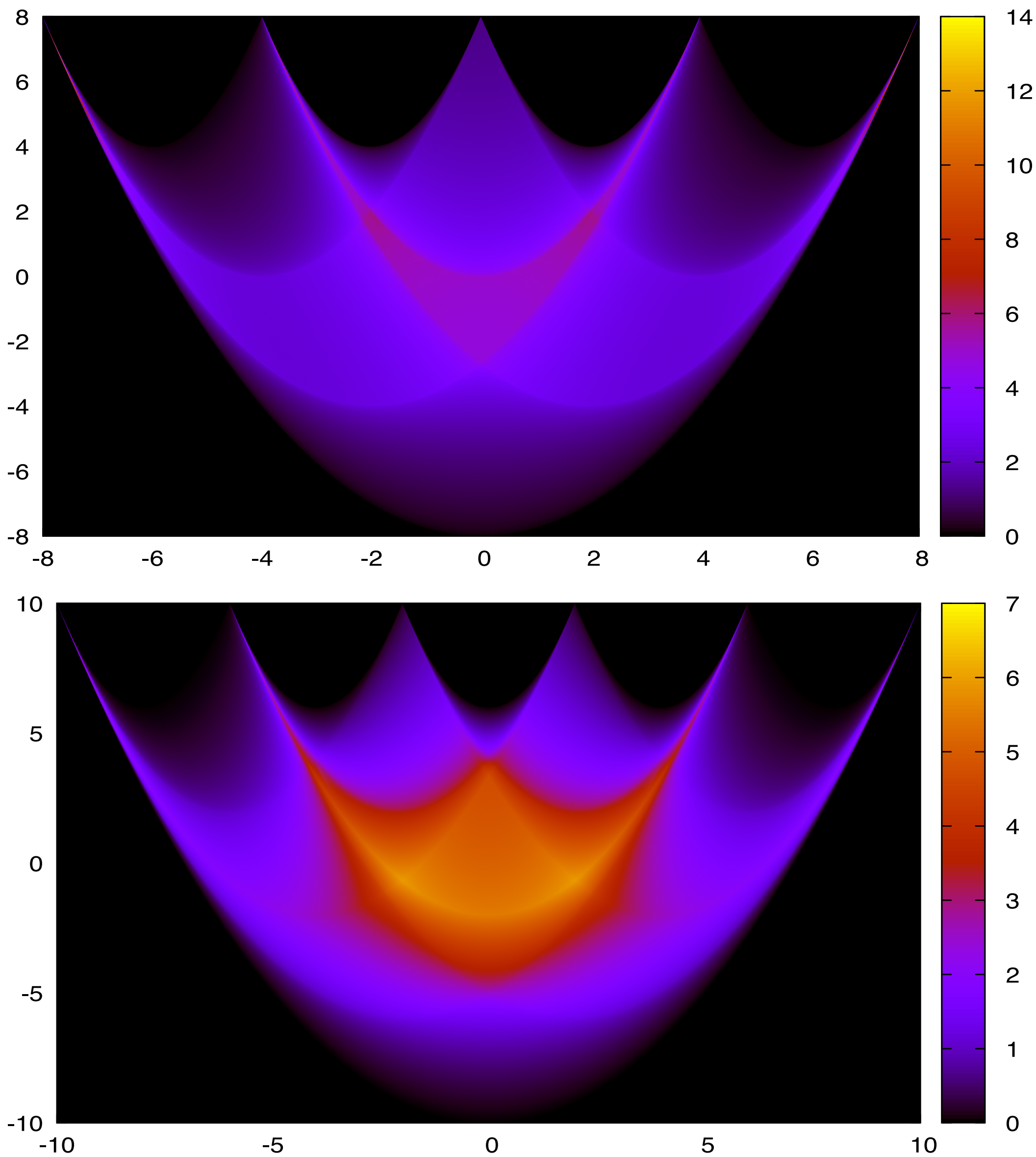


FIGURE 4. Heat maps depicting the distribution of points in the sets $A(2,4), A(2,5)$.

Then, for $u = c(t_1) + c(t_2) + \ldots + c(t_k) \in A(k,n)$,

$$v(u) = \sum_{j=1}^{n} f(t_j). \tag{9}$$

Hence $v(u) \leq 0$ for all $u \in A(k,n)$ iff $f(t) \leq 0$ for all t.

