# Completely Log-Concave Polynomials and Distributions

Nima Anari

Stanford University / Simons Institute for the Theory of Computing

based on joint works with

Shayan Oveis Gharan

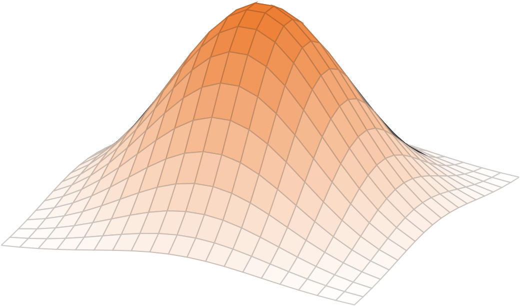
Kuikui Liu

Cynthia Vinzant

# Measures and Distributions

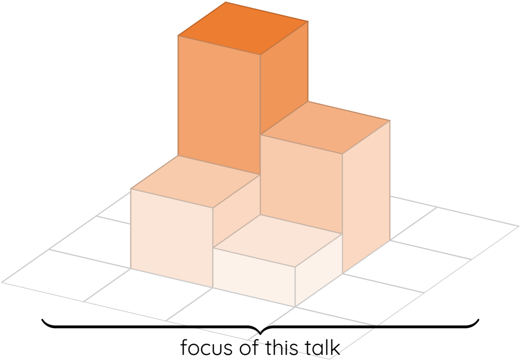
## Continuous

$$\mu : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$$

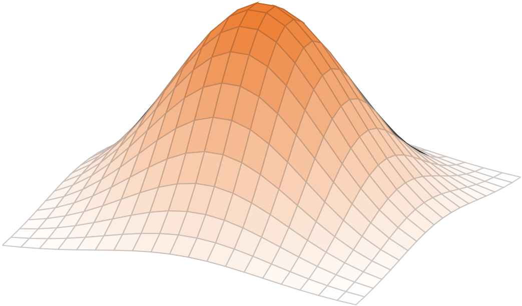


## Discrete

$$\mu : \{0,1\}^n \text{ or } \mathbb{Z}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$$



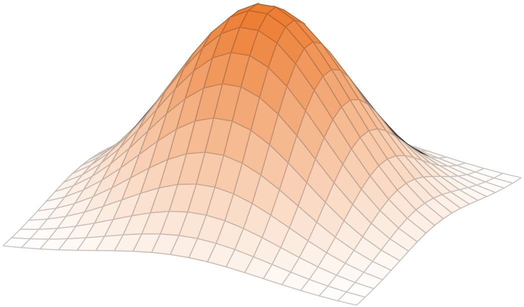
# Continuous Land



# Algorithmic Primitives

Unnormalized density $\mu : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ gives rise to probability distribution:

$$\mathbb{P}[A] \propto \mu(A) = \int_A \mu(x)\,dx.$$



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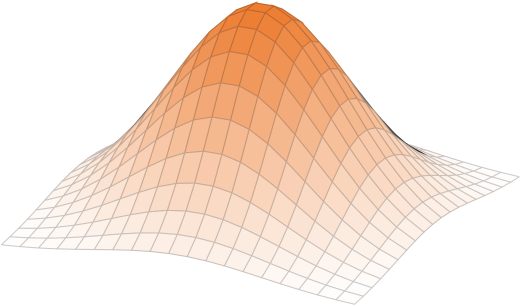

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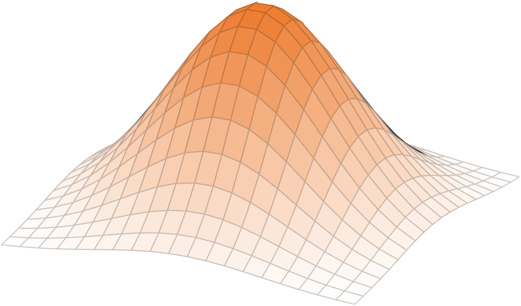

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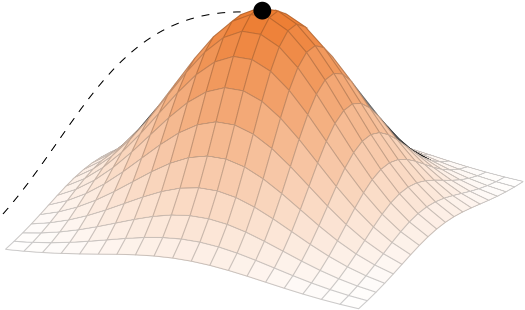
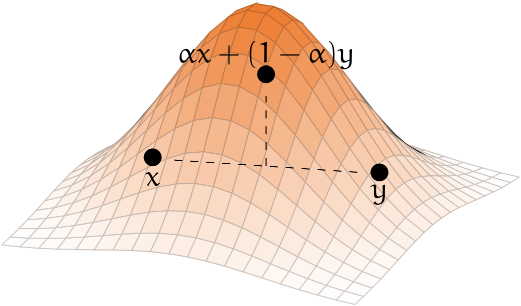
- Optimization: Find the mode?



