Recent Progress on Representation Theoretic Multiplicities in GCT

Christian Ikenmeyer
1. Weakness of occurrence obstructions (with Bürgisser and Panova)

2. Multiplicities are strictly stronger than occurrences (with Dörfler and Panova)

3. Using multiplicities: connecting orbits with their closures (with Kandasamy)
Orbit closures of determinant and permanent

- \( \det_n := \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^{n} x_{i, \pi(i)} \), \( \text{per}_m := \sum_{\pi \in S_m} \prod_{i=1}^{m} x_{i, \pi(i)} \)
- For a linear map \( g : \mathbb{C}^{n^2} \rightarrow \mathbb{C}^{n^2} \) define \( g \det_n \) via \( (g \det_n)(x) := \det_n(g^t(x)) \)
- \( \mathbb{C}^{n^2 \times n^2} \det_n = \{ \text{determinants of } n \times n \text{ matrices whose entries are homog. lin. polyn.} \} \)

Example:

\[
\begin{pmatrix}
  x_{1,1} + x_{1,2} & x_{1,2} - 2x_{2,2} \\
  x_{2,1} & x_{1,1} + x_{1,2}
\end{pmatrix}
\Rightarrow x_{1,1}^2 + 2x_{1,1}x_{1,2} + x_{1,2}^2 - x_{1,2}x_{2,1} + 2x_{2,1}x_{2,2} \in \mathbb{C}^{4 \times 4} \det_2
\]

- Valiant 1979: For all \( m \) there exists \( n \geq m \) such that \( x_{1,1}^{n-m} \text{per}_m \in \mathbb{C}^{n^2 \times n^2} \det_n \).
- Closure: \( \overline{\mathbb{C}^{n^2 \times n^2} \det_n} = \overline{\text{GL}_{n^2} \det_n} \)
- Define \( \text{dc}(\text{per}_m) \) to be the smallest \( n \) such that \( x_{1,1}^{n-m} \text{per}_m \in \overline{\text{GL}_{n^2} \det_n} \).
- GCT Conjecture: \( \text{dc}(\text{per}_m) \) grows superpolynomially.

Observation:

\[
x_{1,1}^{n-m} \text{per}_m \in \overline{\text{GL}_{n^2} \det_n} \iff \overline{\text{GL}_{n^2}(x_{1,1}^{n-m} \text{per}_m)} \subseteq \overline{\text{GL}_{n^2} \det_n}.
\]

Example of a (weak) lower bound technique:
If \( \dim \overline{\text{GL}_{n^2}(x_{1,1}^{n-m} \text{per}_m)} > \dim \overline{\text{GL}_{n^2}(\det_n)} \), then \( \text{dc}(\text{per}_m) > n \).
Coordinate rings

- \( \text{Poly}^n \mathbb{C}^N := \text{homog. degree } n \text{ polyn. in } N \text{ variables.} \)
- \( \text{dim } \text{Poly}^n \mathbb{C}^N = \binom{N+n-1}{n} \)
- \( \mathbb{C}[\text{Poly}^n \mathbb{C}^N]_d := \text{homog. degree } d \text{ polyn. in } \binom{N+n-1}{n} \text{ many variables} \)
- **Example:** \( n = N = 2 \)
  - \( \text{Poly}^2 \mathbb{C}^2 \) has basis \( \{x^2, xy, y^2\} \).
  - Every element in \( \text{Poly}^2 \mathbb{C}^2 \) can be expressed as \( ax^2 + bxy + cy^2 \)
  - \( \mathbb{C}[\text{Poly}^2 \mathbb{C}^2]_2 \) has basis \( \{a^2, ab, ac, b^2, bc, c^2\} \)
  - The discriminant \( b^2 - 4ac \in \mathbb{C}[\text{Poly}^2 \mathbb{C}^2]_2 \)
  - \( b^2 - 4ac = 0 \) iff \( ax^2 + bxy + cy^2 = (\alpha x + \beta y)^2 \) for some \( \alpha, \beta \in \mathbb{C} \)

- **Action of \( \text{GL}_N \) on \( \mathbb{C}[\text{Poly}^n \mathbb{C}^N]_d \):** Define \( (gf)(p) := f(g^t p) \)