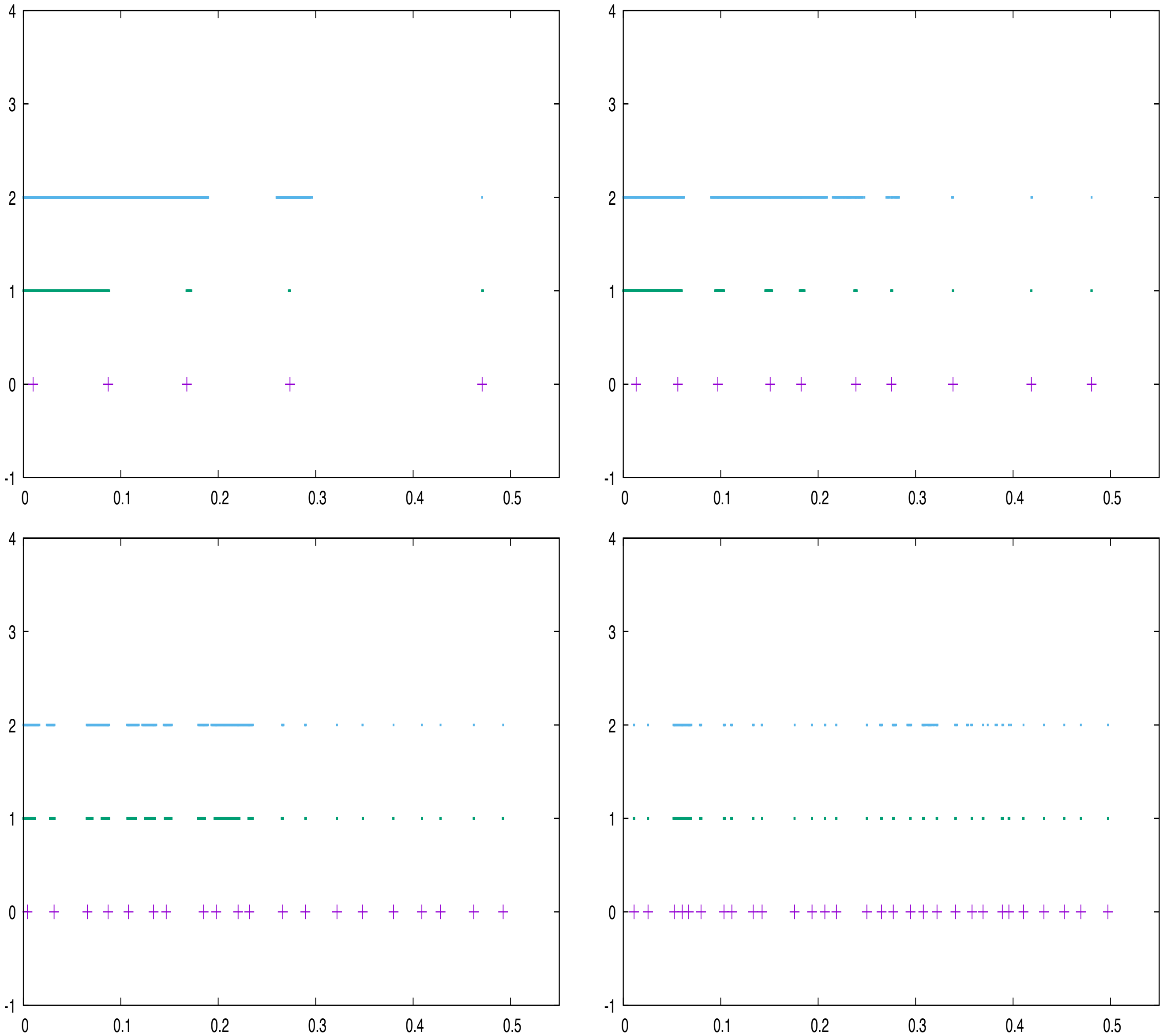


FIGURE 1. Plots depicting the admissible sets (i.e. complement of $F(y)$), for 2 Haar sampled matrices in $SO(2n)$, with $n = 5, 10, 20, 30$, and $k = n - j$, with $j = 0, 1, 2$. The horizontal depicts the points $t \in [0, 1/2]$ in the admissible set, while the vertical axis is j. ($j = 2$ needs to be redrawn to higher resolution. will include $n = 5, 30$ later today)

OUR (EFFICIENT) ALGORITHM INVOLVES AN ITERATIVE CONVEXIFICATION OF THE BASIC PROBLEM COUPLED WITH A SECOND LINEAR PROGRAM. IT YIELDS (GOOD IN TYPICAL SITUATIONS) LOWER BOUNDS FOR $F(y)$ AND $E(y)$.

MORE GENERALLY; FOR $G(y) \subset [-1,1]$ LET

$$C_G = M(G) \subset \mathbb{R}^k \quad \text{AND}$$

$$A_G(k,n) = C_G + C_G + \cdots + C_G$$

OUR INTEREST IS G A UNION OF INTERVALS (FINITELY MANY)

THE QUESTION OF WHETHER $\xi \in \mathbb{R}^k$ IS IN $A_G(k,n)$ IS ALREADY HARD SINCE FOR $k=1$ AND G SAY n POINTS, IT CONTAINS THE SUB-SUM PROBLEM WHICH IS NP COMPLETE (KARP)

WE RELAX THE PROBLEM TO DETERMINE IF z IS IN THE CONVEX HULL OF A:

$z \notin CH(A_G(k, n))$ IFF

THERE IS A SEPARATING HYPERPLANE

$$\left.\begin{array}{l} \min_{b} (b_0 + b_1 z_1 + \cdots b_k z_k) < 0 \\ \text{SUBJECT TO } b_0 + b_1 t + \cdots + b_k t^k \geq 0 \text{ FOR } t \in G \end{array}\right\} LP1$$

FOR EXACT COUNTING $J \subset [-1, 1]$ INTERVAL

$y \in \mathbb{R}^k$, ESTIMATE

$$S_y(J) = \#\{\theta's, \det A \text{ IN } J : M(A) = y\}$$

$$n b_0 + \cdots b_k y_k = L(y) \leq S_y(J) \leq U(y) = n c_0 + c_1 y_1 + \cdots c_k y_k$$

SATISFYING: $\sum_{j=0}^{k} b_j t^j \leq \chi_J(t) \leq \sum_{j=0}^{k} c_j t^j$; $t \in J \cap G$

LINEAR PROGRAM

$$\Delta_y(J) = \min_{b, c} [U(y) - L(y)] \quad \Big\} LP2$$

• IF $\Delta_y(J) < 1$ THEN $S_y(J)$ WHICH IS AN INTEGER IS DETERMINED!

ITERATION:

INITIALIZE $G = [-1, 1]$

FOR EACH $t \in [-1, 1]$ RUN LP1 TO CHECK IF

$$y - M(t) \in CH(A(k, n-1))$$

IF NOT $t \in F_1(y)$.

FOR EACH $t \in F_1(y)$ RUN LP2 TO OBTAIN $E_1(y)$ WHEN $S_y([-1, t])$ IS DETERMINED.

ITERATE: $G_1(y) = [-1, 1] \setminus F_1(y)$,

... $F_2(y), G_2(y), E_2(y), \ldots$

$$F_1(y) \subset F_2(y) \ldots F_\infty(y) \subset F(y)$$

$$E_1(y) \subset E_2(y) \ldots E_\infty(y) \subseteq E(y)$$

$$S_y(t) \qquad t \in E_\infty(y) .$$

NB: AT THE FIRST STEP $G=[-1,1]$

LP1 AND LP2 HAVE EXPLICIT SOLUTIONS BY HAMBURGER, CHEBYSHEV AND MARKOV. WE USE THESE AT THIS STEP AND IT ALSO IS IMPORTANT FOR ANALYSIS

THEOREM: $\xi \in \mathbb{R}^k$ IS IN $CH(C)$ IFF THE HANKEL MATRICES

$k=2\nu$

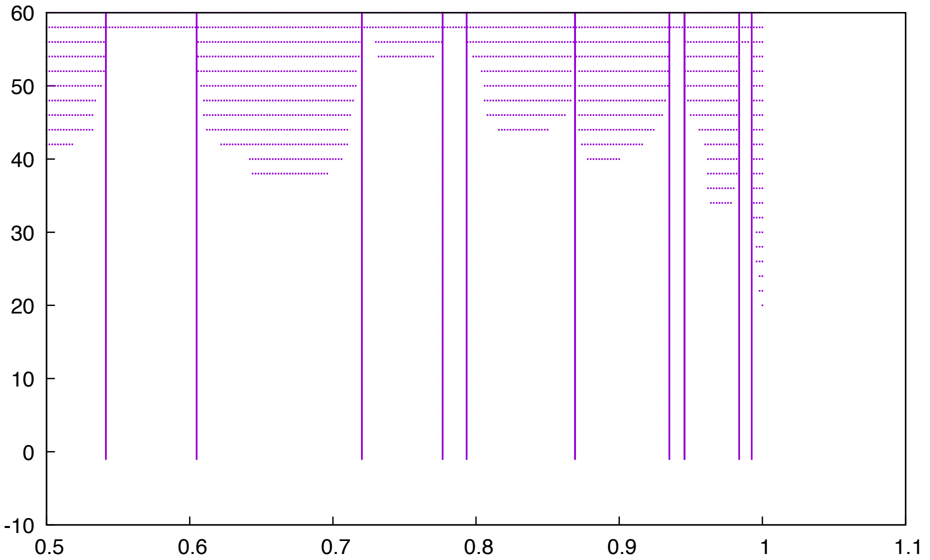
$$\left(\xi_{i+j}\right)_{\substack{i=0,1,\dots\nu \\ j=0,1,\dots\nu}} \qquad \xi_0 = 1$$

AND

$$\left(2\xi_{i+j+1} - \xi_{i+j} - \xi_{i+j+2}\right)_{\substack{i=0,1,\dots,\nu-1 \\ j=0,1,\dots,\nu-1}}$$

ARE NONNEGATIVE.