# Log-Concave Distributions

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$$\mu(x)^{\alpha}\mu(y)^{1-\alpha} \leqslant \mu\left(\alpha x + (1-\alpha)y\right)$$

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Efficiently sample $\kappa$ approximately satisfying

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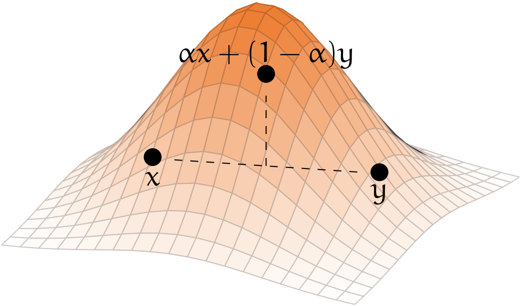
using MCMC methods.

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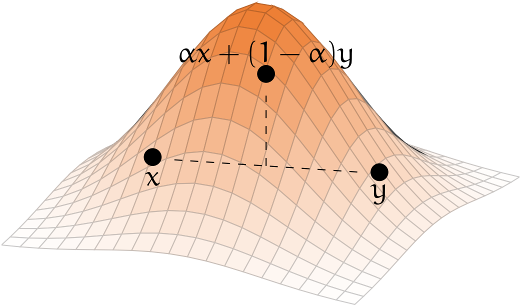
The mode of a log-concave distribution can be found by convex programming:

$$\max_{\kappa} \log(\mu(\kappa)).$$

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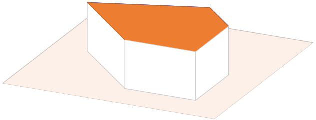
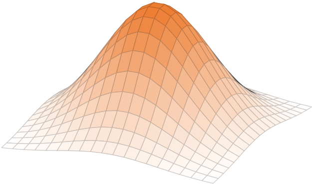
# Examples of Log-Concave Distributions



indicator of convex set



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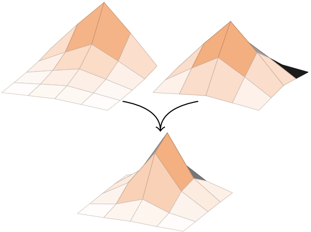
e.g., Gaussian density



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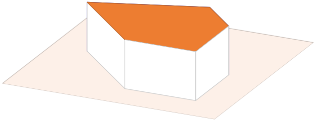

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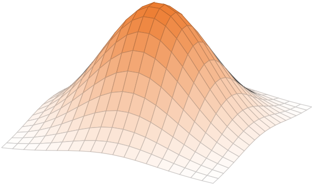

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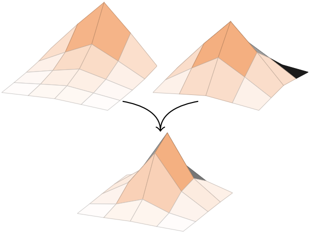

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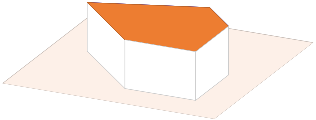

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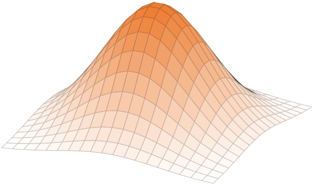

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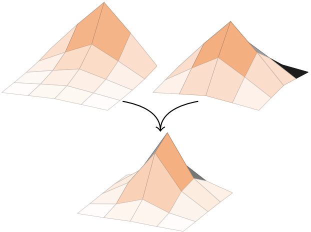

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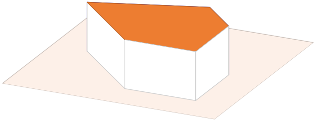

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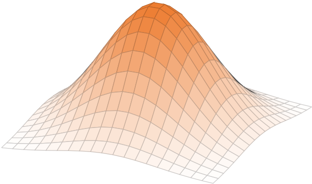

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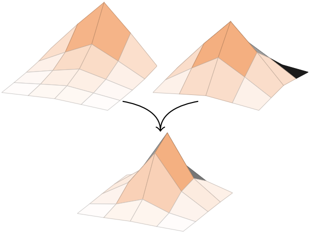

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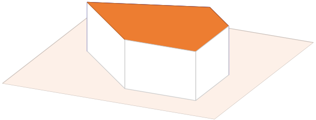

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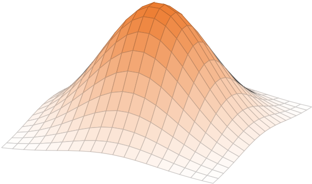

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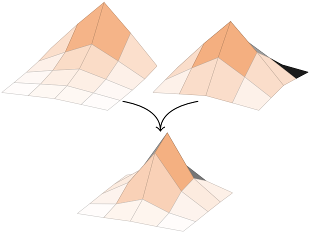

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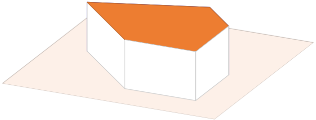

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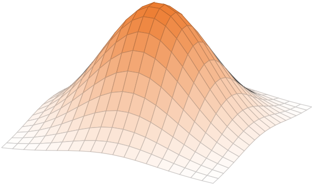

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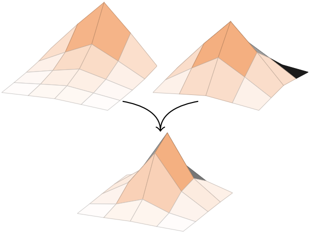