For \( Z \subseteq \text{Poly}^n \mathbb{C}^N \), define the coordinate ring:

\[
\mathbb{C}[\overline{Z}] := \mathbb{C}[\text{Poly}^n \mathbb{C}^N]|_{\overline{Z}} \quad \text{(restrict domain of definition to } \overline{Z})
\]

If \( \overline{Y} \subseteq \overline{Z} \), then this gives a natural surjection:

\[
\mathbb{C}[\overline{Z}] \twoheadrightarrow \mathbb{C}[\overline{Y}]
\]

If \( \overline{Z} \) is closed under the action of \( \text{GL}_N \), then \( \mathbb{C}[\overline{Z}] \) inherits the action of \( \text{GL}_N \).
Obstructions based on representation theoretic multiplicities

- Goal: To prove $\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m) \not\subseteq \text{GL}_{n^2}\det_n$
- If $\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m) \subseteq \text{GL}_{n^2}\det_n$, then $\mathbb{C}[\text{GL}_{n^2}\det_n]_d \twoheadrightarrow \mathbb{C}[\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m)]_d$

The group action of $\text{GL}_{n^2}$ lets us decompose into irreducibles:

- $\mathbb{C}[\text{GL}_{n^2}\det_n]_d = \bigoplus_{\lambda} \mathcal{V}^{\oplus \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\det_n]_d)}$
- $\mathbb{C}[\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m)]_d = \bigoplus_{\lambda} \mathcal{V}^{\oplus \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m)]_d)}$

Since the surjection is $\text{GL}_{n^2}$-equivariant, Schur’s lemma implies:

$$\text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\det_n]_d) \geq \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m)]_d)$$

**Multiplicity obstruction:**

If $\exists \lambda$ with $\text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\det_n]_d) < \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m)]_d)$, then $d\text{c}(\text{per}_m) > n$.

**Occurrence obstruction:**

If $\exists \lambda$ with $\text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\det_n]_d) = 0 < \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(x_{1,1}^{n-m}\text{per}_m)]_d)$, then $d\text{c}(\text{per}_m) > n$.

**Theorem [Bürgisser, I, Panova 2016]**, disproving a conj. by Mulmuley and Sohoni

There are no occurrence obstructions that prove $d\text{c}(\text{per}_m) \geq m^{25}$.

Proof relies on the padding of the permanent. Replace $\det_n$ by homogeneous iterated matrix multiplication to avoid this: Boot camp talk
Summary of part 1

- If \( \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\text{det}_n]^d) < \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(x_1^{n-m}\text{per}_m)]^d) \), then \( \text{dc}(\text{per}_m) > n \).

- Occurrence obstruction: \( \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\text{det}_n]^d) = 0 < \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}(x_1^{n-m}\text{per}_m)]^d) \)

- But \( \text{mult}_\lambda(\mathbb{C}[\text{GL}_{n^2}\text{det}_n]^d) > 0 \) in all relevant cases, so that \( \text{dc}(\text{per}_m) > m^{25} \) cannot be proved using occurrence obstructions.

- The proof works in all computational models that involve padding.
Good news: There are group varieties that
- cannot be separated with occurrence obstructions, but
- can be separated with multiplicity obstructions.

(no padding involved)
Multiplicities are strictly stronger than occurrences

Factorizing power sums

Two GL$_m$-varieties:

- Product of homogeneous linear forms:
  \[ \text{Ch}_m^n := \{ \ell_1 \cdots \ell_n \mid \ell_i \in \text{Poly}^1 \mathbb{C}^m \} \subseteq \text{Poly}^n \mathbb{C}^m. \]

- Border Waring rank \( \leq k \) polynomials:
  \[ \text{Pow}_{m,k}^n := \{ \ell_1^n + \cdots + \ell_k^n \mid \ell_i \in \text{Poly}^1 \mathbb{C}^m \} \subseteq \text{Poly}^n \mathbb{C}^m. \]

**Theorem [Dörfler, I, Panova 2019]**

For any \( m \geq 3, n \geq 2 \), let \( k = d = n + 1 \), \( \lambda = (n^2 - 2, n, 2) \). Then
\[ \text{mult}_\lambda (\mathbb{C}[\text{Ch}_m^n]_d) < \text{mult}_\lambda (\mathbb{C}[\text{Pow}_{m,k}^n]_d), \]
i.e., \( \lambda \) is a multiplicity obstruction that shows \( \text{Pow}_{m,k}^n \not\subseteq \text{Ch}_m^n \).