THE SOLUTION OF LP2 FOR $G=[-1,1]$ BY CHEBYSHEV AND MARKOV USES THEIR THEORY OF NODES AND WEIGHTS (VIA PADE APPROXIMATIONS)



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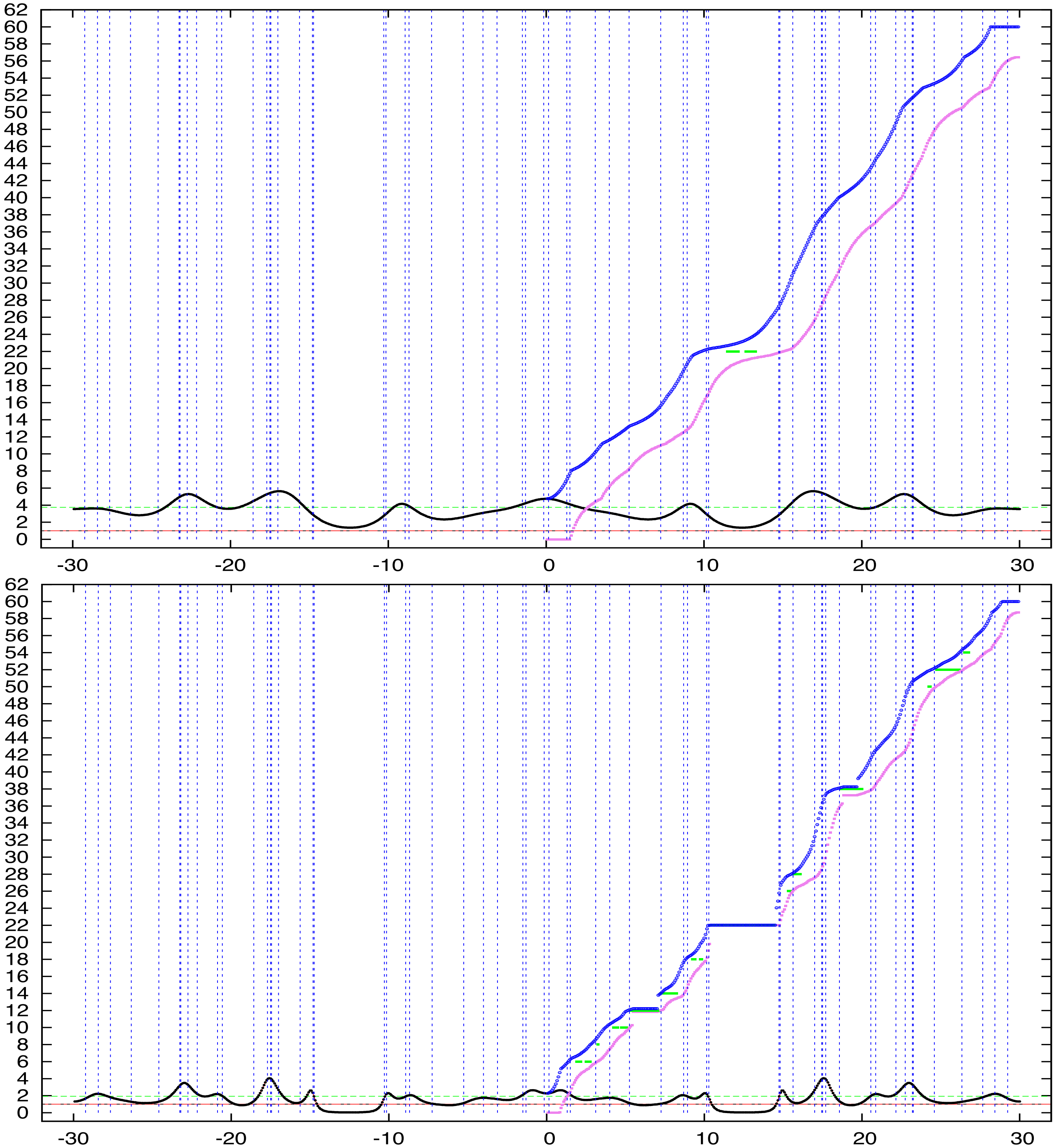


FIGURE 6. The same as the previous figure, but for 30 pairs of zeros, symmetric about 0, chosen uniformly and independently at random, for $k = 15$ (top) and $k = 30$ (bottom). We notice that there are larger gaps (as well as many smaller gaps) in comparison to the SO(60) example, and that more of the gaps are detected sooner.

- $F_1(y)$ MAY BE EMPTY IN WHICH CASE WE LEARN NOTHING.

THE KEY POINT THAT WE ESTABLISH IS THAT IF $2\alpha = k/m$ GOES TO 0 SLOWLY THEN BOTH $F_1(y)$ AND $E_1(y)$ ARE NONEMPTY (IN FACT QUITE LARGE) IF $y = M(A)$ IS DRAWN AT RANDOM W.R.T. HAAR MEASURE ON $O(2n+1)$.

- NOTE THAT ONCE $E_\infty(y) \neq \phi$ AND $-1 < t < 1$ IS IN $E_\infty(y)$ THEN THE PARITY OF $S_y(t)$ IS EVEN IFF $\det A = 1$, THAT IS WE DETERMINE $W(A)$

$\Rightarrow$ SUBEXPONENTIAL COMP. OF $W(E)$.

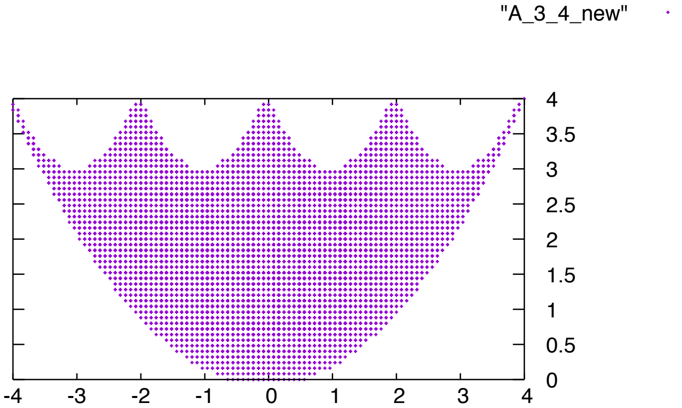
WHAT IS THE LIMIT OF THIS METHOD — HOW SMALL CAN k/n BE FOR $W(A)$ TO BE COMPUTED?

A LIMIT IS SET BY THE VERY STRONG SZEGO LIMIT TYPE THEOREM OF JOHANSSON / LAMBERT (2019):

LET $\nu_+(k,n)$, $\nu_-(k,n)$ BE THE PUSH FORWARD MOMENT MEASURES (ON $\mathbb{R}^k$) OF HAAR MEASURE ON $O^+(2n+1)$ AND $O^-(2n+1)$. THEN FOR $k = n^a$, $0 \leq a < 1/3$

$$\text{TOTAL VARIATION}\left[\nu_+(k,n), \nu_-(k,n)\right] \leq C e^{-(1-a) n^{1-a}}$$

SO CERTAINLY THERE IS NO POLYNOMIAL TIME COMPUTATION OF $W(N_E)$ BY THIS METHOD.