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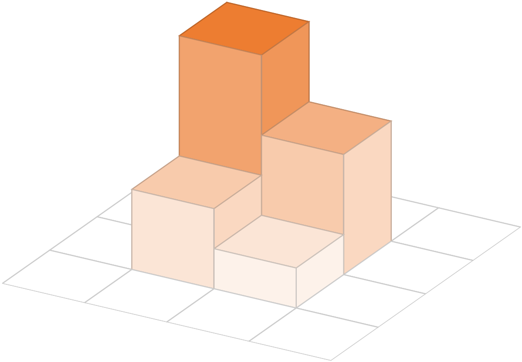
# Discrete Land



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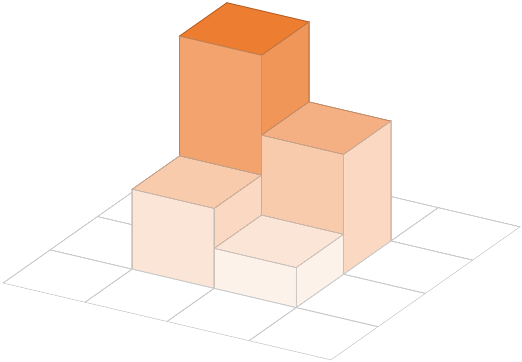

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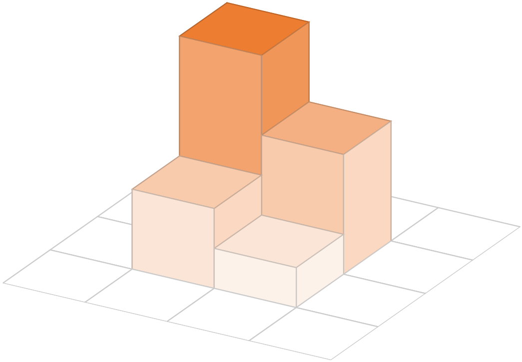

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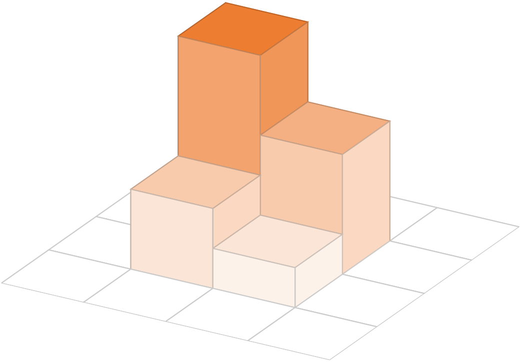

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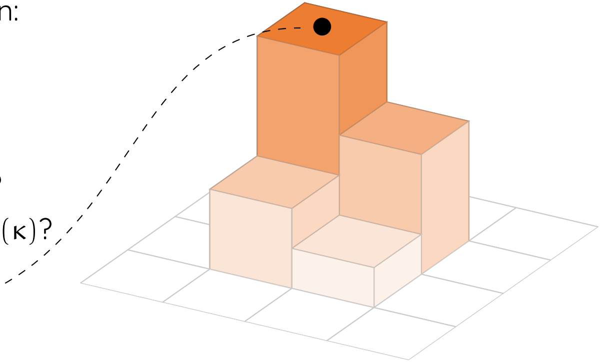
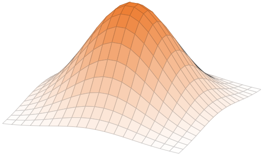
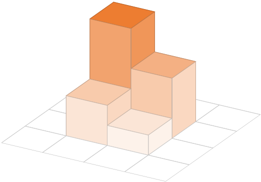
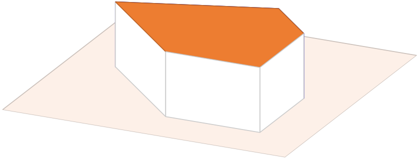
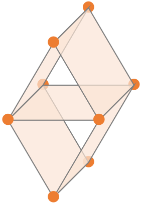


What should be the analog of log-concavity in discrete distributions?

# Analogy Between Continuous and Discrete

| | Continuous | Discrete |
|---|---|---|
| Distributions: |  |  |
| Supports: | <br>convex set | <br>matroid(-like) |

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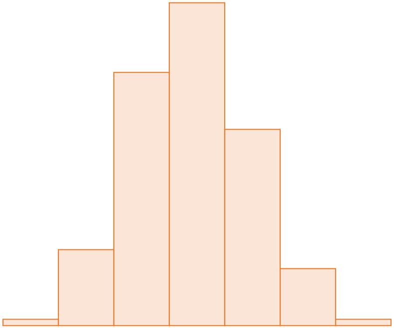
**First Proposal**

$$\mu(\kappa_1)^{\alpha_1} \ldots \mu(\kappa_m)^{\alpha_m} \leqslant \mu(\alpha_1 \kappa_1 + \cdots + \alpha_m \kappa_m),$$

for $\alpha_1 + \cdots + \alpha_m = 1$, whenever it makes sense.



