(P1) **Homogenization of Completely Log-Concave Polynomials**

Suppose that $f(z_1, \ldots, z_n) = f_0 + f_1 + \cdots + f_d$, where $f_k$ is a $k$-homogeneous polynomial in $z_1, \ldots, z_n$. Prove that $f$ is completely log-concave if and only if the following polynomial in the variables $y$ and $z_1, \ldots, z_n$ is completely log-concave:

$$\frac{y^d}{d!} f_0 + \frac{y^{d-1}}{(d-1)!} f_1 + \cdots + \frac{y^0}{0!} f_d.$$

(P2) **Universality of Log-Concavity for Homogeneous Polynomials**

Suppose that $f(z_1, \ldots, z_n)$ is a $d$-homogeneous polynomial with nonnegative coefficients. Prove that the following are equivalent:

(a) $f$ is log-concave over $\mathbb{R}^n_{\geq 0}$.

(b) $f^{1/d}$ is concave over $\mathbb{R}^n_{\geq 0}$.

(c) $f$ is quasi-concave over $\mathbb{R}^n_{\geq 0}$, that is $f^{-1}([1, \infty)) \cap \mathbb{R}^n_{\geq 0}$ is a convex set.

(P3) **Univariate and Multiaffine Bivariate Polynomials**

Prove that the polynomial $a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$ is completely log-concave if and only if $0! \cdot a_0, 1! \cdot a_1, \ldots, d! \cdot a_d$ is a log-concave sequence.

For what $a, b, c, d$ is the polynomial $a + bz_1 + cz_2 + dz_1 z_2 \in \mathbb{R}[z_1, z_2]$ completely log-concave?

Prove that if $S \subseteq [n]$ is a random subset whose distribution is completely log-concave, then for every pair of distinct elements $i, j \in [n]$

$$\mathbb{P}[i, j \in S] \leq 2 \mathbb{P}[i \in S] \mathbb{P}[j \in S].$$

(P4) **Coefficient Products**

Suppose that $f, g \in \mathbb{R}[z_1, \ldots, z_n]$ are completely log-concave polynomials. Let $h$ be defined such that for every $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\partial_{z_1}^{a_1} \cdots \partial_{z_n}^{a_n} h|_{z=0} = \left( \partial_{z_1}^{a_1} \cdots \partial_{z_n}^{a_n} f|_{z=0} \right) \cdot \left( \partial_{z_1}^{a_1} \cdots \partial_{z_n}^{a_n} g|_{z=0} \right).$$

Prove that $h$ is completely log-concave in the following cases: $n = 1$, or when $n = 2$ and $f, g$ are homogeneous. Give an example outside of these cases where $h$ is not completely log-concave.
(P1) **Products**

Prove that when \( f, g \in \mathbb{R}[z_1, \ldots, z_n] \) are completely log-concave, then so is \( f \cdot g \).

(P2) **Operators Preserving Complete Log-Concavity**

Prove that if \( T : \mathbb{R}[z_1, \ldots, z_n]_{\leq 1} \rightarrow \mathbb{R}[z_1, \ldots, z_n] \) is a linear map that preserves real-stability and maps polynomials with nonnegative coefficients to polynomials with nonnegative coefficients, then \( T \) preserves complete log-concavity.

**Hint:** Prove that if the symbol of \( T \) is completely log-concave, then \( T \) preserves complete log-concavity.