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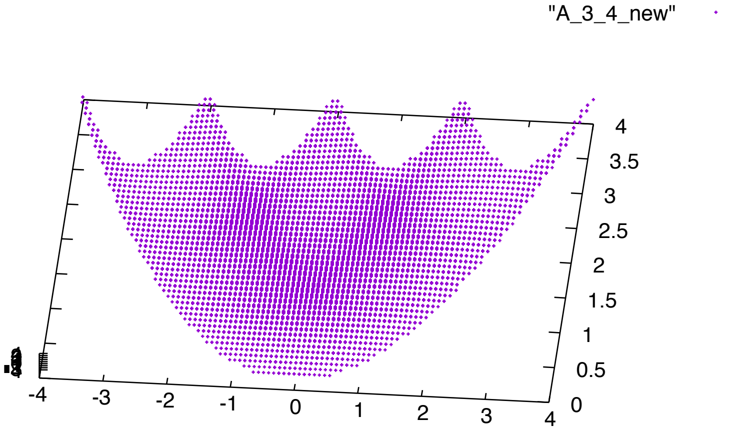
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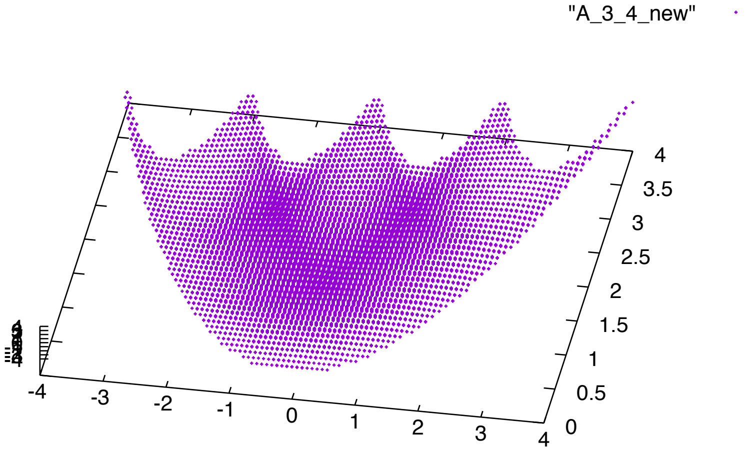
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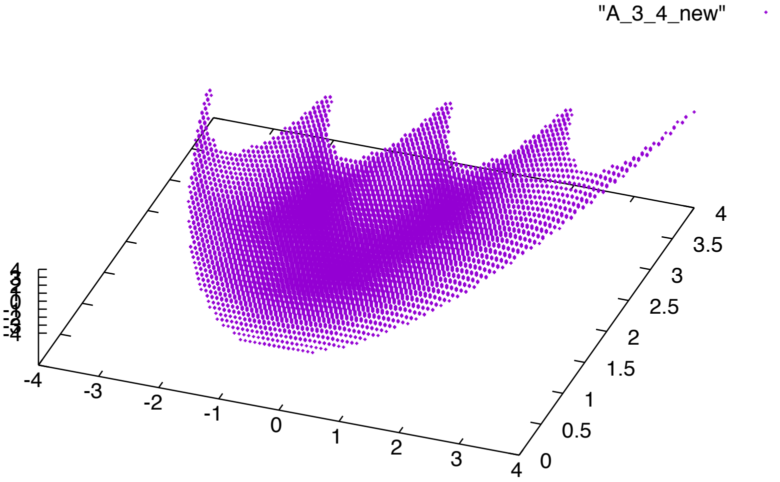
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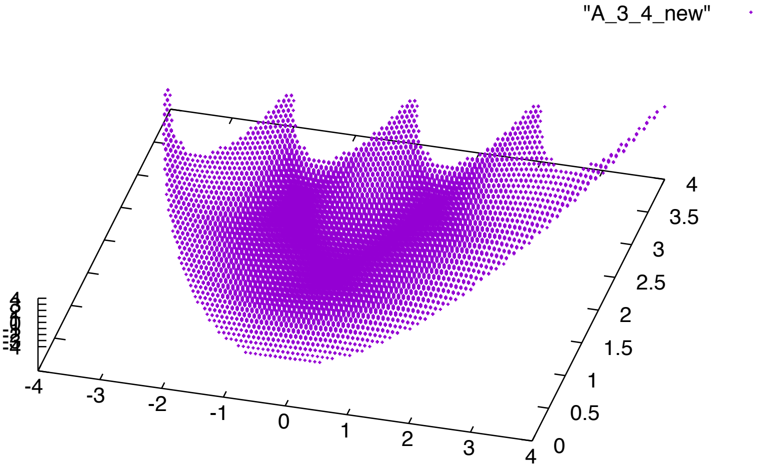

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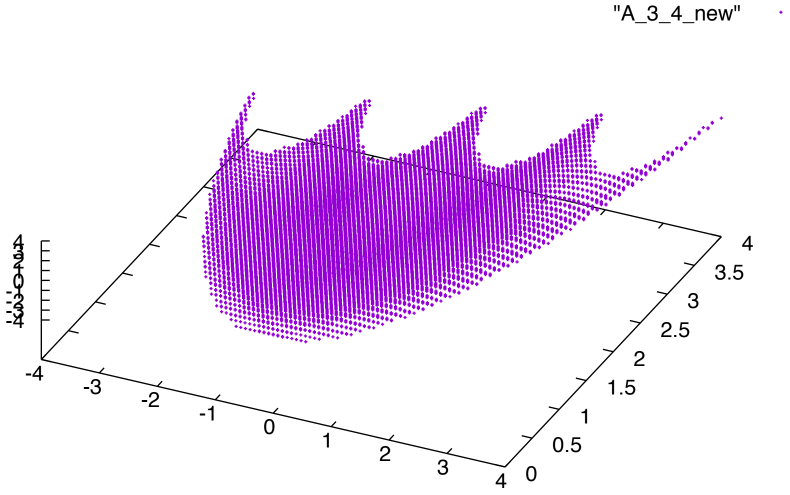
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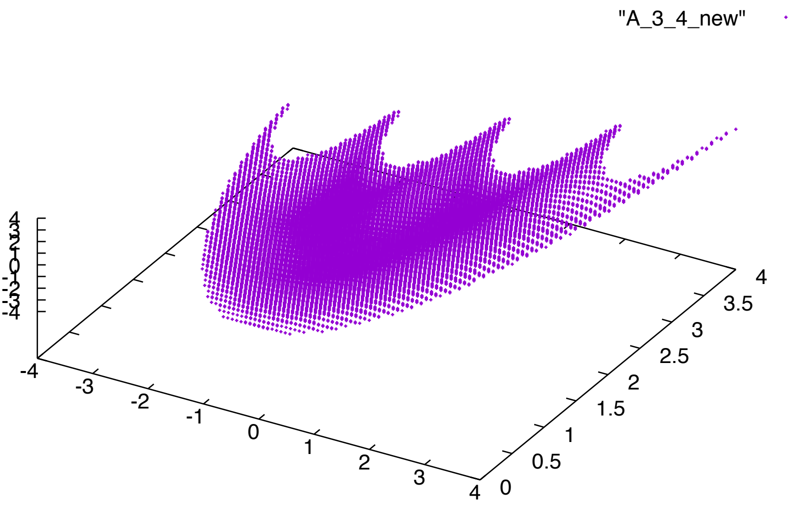