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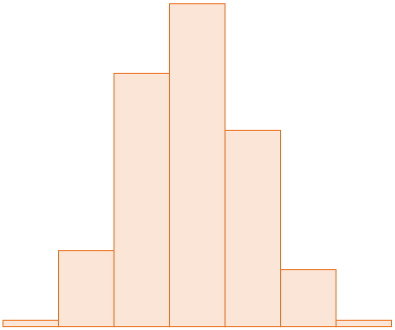

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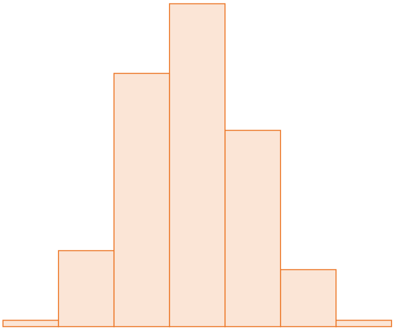

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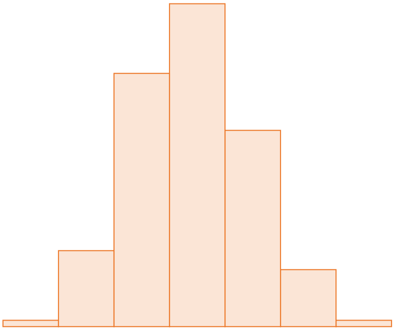

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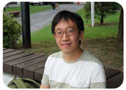
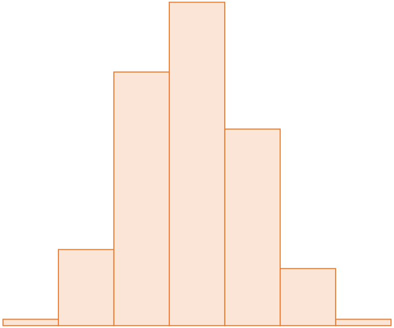
- $\mu =$ coefficients of chromatic polynomial.



[Huh'10]



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[McMullen'89]

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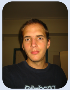
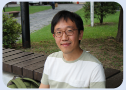
## Rota’s Conjecture

$\mu$ = coefficients of matroid characteristic polynomial.


[Adiprasito-Huh-Katz’17]

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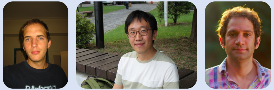

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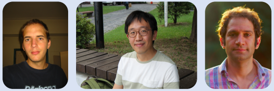

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- Kazhdan-Lusztig Conjecture: Certain objects in representation theory [Elias-Williamson’14].

# Second Attempt: Real-Rootedness

## Coefficients of Real-Rooted Polynomials [Newton]

If $\mu(0)z^0 + \cdots + \mu(d)z^d \in \mathbb{R}[z]$ is real-rooted, then $\mu(0), \ldots, \mu(d)$ is log-concave.
In fact, the following is also log-concave (ultra-log-concavity):

$$\frac{\mu(0)}{\binom{d}{0}}, \frac{\mu(1)}{\binom{d}{1}}, \ldots, \frac{\mu(d)}{\binom{d}{d}}.$$

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In fact, the following is also log-concave (ultra-log-concavity):

$$\frac{\mu(0)}{\binom{d}{0}}, \frac{\mu(1)}{\binom{d}{1}}, \ldots, \frac{\mu(d)}{\binom{d}{d}}.$$

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# Second Attempt: Real-Rootedness

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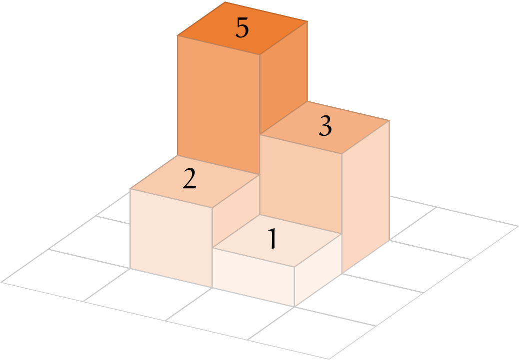
- For 1-D: If $g_\mu$ has real roots, then $\mu$ is log-concave.

# Strongly Rayleigh Distributions

**[Borcea-Brändén-Liggett'07]**

Call $\mu : \{0,1\}^n \to \mathbb{R}_{\geq 0}$ Strongly Rayleigh when $g_\mu$ is real stable.

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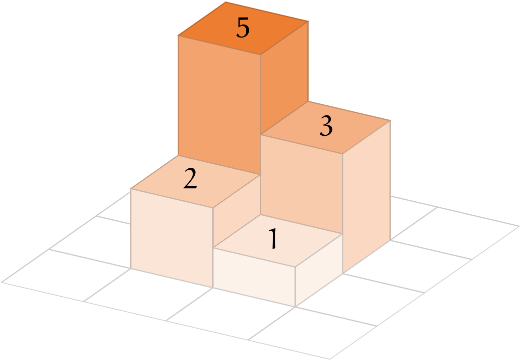


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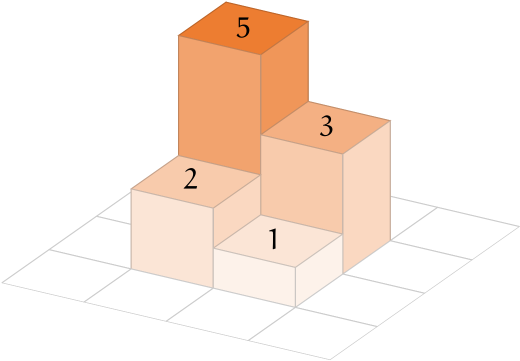
- Spanning trees in a graph:

$$\mu(\mathbb{1}_S) = \begin{cases} 1 & S \text{ forms a spanning tree,} \\ 0 & \text{otherwise.} \end{cases}$$