In a finite case we can rule out the existence of occurrence obstructions:

**Theorem [Dörfler, I, Panova 2019]**

Let \( k = 4, n = 6, m = 3, d = 7, \lambda = (n^2 - 2, n, 2) = (34, 6, 2) \). Then
\[ \text{mult}_\lambda (\mathbb{C}[\text{Ch}_m^n]_d) = 7 < 8 = \text{mult}_\lambda (\mathbb{C}[\text{Pow}_{m,k}^n]_d) \]
and hence
\[ \text{Pow}_{m,k}^n \not\subseteq \text{Ch}_m^n. \]

For all \( \mu \): If \( \text{mult}_\mu (\mathbb{C}[\text{Pow}_{m,k}^n]_d) > 0 \), then \( \text{mult}_\mu (\mathbb{C}[\text{Ch}_m^n]_d) > 0. \)
No occurrence obstructions

- Goal: If \( \text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0 \), then \( \text{mult}_\mu(\mathbb{C}[\text{Ch}_3^6]) > 0 \).
- Partitions: \( \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3 \), \( \mu_1 \geq \mu_2 \geq \mu_3 \)

**Proposition (Semigroup properties)**

Let \( \mu \) and \( \nu \) be partitions with \( \text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0 \) and \( \text{mult}_\nu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0 \). Then \( \text{mult}_{\mu+\nu}(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0 \).

Let \( \mu \) and \( \nu \) be partitions with \( \text{mult}_\mu(\mathbb{C}[\text{Ch}_3^6]) > 0 \) and \( \text{mult}_\nu(\mathbb{C}[\text{Ch}_3^6]) > 0 \). Then \( \text{mult}_{\mu+\nu}(\mathbb{C}[\text{Ch}_3^6]) > 0 \).

**Conclusion:** \( \{\mu \mid \text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0\} \) and \( \{\mu \mid \text{mult}_\mu(\mathbb{C}[\text{Ch}_3^6]) > 0\} \) are semigroups.

\( \{\mu \mid \text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0\} \) has 89 generators:

(6), (6, 6), (8, 4), (10, 2), (6, 6, 6), (8, 6, 4), (10, 4, 4), (9, 6, 3), (8, 8, 2), (10, 6, 2), (11, 5, 2), (10, 7, 1), (12, 4, 2), (11, 6, 1), (10, 8), (14, 2, 2), (13, 4, 1), (13, 5), (15, 3), (8, 8, 8), (10, 8, 6), (11, 7, 6), (10, 9, 5), (11, 8, 5), (10, 10, 4), (12, 7, 5), (11, 9, 4), (13, 6, 5), (12, 8, 4), (11, 10, 3), (13, 7, 4), (12, 9, 3), (13, 8, 3), (12, 10, 2), (15, 5, 4), (14, 7, 3), (13, 9, 2), (13, 10, 1), (16, 5, 3), (15, 7, 2), (14, 9, 1), (17, 4, 3), (15, 8, 1), (15, 9), (19, 3, 2), (18, 5, 1), (17, 7), (10, 10, 10), (11, 10, 9), (12, 10, 8), (13, 9, 8), (12, 11, 7), (13, 10, 7), (14, 9, 7), (13, 11, 6), (15, 8, 7), (13, 12, 5), (16, 7, 7), (15, 9, 6), (14, 11, 5), (13, 13, 4), (15, 10, 5), (15, 11, 4), (14, 13, 3), (16, 11, 3), (15, 13, 2), (15, 14, 1), (17, 13), (13, 12, 11), (14, 11, 11), (13, 13, 10), (15, 11, 10), (14, 13, 9), (16, 11, 9), (15, 13, 8), (15, 14, 7), (18, 9, 9), (15, 15, 6), (17, 17, 2), (18, 17, 1), (26, 5, 5), (15, 14, 13), (16, 13, 13), (15, 15, 12), (17, 17, 8), (18, 15, 15), (17, 17, 14), (25, 23), (45, 45).

