The GCT Program: Recent developments and concrete open problems

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The main reference for this talk
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[GCT5][M.]: Geometric Complexity Theory V: Efficient algorithms for Noether Normalization.
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The basic set up

▶ $K$: An algebraically closed field of characteristic zero.
▶ $VP$: The class of families of polynomials that can be computed by algebraic circuits over $K$ of polynomial degree and size.
▶ $VP_{ws}$: The class of families of polynomials that can be computed by symbolic determinants over $K$ of polynomial size.
▶ $VNP$: The class of families of $p$-definable polynomials (e.g. the permanent).
▶ $VP$: The class of families of polynomials that can be approximated infinitesimally closely by algebraic circuits over $K$ of polynomial degree and size.
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The battleground of GCT: The $\text{VP}$ vs. $\overline{\text{VP}}$ problem

Conjecture (Valiant: 1979):

$\text{VP} \neq \overline{\text{VNP}}$.

The hardness hypothesis of GCT (GCT1:MS2001):

$\overline{\text{VNP}} \not\subseteq \text{VP}$.

Any realistic approach to the $\text{VP}$ vs. $\overline{\text{VNP}}$ problem can be expected to prove this stronger form of Valiant's conjecture.

Question [GCT1,B,BLMW]: Is $\text{VP} = \overline{\text{VP}}$?

▶ Related to foundational issues in algebraic geometry and representation theory [GCT6]. [Not covered in this lecture].

▶ The cause of a deep difficulty at the interface of algebraic geometry, representation theory and complexity theory, called the GCT chasm, which arises in the context of the $\text{VP}$ vs. $\overline{\text{VNP}}$ problem, regardless of whether the answer to this question is affirmative or negative [GCT5] [This lecture].

▶ This difficulty has to be overcome by any approach to the $\text{VP}$ vs. $\overline{\text{VNP}}$ problem that seeks to separate $\overline{\text{VNP}}$ from $\text{VP}$.

We call any such approach a GCT approach in a broad sense.
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Conjecture (Valiant: 1979): $\text{VP} \neq \text{VNP}$.
The battleground of GCT: The VP vs. VP problem

Conjecture (Valiant: 1979): VP \neq VNP.

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- Let $Y$ be an $m \times m$ variable matrix, $X$ an $n \times n$ submatrix of $Y$, $n < m$, and $z$ any entry of $Y$ outside $X$. 

$V = K[Y]^m$: The space of homogeneous forms of degree $m$ in the entries of $Y$.

$P(V)$: The projective space associated with $V$.

$\Sigma[\det, m] \subseteq P(V)$: The set of all points in $P(V)$ corresponding to non-zero homogeneous polynomials in the entries of $Y$, which can be expressed as determinants of symbolic $m \times m$ matrices, whose entries are homogeneous linear functions of the entries of $Y$ (a constructible set).

$\Delta[\det, m] = \Sigma[\det, m] \subseteq P(V)$: The Zariski closure of $\Sigma[\det, m]$ in $P(V)$ (a variety).

$\Delta[\det, m]$ also equals the $\text{GL}_m(2)$-orbit-closure of $\det(Y) \in P(V)$ under the natural action of $\text{GL}_m(2)$ on $P(V)$. 

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Reformulation of the $\overline{VP_{ws}}$ vs. $VNP$ problem (continued)

The $VNP \not\subseteq VP_{ws}$ conjecture [Valiant] is equivalent to saying that

$$z^m - n \text{perm}(X) \not\in \Sigma[\text{det}, m].$$

The $VNP \not\subseteq VP_{ws}$ conjecture [GCT1] is equivalent to saying that

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(A variety) $\Sigma[\text{det}, m]$ $\Delta[\text{det}, m]$ (A constructible set)

The geometry of $\Sigma[\text{det}, m]$ is controlled by the singularities of $\Delta[\text{det}, m]$. Hence their structure is important in the context of the $VP_{ws}$ vs. $VP_{ws}$ and $VP_{ws}$ vs. $VNP$ problems.