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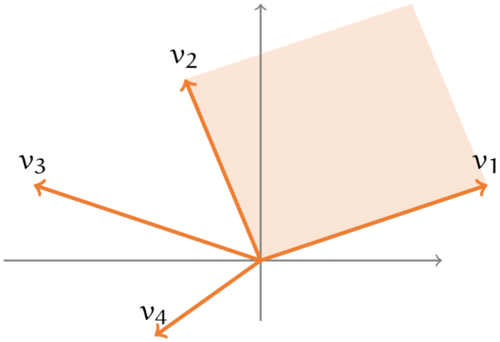
# Main Example: Determinantal Point Process

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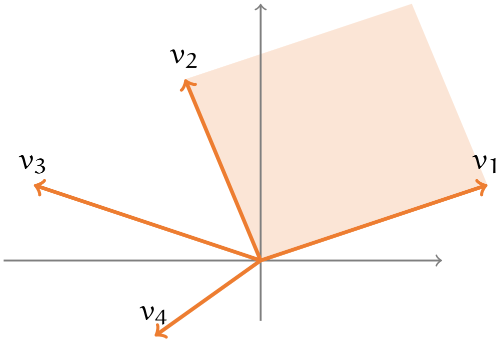


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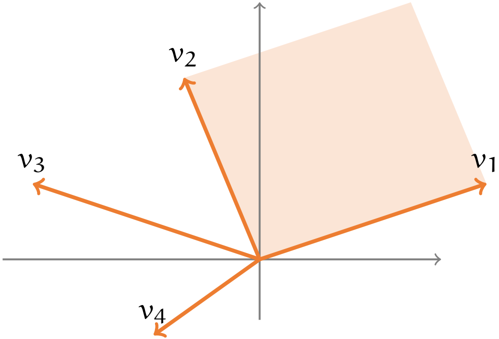


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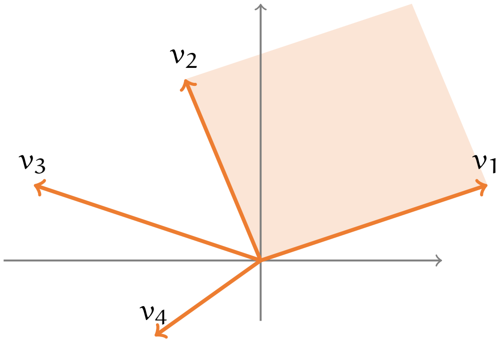


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$$g_\mu(z_1, \dots, z_n) = \det(I + \mathrm{diag}(z_1, \dots, z_n) L)$$

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# Algorithms for Strongly Rayleigh Distributions

- Sampling: Local Markov chains mix in polynomial time [A-Oveis Gharan-Rezaei'16, Li-Jegelka-Sra'17].

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- Counting: Given oracle for $g_\mu$ can $2^{O(n)}$-approximate coefficients of $g_\mu$ in polynomial time [Gurvits'04]. Given oracles for $g_{\mu_1}, g_{\mu_2}$ can $2^{\min\{\deg g_{\mu_1}, \deg g_{\mu_2}\}}$-approximate $\sum_S \mu_1(S)\mu_2(S)$ [A-Oveis Gharan'17]. Similar results [Nikolov-Singh'16, Straszak-Vishnoi'17].

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Supports are matroids [Choe-Oxley-Sokal-Wagner'04], but not all matroids are possible supports [Brändén'07].

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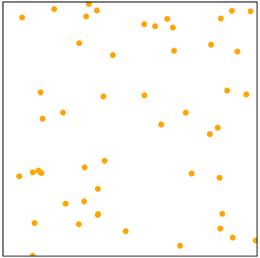
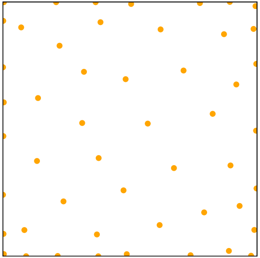
- Next we will see illustrative applications of log-concavity in optimization and deterministic counting. More throughout the semester.

# Log-Concavity $\implies$ Optimization

## Optimization Problem

For $d$-homogeneous $g_\mu$ find $S \in \binom{[n]}{d}$ such that $\mu(S)$ is maximized.





Relax and solve the following [on board ...]

$$\max\{g_\mu(z_1, \ldots, z_n) \mid z_1, \ldots, z_n \geqslant 0, z_1 + \cdots + z_n = d\}.$$

# Log-Concavity $\Longrightarrow$ Deterministic Counting

If $\mu$ is an arbitrary distribution and $\mu_1, \ldots, \mu_n$ are the marginals:

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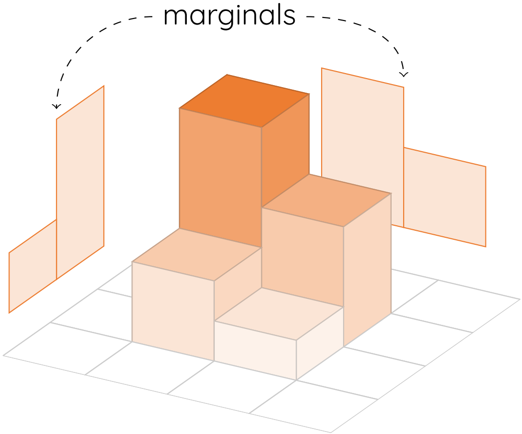