For each generator \( \mu \) we construct an occurrence of \( V_\mu \) in \( \mathbb{C}[\text{Ch}_3^6] \) by computer.
Multiplicities are strictly stronger than occurrences

Multiplicity obstructions exist

**Theorem [Dörfler, I, Panova 2019]**

For any \( m \geq 3, \ n \geq 2 \), let \( k = d = n + 1 \), \( \lambda = (n^2 - 2, n, 2) \). Then
\[
\text{mult}_\lambda(\mathbb{C}[\text{Ch}_m^n]_d) < \text{mult}_\lambda(\mathbb{C}[\text{Pow}_{m,k}^n]_d),
\]
i.e., \( \lambda \) is a multiplicity obstruction that shows \( \text{Pow}_{m,k}^n \not\subseteq \text{Ch}_m^n \).

**Proof:**

The **plethysm coefficient** \( a_\lambda(d, n) := \text{mult}_\lambda(\mathbb{C}[\text{Poly}^n \mathbb{C}^N]_d) \)

**Proposition [Bürgisser, I, Panova 2016]**

If \( k \geq d \), then \( \text{mult}_\lambda(\mathbb{C}[\text{Pow}_{m,k}^n]_d) = a_\lambda(d, n) \).

In other words: \( \text{Pow}_{m,k}^n \) is a hitting set for degree \( \leq k \) polynomials

Remains to show: \( \text{mult}_\lambda(\mathbb{C}[\text{Ch}_m^n]_d) < a_\lambda(d, n) \)
Multiplicities are strictly stronger than occurrences

Remains to show: \( \text{mult}_{\lambda}(\mathbb{C}[\text{Ch}_m^n]) < a_\lambda(d, n) \) for \( d = n + 1, \lambda = (n^2 - 2, n, 2) \)

Use inheritance theorem: \( \text{mult}_{\lambda}(\mathbb{C}[\text{Ch}_m^n]) = \text{mult}_{\lambda}(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) \)

\( \mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)] := \) rational functions that are defined everywhere on \( \text{GL}_n(x_1 \cdots x_n) \).

\( \mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)] \subseteq \mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)] \), in particular

\[
\text{mult}_{\lambda}(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) \leq \text{mult}_{\lambda}(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]).
\]

\[
\text{mult}_{\lambda}(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) = \dim \mathcal{V}_{\lambda}^H \text{ for } |\lambda| = nd,
\]

where \( H \subseteq \text{GL}_n \) is the stabilizer of \( x_1 \cdots x_n \).

Proof:

\[
\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)] = \mathbb{C}[	ext{GL}_n/H] = \mathbb{C}[	ext{GL}_n]^H \text{ Algebraic Peter-Weyl} = \bigoplus_{\lambda} \mathcal{V}_\lambda \otimes \mathcal{V}_{\lambda}^H \quad \Box
\]

Proposition (proof based on symmetric functions)

For \( \lambda = (n^2 - 2, n, 2) \): \( a_\lambda(n + 1, n) = 1 + a_\lambda(n, n + 1) \).

\[
\text{mult}_{\lambda}(\mathbb{C}[\text{Ch}_m^n]) = \text{mult}_{\lambda}(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) \leq \text{mult}_{\lambda}(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) = a_\lambda(n, d) < a_\lambda(d, n). \Box
\]
Summary of part 2

- \( \text{mult}_\lambda(\mathbb{C}[\text{Ch}_m^n]_d) < \text{mult}_\lambda(\mathbb{C}[\text{Pow}_{m,k}^n]_d) \), therefore \( \text{Pow}_{m,k}^n \not\subseteq \text{Ch}_m^n \).

- Proof based on relationship “orbit vs orbit closure”:
  \( \text{mult}_\lambda(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) \leq \text{mult}_\lambda(\mathbb{C}[	ext{GL}_n(x_1 \cdots x_n)]) \).

- In a finite case we verified by computer:
  there are no occurrence obstructions showing \( \text{Pow}_{m,k}^n \not\subseteq \text{Ch}_m^n \), but multiplicity obstructions work
1. Weakness of occurrence obstructions (with Bürgisser and Panova)

2. Multiplicities are strictly stronger than occurrences (with Dörfler and Panova)

3. Using multiplicities: connecting orbits with their closures (with Kandasamy)
[Bürgisser, I 2017] connects orbit and orbit closure more closely:

- Let $0 \neq \Phi \in \mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]$ be invariant under $\text{SL}_m$ ($x_1^n + \cdots + x_m^n$ is not in the null cone)
- Then $\Phi$ is nonzero everywhere on $\text{GL}_m(x_1^n + \cdots + x_m^n)$
- It turns out: $\Phi$ vanishes on the boundary $\overline{\text{GL}_m(x_1^n + \cdots + x_m^n)} \setminus \text{GL}_m(x_1^n + \cdots + x_m^n)$
  (proof uses Hilbert-Mumford criterion and refinement by Luna and Kempf)
- As a result:
  $$\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)] = \mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_{\Phi}$$
  is the localization at $\Phi$.