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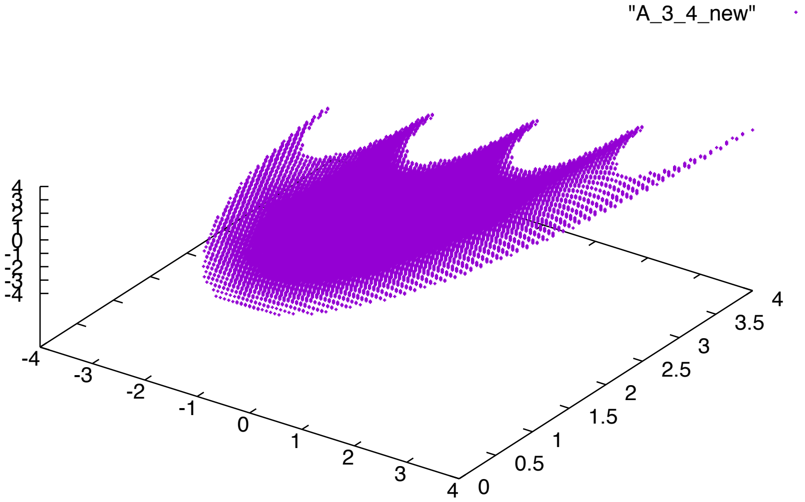

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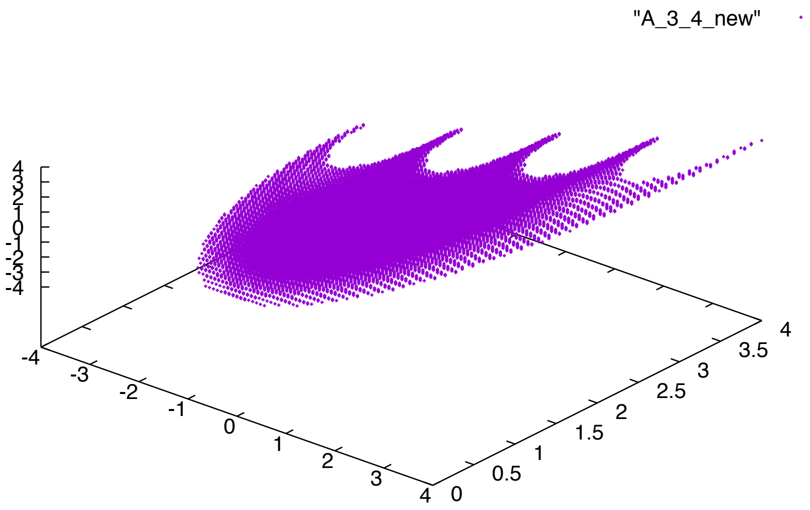
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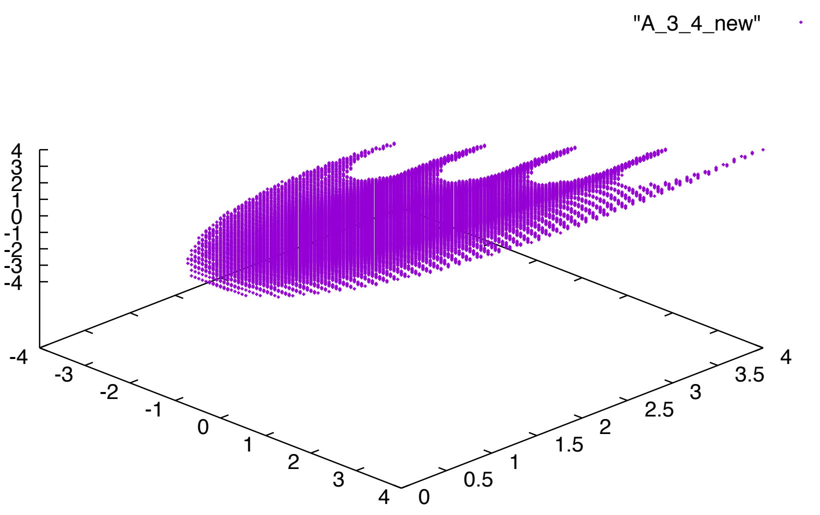

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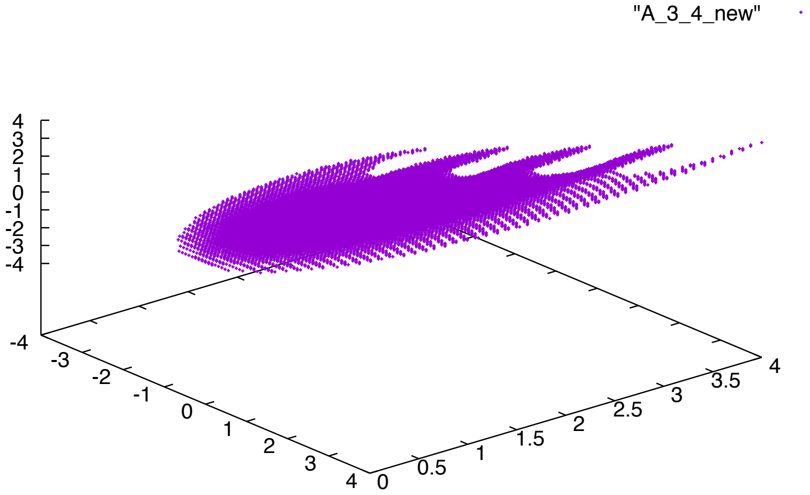
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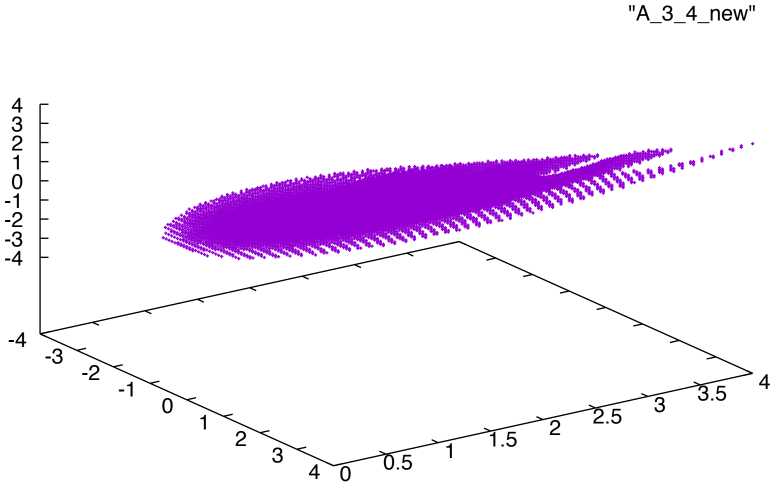