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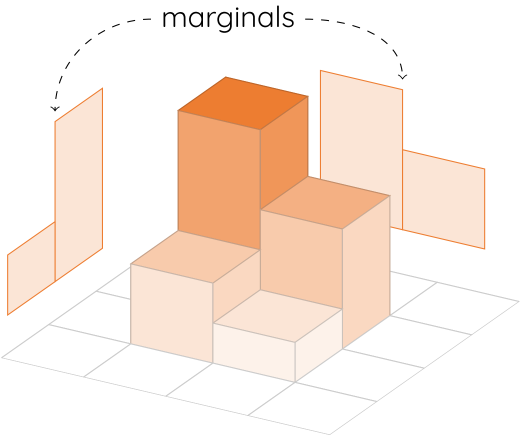

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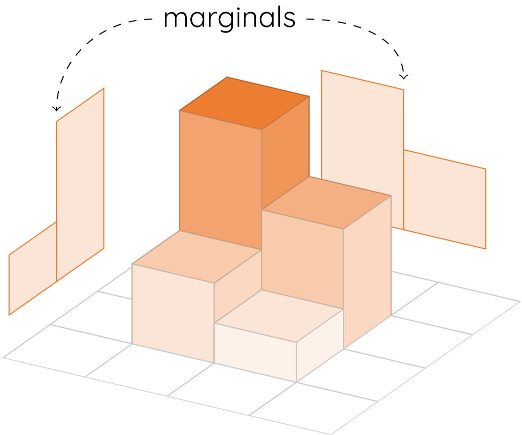


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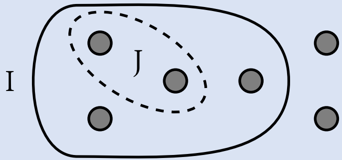
- A deterministic efficient algorithm to $2^{O(\text{rank})}$-approximately count bases of a matroid or common bases of two matroids [A-Oveis Gharan-Vinzant'18].

# Matroids

A matroid is a family $\mathcal{I}$ of subsets of $\{1, \ldots, n\}$, called independent sets:

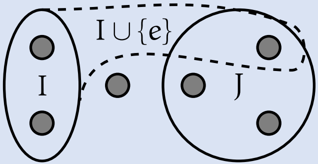
## Downward Closed

If $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$.




## Exchange Axiom

If $I, J \in \mathcal{I}$ and $|J| > |I|$, there is $e \in J - I$ such that $I \cup \{e\} \in \mathcal{I}$.



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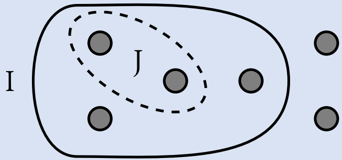


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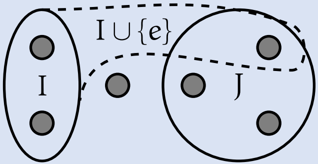


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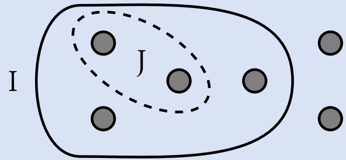


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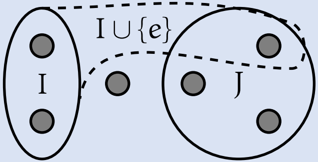



- Bases: Maximal independent sets $\mathcal{B}$. They all have size $\mathsf{rank}$.
- Examples: Uniform, Laminar, Graphic, Linear, Algebraic, Paving, etc.

# Matroid in Real Life 1: Erasures in Linear Codes

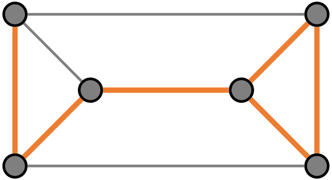
For linear code $\{(x_1, \ldots, x_n) \in \mathbb{F}_2^n \mid Mx = 0\}$, can recover from erasures iff

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & \ldots & M_{1n} \\ M_{21} & M_{22} & M_{23} & M_{24} & \ldots & M_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{k1} & M_{k2} & M_{k3} & M_{k4} & \ldots & M_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ \bullet \\ x_3 \\ \bullet \\ \vdots \\ x_n \end{bmatrix}$$

columns corresponding to erased bits are linearly independent.

# Matroid in Real Life 2: Graph Reliability

For graph $G = (V, E)$ and number $k$, connected $k$-subsets of $E$ form bases of a matroid.



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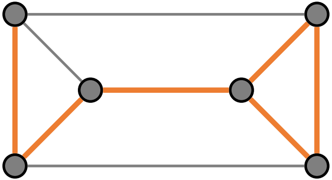

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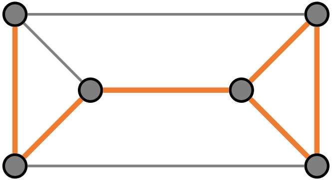
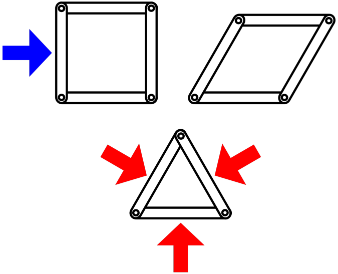


# Matroid in Real Life 3: Rigidity Matroids



Link failure probabilities known. What is the chance the structure remains rigid?