**Theorem [Bürgisser, I 2017]**

For all $d$ there is $e$:

$$\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_d \xrightarrow{\gamma} \mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_{d+em} \subseteq \mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_{d+em},$$

where $\gamma(f) := \Phi^e f$.

**Theorem [I, Kandasamy 2019]**

For even $n$, an upper bound on the required $e$ is $m + 4 \frac{d}{n}$.
Using multiplicities: connecting orbits with their closures

**Theorem [Bürgisser, I 2017]**

For all $d$ there is $e$:
\[
\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_d \otimes^{\Phi_e} \mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_{d+em} \subseteq \mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]_{d+em}.
\]

**Theorem [I, Kandasamy 2019]**

For even $n$, an upper bound on the required $e$ is $m + 4\frac{d}{n}$.

- Given a Young tableau $T$, we can explicitly construct a function $f_T$ in $\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]$.
- All highest weight functions in $\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]$ can be constructed in this way.
- We have a combinatorial/linear algebra way of evaluating at points.

We have a similar situation in $\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]$:

- Given a Young tableau $S$, we can explicitly construct a function $f_S$ in $\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)] \simeq \mathbb{V}_\lambda^H$.
- All highest weight functions in $\mathbb{C}[\text{GL}_m(x_1^n + \cdots + x_m^n)]$ can be constructed in this way.
- We have a combinatorial/linear algebra way of evaluating at points.

- Proof idea of I-Kandasamy: Given a tableau $S$, construct a slightly larger tableau $T$ such that $f_T$ and $f_S$ coincide on $\text{SL}_m(x_1^n + \cdots + x_m^n)$. 
Summary of part 3

- The representation theory of $\mathbb{C}[\text{GL}_N\mathbb{F}_p]$ can usually be much better understood than the representation theory of $\mathbb{C}[\text{GL}_N\mathbb{F}]$

- In many cases of interest: the representation theory of $\mathbb{C}[\text{GL}_N\mathbb{F}_p]$ and $\mathbb{C}[\overline{\text{GL}_N\mathbb{F}_p}]$ is connected by a fundamental invariant $\Phi$

- In the case of power sums, this connection is very close

- The hope is that $\mathbb{C}[\text{GL}_N\mathbb{F}_p]$ and $\mathbb{C}[\overline{\text{GL}_N\mathbb{F}_p}]$ are closely related in more involved cases
Where does the hope for multiplicities come from?

Let $H \subseteq \text{GL}_N$ be the stabilizer of $p$.

$$\text{mult}_\lambda(\mathbb{C}[\text{GL}_N p]) = \dim \mathcal{V}_\lambda^H$$

**Theorem [Larsen and Pink 1990, Inventiones math.]**

$H \subseteq \text{GL}_N$. Under reasonable assumptions, the group $H$ is determined (up to group isomorphism) by the dimensions $\dim \mathcal{V}_\lambda^H$.

Pick $H$ to be the stabilizer of a point $p$ that is **characterized by its stabilizer**:

- determinant
- permanent
- iterated matrix multiplication polynomial
- power sum polynomial
- multilinear monomial
- matrix multiplication tensor
- unit tensor

Conclusion: A strengthening of this theorem would yield that $p$ is characterized by its multiplicities.
Summary

- In the computational models with padding there are no occurrence obstructions that prove strong lower bounds.

- The padding can be removed: Iterated matrix multiplication.

- But even in small explicit unpadded cases: multiplicity obstructions are stronger than occurrence obstructions.

- Multiplicities in $\mathbb{C}[\text{GL}_N]$ can be studied with algebraic combinatorics. The connection to $\mathbb{C}[\text{GL}_N]$ is hopefully close. (This works for power sums)

- Larsen and Pink: Give hope for multiplicity obstructions.

Thank you for your attention!