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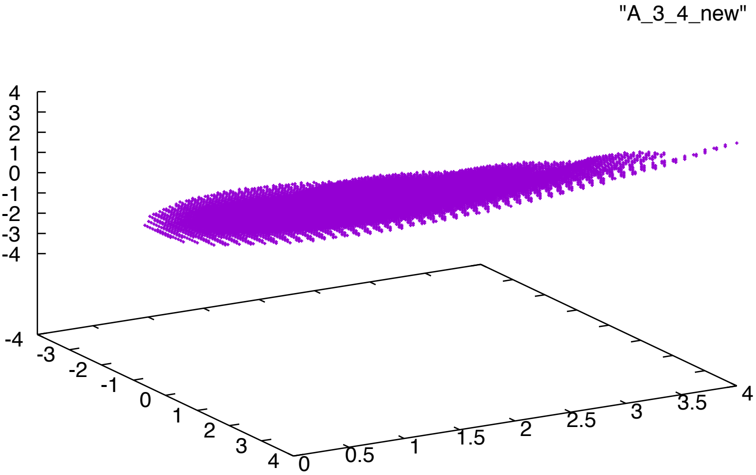

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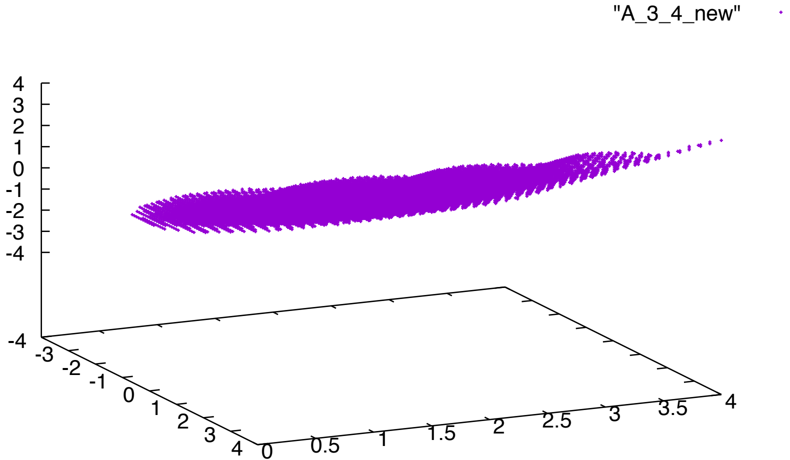
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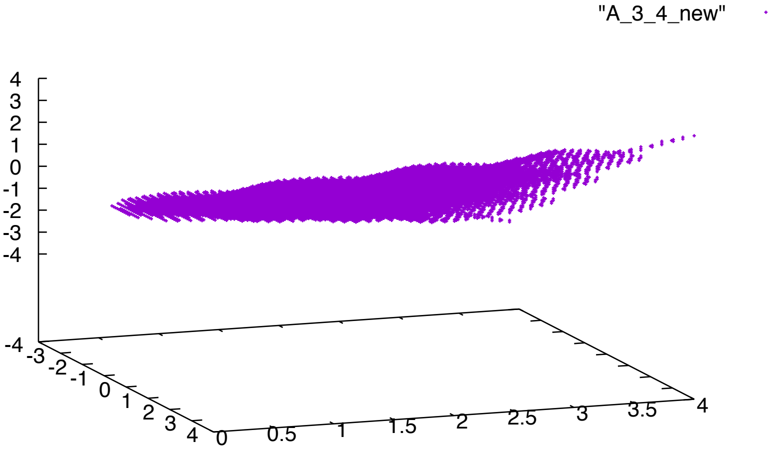