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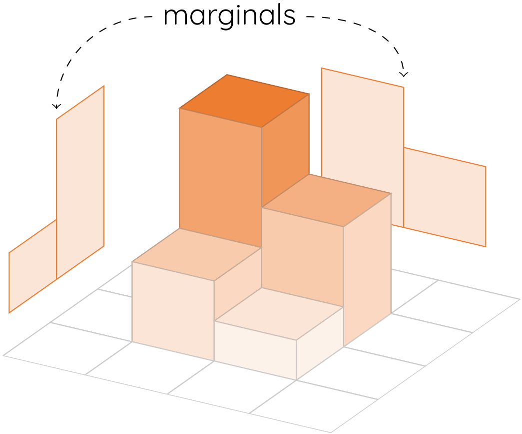

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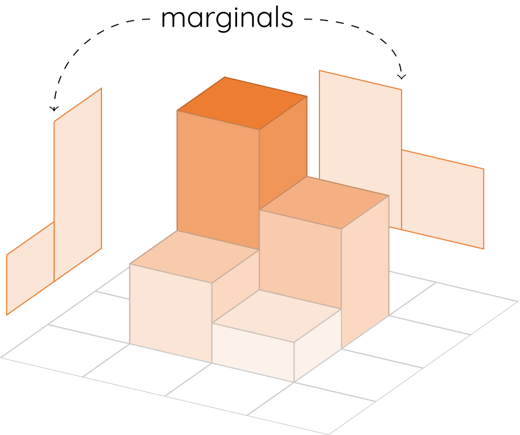


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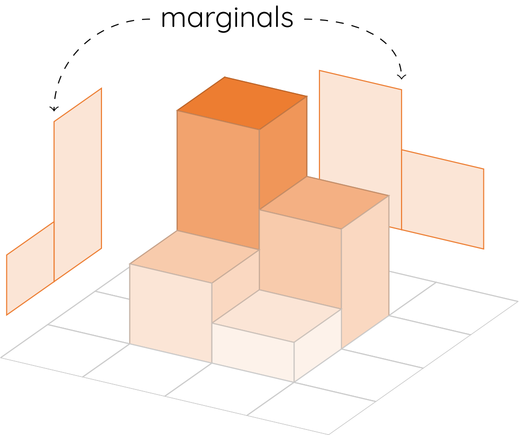


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# Third Attempt: Complete Log-Concavity

Real stable polynomials and strongly Rayleigh measures

- have negative correlation. Matroids were conjectured to have this property [Seymour-Welsh'75], but the same people found a counterexample.

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**Complete Log-Concavity** [A-Oveis Gharan-Vinzant'18 inspired by Gurvits'06]

A polynomial $g \in \mathbb{R}[z_1, \ldots, z_n]$ is completely log-concave iff for any $k \geqslant 0$ and any $v_1, \ldots, v_k \in \mathbb{R}^n_{\geqslant 0}$, the following function is log-concave on $\mathbb{R}^n_{\geqslant 0}$

$$D_{v_1} D_{v_2} \ldots D_{v_k} g.$$

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Based on Hodge theory for matroids [Adiprasito-Huh-Katz'17]:

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- "Perfect" high-dimensional expansion of [Kaufman-Oppenheim'17].

# Déjà-Vu

Based on Hodge theory for matroids [Adiprasito-Huh-Katz'17]:

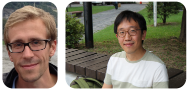
**Matroids are Completely Log-Concave [A-Oveis Gharan-Vinzant'18]**

If $\mu$ is the indicator of bases of a matroid, then $g_\mu$ is completely log-concave:

$$g_\mu(z_1, \ldots, z_m) = \sum_{B \in \mathcal{B}} \prod_{i \in B} z_i.$$

Complete log-concavity is equivalent to:

- Strong log-concavity of [Gurvits'06].
- Mixed order-1 Hodge-Riemann relations in Hodge theory.
- "Perfect" high-dimensional expansion of [Kaufman-Oppenheim'17].
- Notion independently developed by [Brändén-Huh].



# Calculus of Complete Log-Concavity

Complete Log-Concavity: For any $k$ and $v_1, \ldots, v_k \in \mathbb{R}^n_{\geqslant 0}$ the following is log-concave over $\mathbb{R}^n_{\geqslant 0}$:

$$D_{v_1} D_{v_2} \ldots D_{v_k} g(z_1, \ldots, z_n).$$

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**[A-Liu-Oveis Gharan-Vinzant]**

For $d$-homogeneous "connected-support" $g_\mu$ enough to check $k = d - 2$ and

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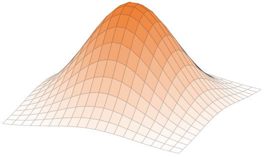
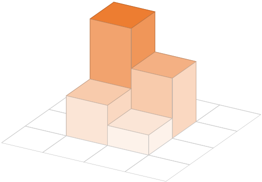
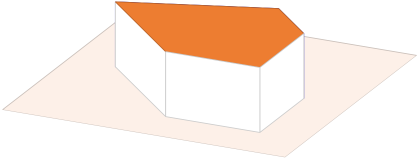
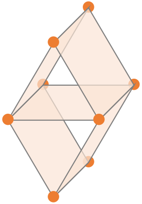
[matroids and bivariate polynomials on board ...]