Unfortunately, the singularities of $\Delta[\text{det}, m]$ are not even normal [Kumar]. This is the beginning of difficulties [Next].
Reformulation of the $\overline{\text{VP}}_{\text{ws}}$ vs. $\text{VNP}$ problem (continued)

▶ The $\text{VNP} \not\subset \overline{\text{VP}}_{\text{ws}}$ conjecture [Valiant] is equivalent to saying that $z^{m-n}\text{perm}(X) \notin \Sigma[\text{det}, m]$.

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Noether’s Normalization Lemma (NNL)

Hilbert: There exists a homogeneous linear map $\psi: V \rightarrow K^k$, for any $k > \dim(\Delta_{\det, m})$, such that $\psi$ does not vanish on any non-zero point in the affine cone $\hat{\Delta}_{\det, m} \subseteq V$ of $\Delta_{\det, m} \subseteq \mathbb{P}(V)$. This means the rational map $\hat{\psi}: \mathbb{P}(V) \otimes K^k \mathbb{P}(K^k)$ is regular (well-defined) on $\Delta_{\det, m} \subseteq \mathbb{P}(V)$.

We call such a homogeneous, linear map $\psi: V \rightarrow K^k$ a normalizing map for $\Delta_{\det, m}$.

The Problem NNL: Given $\Delta_{\det, m}$, with a succinct specification, construct a normalizing map $\psi: V \rightarrow K^k$, with $k = \text{poly}(m)$, with a succinct specification. Succinct means of $\text{poly}(m)$ size. The usual specifications of $\Delta_{\det, m} \subseteq \mathbb{P}(V)$ by its equations or of $\psi$ by its matrix are not succinct, since $\dim(V) = 2\text{poly}(m)$. 
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Given $\Delta[\det, m]$, with a succinct specification, construct a normalizing map $\psi : V \to K^k$, with $k = \text{poly}(m)$, with a succinct specification.
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The current complexity status of the problem NNL

The problem NNL for $\det_{m}$ is equivalent to the problem of constructing a hitting set for $\text{VP}_{ws}$. This, in conjunction with Gröbner basis theory, implies that NNL is in EXPSPACE [GCT5]. Recent development [Forbes and Shpilka; Guo, Saxena, Sinhababu]: NNL is in PSPACE. This is how far we can go without knowing the relationship between $\text{VP}_{ws}$ and $\text{VP}_{ws}$. If $\text{VP}_{ws} = \text{VP}_{ws}$, then NNL is in PH, assuming generalized Riemann hypothesis. Where is NNL?
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NNL vs. the Hardness Hypothesis of GCT

Theorem (GCT5)

NNL is P (ignoring a quasi prefix) iff a variant of the hardness hypothesis of GCT holds.

▶ The variant:

Some exponential-time computable multilinear polynomial cannot be approximated infinitesimally closely by sub-exponential-size algebraic circuits.

▶ The proof:

Classical algebraic geometry [Hilbert, ...] + algebraic complexity theory [Kaltofen and Trager (the crux of the proof), Heintz and Schnorr, Kabanets and Impagliazzo, Nisan and Wigderson].

▶ Analogous result holds, in general, for any explicit variety in place of $\Delta[\det, m]$.

▶ By an explicit variety, we mean any variety whose coordinate ring has a set of generators that can be encoded succinctly and uniformly by algebraic circuits of size polynomial in the dimension of the variety.
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An Algorithmic Challenge

An Intermediate Problem: Show that NNL is in PH (assuming only GRH).

This is a challenge regardless of whether \( VP_{ws} = VP \) or not.

If \( VP_{ws} = VP \), the challenge is to show this equality.

If not, the task gets even harder.

Hence bringing NNL from PSPACE to PH would need overcoming the \( VP_{ws} vs. VP \) problem [the battleground of GCT], one way or the other.

[GCT6]: The \( VP_{ws} vs. VP \) problem is related to foundational issues in algebraic geometry and representation theory.

Hence bringing NNL to PH may need a deep synthesis and extension of the existing techniques of algebraic geometry, representation theory, and complexity theory.
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The GCT chasm

We call the existing PSPACE vs. P gap in the complexity of NNL the GCT chasm (revising the earlier definition in GCT5, thanks to [FS,GSS]).

battleground of GCT

PSPACE

NNL

P

PH

NP

The entry to the GCT Chasm (the VP vs. VP problem)

▶ This GCT chasm will have to be crossed by any approach to the VP vs. VNP which also separates VNP from VP in the process.