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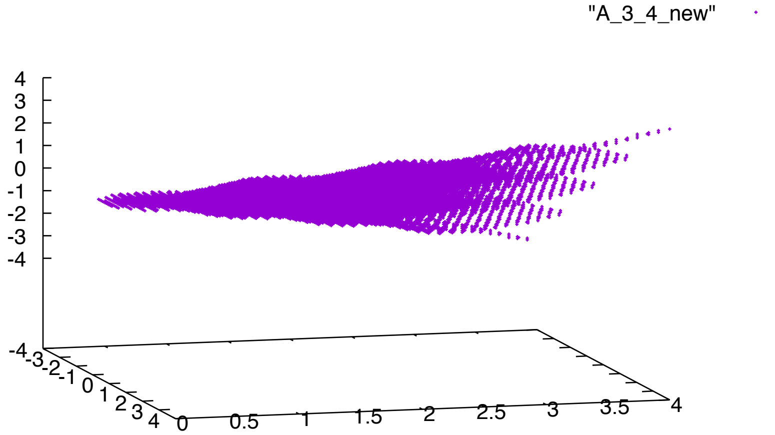
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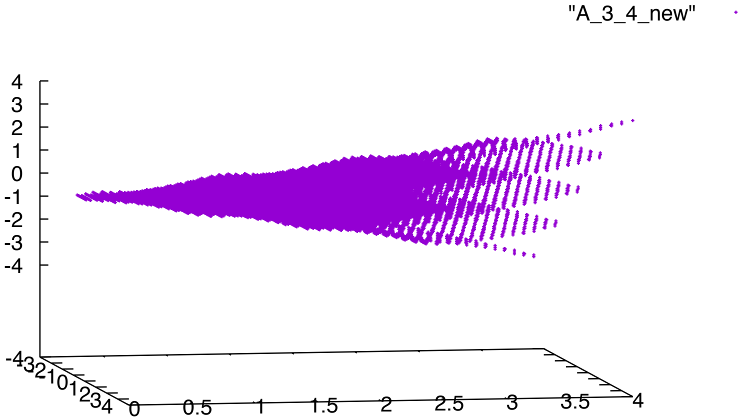
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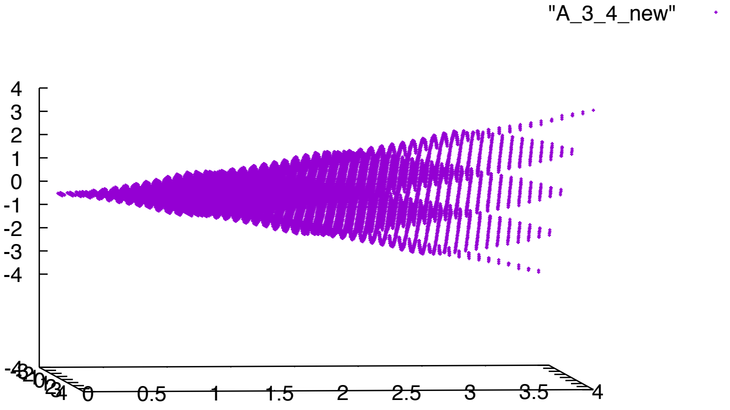
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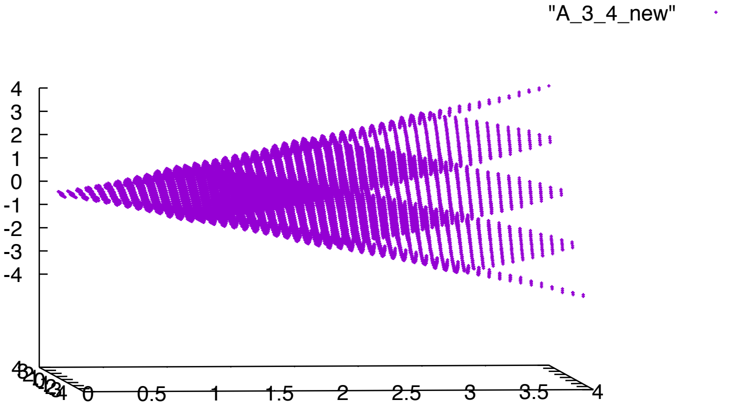
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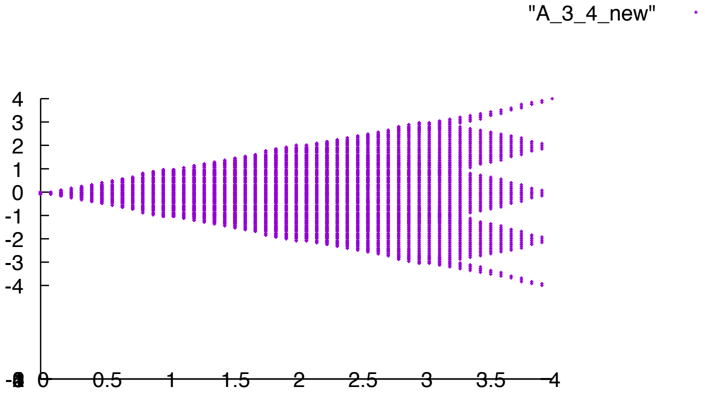
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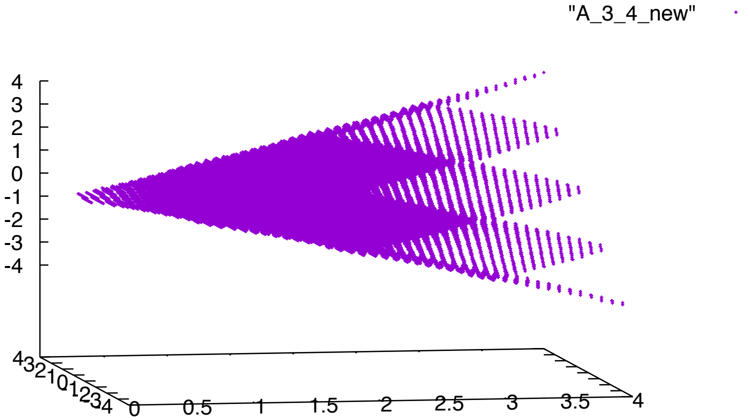
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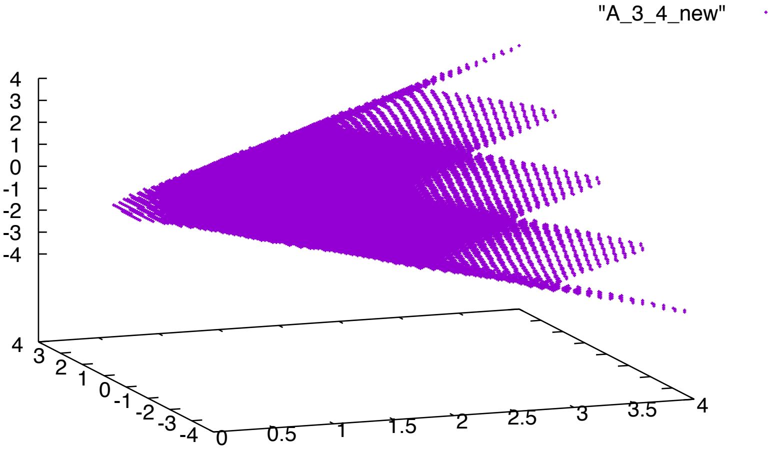

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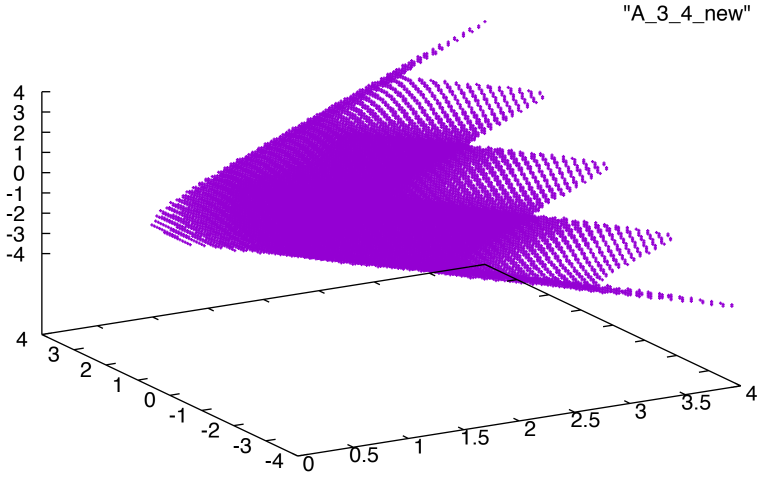
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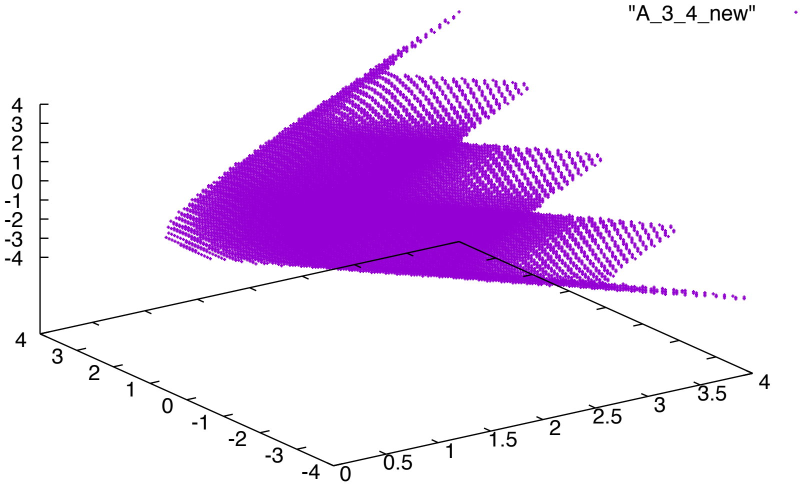