# Analogy Between Continuous and Discrete

| | Continuous | Discrete |
|---|---|---|
| Distributions: |  |  |
| Supports: |  convex set |  matroid(-like) |

# Mason's Conjecture

**[A-Liu-Oveis Gharan-Vinzant, equivalent form by Brändén-Huh]**

Suppose that $\mathcal{I}$ is the collection of independent sets of a matroid on $\{1, \ldots, n\}$ elements. Then the following is completely log-concave:

$$g(y, z_1, \ldots, z_n) = \sum_{I \in \mathcal{I}} y^{n-|I|} \prod_{i \in I} z_i.$$

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- This finally resolves the strongest form of Mason's conjecture [Mason'72]:

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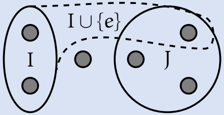
- Weaker form was solved by matroid Hodge theory [Huh-Schröter-Wang'18]:

$$0! \cdot |\mathcal{I}^0|, 1! \cdot |\mathcal{I}^1|, \ldots, \text{rank}! \cdot |\mathcal{I}^{\text{rank}}|.$$

# New World of Complete Log-Concavity

## Matroids

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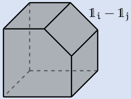
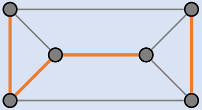
## Submodular Polytopes



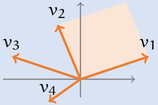

$$\sum_{(\alpha_1, \ldots, \alpha_n) \in P \cap \mathbb{Z}^n_{\geqslant 0}} \frac{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}{\alpha_1! \ldots \alpha_n!}.$$

## Random Cluster Model

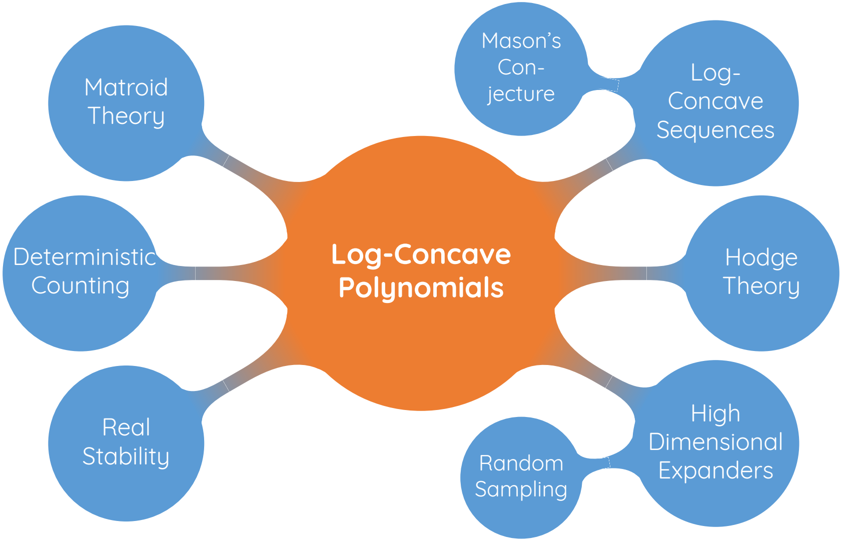


$\mathbb{P}[S] \propto q^{\#cc} p^{|S|} (1-p)^{|\overline{S}|}$ for $q \leqslant 1$.

## Fractional DPPs




$\mathbb{P}[S] \propto |\det([v_i]_{i \in S})|^{\alpha}$ for $\alpha \leqslant 2$.


Log-Concave Polynomials
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Deterministic Counting
Real Stability
Mason's Con-jecture
Log-Concave Sequences
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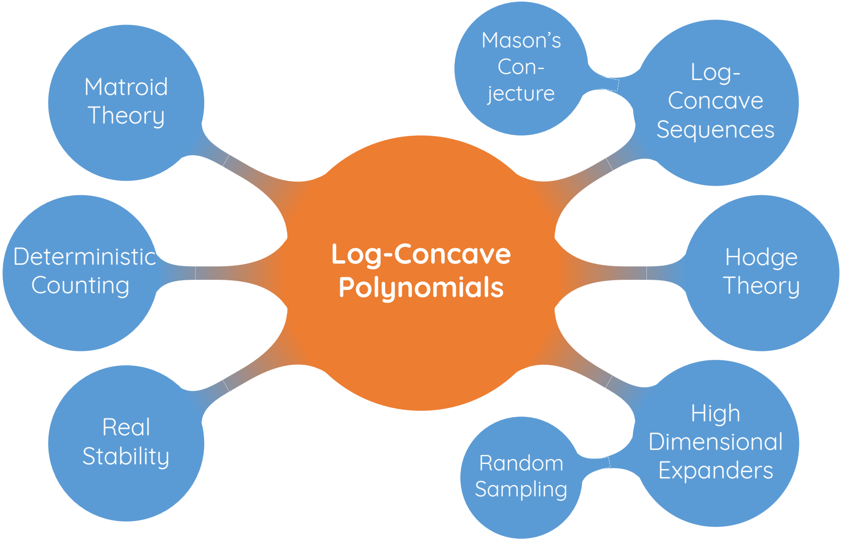
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Thank you!