Recall: By definition, any such approach is a GCT approach in a broad sense.

▶ GCT5, GCT6, and GCT7 provide a concrete GCT program to cross the GCT chasm.
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The GCT chasm

We call the existing PSPACE vs. P gap in the complexity of NNL the GCT chasm (revising the earlier definition in GCT5, thanks to [FS,GSS]).

This GCT chasm will have to be crossed by any approach to the VP vs. VNP which also separates VNP from $\overline{VP}$ in the process. Recall: By definition, any such approach is a GCT approach in a broad sense.

GCT5, GCT6, and GCT7 provide a concrete GCT program to cross the GCT chasm.
The first step of the GCT program to cross the GCT chasm (The Orbit Closure Intersection Problem)
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Given $(V, G)$, and rational points $v, w \in V$, decide if the $G$-orbit-closures of $v$ and $w$ intersect.
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The orbit-closure intersection problem is in $P$, for any finite-dimensional representation $V$ of a reductive group $G$, if (1) the categorical quotient $V \sslash G = \text{spec}(K[V]^G)$ is explicit, and (2) the white-box PIT is in $P$. 
The Orbit-Closure-Intersection Hypothesis

The orbit-closure intersection problem is in P, for any finite dimensional representation $V$ of a reductive group $G$ (possibly disconnected).

Expected to be an inherent difficulty underneath white-box PIT.

The status of the hypothesis:

▶ Holds if $G$ is connected and $\dim(G)$ is constant [GCT5].

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This special case of the GCT hypothesis above implies a polynomial time algorithm for non-commutative rational identity testing.

▶ Holds if $V = K(n^2)$ with the natural action of $S_n$ (Weighted Graph Isomorphism): [Babai 2017].
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A special case of the hypothesis for finite groups

The Orbit Equality Problem:
Show that the problem of deciding, given any representation $V$ of a finite group $G$ and two rational points $v, w \in V$, whether $v$ and $w$ lie in the same $G$-orbit belongs to P.

This is a special case of the orbit-closure-intersection problem for finite groups.

The main obstacles:
1. Classification of all finite groups (not just finite simple groups) is not yet known. In fact, this is the most outstanding open problem of finite group theory.
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This is why the techniques such as operator scaling and optimization are unlikely to work for white-box PIT.
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An overview of the GCT program

- Hardest: Prove that \( \text{VNP} \not\subseteq \text{VP} \) (the hardness hypothesis of GCT), using obstructions \([\text{GCT2:MS2008}]\).

- Occurrence obstructions do not exist \([\text{Bürgisser, Ikenmeyer, Panova}]\).

- GCT7: a systematic program to prove existence of multiplicity obstructions.

- Easier: Show that the problem NNL for general explicit varieties is in P. GCT5,6,7: a systematic program for this.

- Much easier \([\text{GCT5}]\) [Not covered in this talk] [An inherent difficulty underneath black-box PIT]: Show that the problem NNL for the categorical quotient \( V//G \) is in P, for any finite dimensional representation \( V \) of any reductive group \( G \).

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- **Hardest**: Prove that $\text{VNP} \not\subseteq \overline{\text{VP}}$ (the hardness hypothesis of GCT), using obstructions [GCT2:MS2008]. (Occurrence obstructions do not exist [Bürgisser, Ikenmeyer, Panova]). GCT7: a systematic program to prove existence of multiplicity obstructions.

- **Easier**: Show that the problem NNL for general explicit varieties is in $\text{P}$. GCT5,6,7: a systematic program for this.

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- **Easier:** Show that the problem NNL for general explicit varieties is in $P$. GCT5,6,7: a systematic program for this.

- **Much easier [GCT5]** [Not covered in this talk] [An inherent difficulty underneath black-box PIT]: Show that the problem NNL for the categorical quotient $V//G$ is in $P$, for any finite dimensional representation $V$ of any reductive group $G$.

- **Easiest [GCT5]** [Covered in this talk] [An inherent difficulty underneath white-box PIT]: Show that the orbit-closure intersection problem is in $P$, for any finite dimensional representation $V$ of any reductive group $G$ (possibly disconnected).