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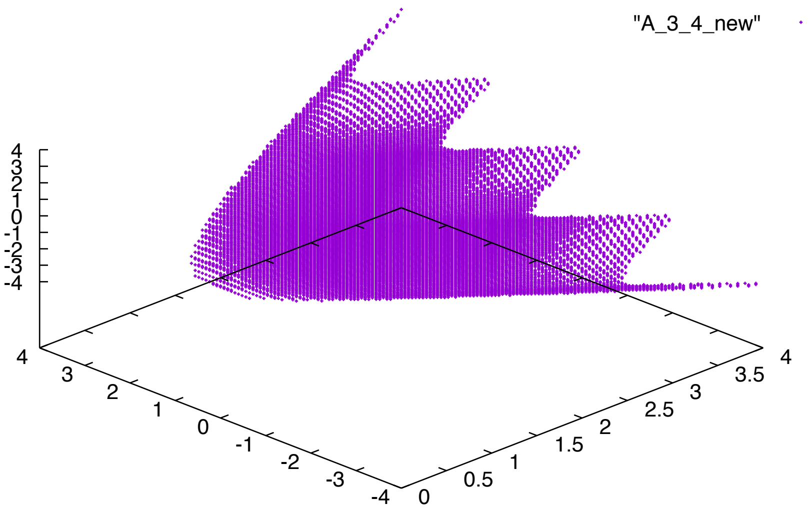
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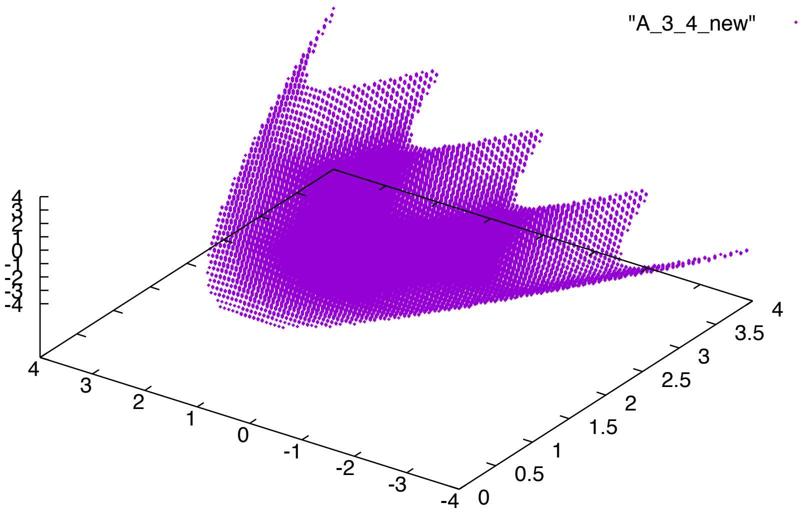
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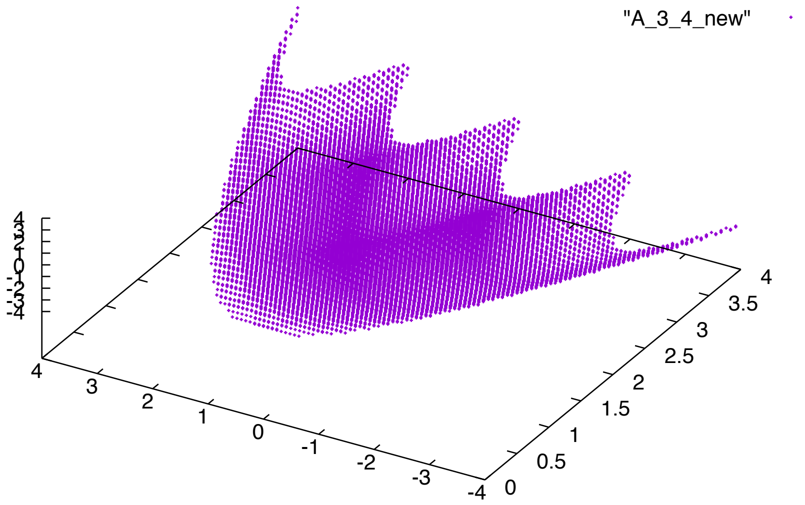
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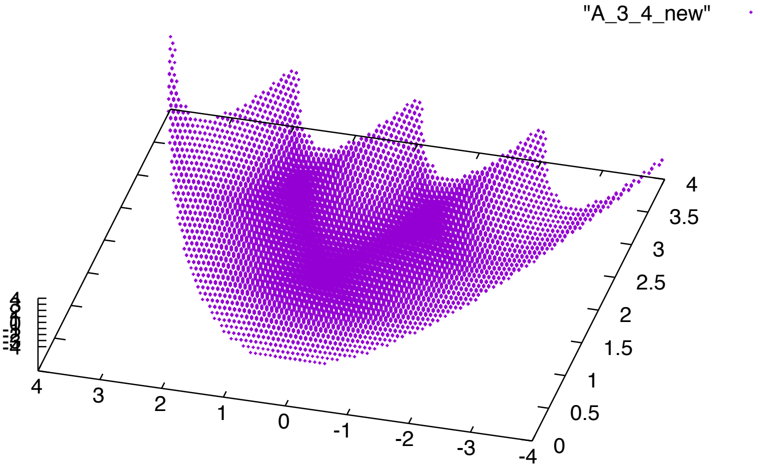

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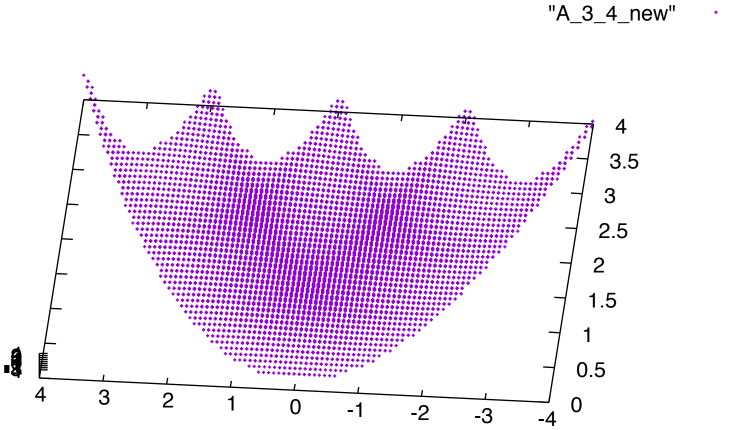

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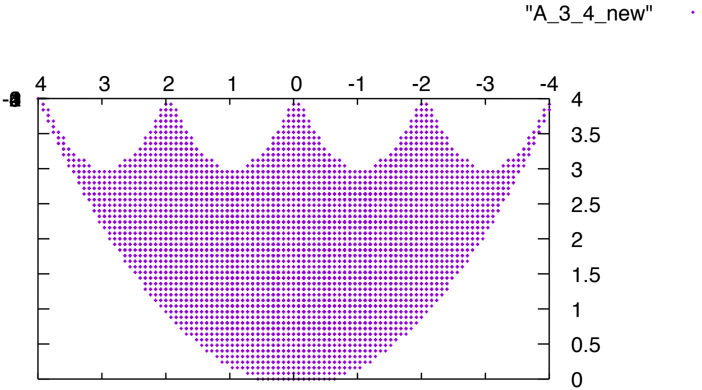
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