Invariant theory-
a gentle introduction
for computer scientists
(optimization and complexity)

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Prehistory

Linial, Samorodnitsky, W 2000  Cool algorithm
Discovered many times before

Kruithof 1937  in telephone forecasting,
Deming-Stephan 1940 in transportation science,
Brown 1959  in engineering,
Wilkinson 1959 in numerical analysis,
Stone 1964 in economics,
Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns
Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

\[
\begin{array}{ccc}
1/3 & 1/3 & 1/3 \\
1/2 & 1/2 & 0 \\
1 & 0 & 0 \\
\end{array}
\]
Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

\[
\begin{array}{ccc}
2/11 & 2/5 & 1 \\
3/11 & 3/5 & 0 \\
6/11 & 0 & 0 \\
\end{array}
\]
Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns
Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

A very different efficient Perfect matching algorithm

We’ll understand it much better with Invariant Theory

Converges (fast) iff Per(A) >0
Outline

Main motivations, questions, results, structure

• Algebraic Invariant theory
• Geometric invariant theory
• Optimization & Duality
• Moment polytopes
• Algorithms
• Conclusions & Open problems
Invariant Theory

symmetries, group actions, orbits, invariants

Physics, Math, CS
Linear actions of groups

Group $G$ acts *linearly* on vector space $V \ (= F^d)$. $[F = C]$

Action: Matrix-Vector multiplication

$M: G \to GL(V) \ (d \times d \text{ matrices})$ group homomorphism.

$M_g: V \to V$ invertible linear map $\forall g \in G$.

$M_{g_1g_2} = M_{g_1}M_{g_2}$ and $M_{id} = id$.

**Ex 1** $G = S_n$ acts on $V = C^n$ by *permuting coordinates*.

$M_\sigma \ (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

**Ex 2** $G = GL_n(C)$ acts on $V = M_n(C)$ by *conjugation*.

$M_A X = AXA^{-1} \ (d = n^2 \gg n \text{ variables})$. 

$G$ reductive.

$g \to M_g$ rational.
Objects of study

Group $G$ acts \textit{linearly} on vector space $V = \mathbb{C}^d$, and also on polynomials $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_d]$

- **Invariant polynomials**: under action of $G$:
  
  $p$ s.t. $p(M_g v) = p(v)$ for all $g \in G$, $v \in V$.

- **Orbits**: Orbit of vector $v$, $O_v = \{M_g v : g \in G\}$

- **Orbit-closures**: An orbit $O_v$ may not be closed. Take its closure in Euclidean topology.
  
  $\overline{O_v} = \text{cl} \{M_g v : g \in G\}$. 
Example 1

\[ G = S_n \text{ acts on } V = \mathbb{C}^n \text{ by permuting coordinates.} \]

\[ M_\sigma (x_1, ..., x_n) \rightarrow (x_{\sigma(1)}, ..., x_{\sigma(n)}) . \]

• **Invariants:** symmetric polynomials.

• **Orbits:** \( x, y \) in same orbit iff they are of *same type*.

\[ \forall c \in \mathbb{C}, |\{ i : x_i = c \}| = |\{ i : y_i = c \}|. \]

• **Orbit-closures:** same as orbits.
Example 2

$G = GL_n(C)$ acts on $V = M_n(C) = C^{n^2}$ by conjugation.

\[ M_A X = AXA^{-1}. \]

• Invariants: trace of powers: $\text{tr}(X^i)$.

• Orbits: Characterized by Jordan normal form.

• Orbit-closures: differ from orbits.

1. $\overline{O}_X \neq O_X$ iff $X$ is not diagonalizable.

2. $\overline{O}_X$ and $\overline{O}_Y$ intersect iff $X, Y$ have the same eigenvalues.
Orbits and orbit-closures in TCS

- **Graph isomorphism**: Whether orbits of two graphs the same.  
  **Group action**: permuting the vertices.

- **Border rank**: Whether a tensor lies in the orbit-closure of the diagonal unit 3-tensor.  
  **Special case**: Matrix Multi exponent  
  **Group action**: Natural action of $GL_n(C) \times GL_n(C) \times GL_n(C)$.

- **PIT** Does an $n \times n$ symbolic determinant on $m$ variables vanish?  
  **Group action**: Natural action of $GL_n(C) \times GL_n(C) \times GL_m(C)$.

- **Property testing**: Graphs – **Group action**: Symmetric group,  
  Codes - **Group action**: Affine group

- **Arithmetic circuits**: The $VP$ vs $VNP$ question via GCT program:  
  Whether permanent lies in the orbit-closure of the determinant.  
  **Group action** = **Reductions**: Action on polynomials induced by linear transformation on variables.
Invariant ring

Group $G$ acts \textit{linearly} on vector space $V$.


[Hilbert 1890, 93]: $C[V]^G$ is \textit{finitely generated}!

Nullstellansatz, \textit{Finite Basis Theorem} etc. proved in these papers as \textit{“lemmas”}! Also origin of \textit{Grobner Basis Algorithm}

1. $G = S_n$ acts on $V = C^n$ by permuting coordinates.
   $C[V]^G$ generated by \textit{elementary symmetric} polynomials.

2. $G = GL_n(C)$ acts on $V = M_n(C)$ by conjugation.
   $C[V]^G$ generated by $\text{tr}(X^i)$, $1 \leq i \leq n$.

[Derksen 2000]: $C[V]^G$ is \textit{generated} by degree $\exp(n)$

Degree bound $\leq n$
Computational invariant theory

Highly *algorithmic field*. Algorithms sought and well developed. Polynomial eq sys solving, ideal bases, comp algebra, FFT, MM via groups,...

Main problems:
• Describe all invariants (*generators, relations*).
• Simpler: *degree bounds* for generating set.
• *Isomorphism/Word problem*: When are two objects the “same”?
  1. Orbit intersection.
  2. Orbit-closure intersection.
  3. Noether normalization, Mulmuley’s GCT5,...
  4. Orbit-closure containment.
  5. Simpler: *null cone*. When is an object ``like'' 0? Is $0 \in \overline{O_v}$?
Geometric invariant theory (GIT)
Null cone  Captures many interesting questions.

Group $G$ acts \textit{linearly} on vector space $V$.

Null cone: Vectors $v$ s.t. $0$ lies in the orbit-closure of $v$.

$$N_G(V) = \{v: \text{ } 0 \in \overline{O_v}\}.$$ 

Sequence of group elements $g_1, \ldots, g_k, \ldots$ s.t. $\lim_{k \to \infty} M_{g_k} v = 0$.

Problem: Given $v \in V$, decide if it is in the null cone.

Optimization/Analytic: Is $\inf_{g \in G} ||M_g v|| = 0$ ?

Algebraic:[Hilbert 1893; Mumford 1965]: $v$ in null cone iff $p(v) = 0$ for \textit{all} homogeneous invariant polynomials $p$.

\begin{itemize}
  \item One direction clear (polynomials are continuous).
  \item Other direction uses \textit{Nullstellensatz} and algebraic geometry.
\end{itemize}

\begin{center}
\textbf{analytic} $\leftrightarrow$ \textbf{algebraic} \\
\textbf{optimization} $\leftrightarrow$ \textbf{complexity}
\end{center}
Example 1

\( G = S_n \) acts on \( V = \mathbb{C}^n \) by permuting coordinates.

\[
M_\sigma(x_1, \ldots, x_n) \rightarrow (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

Null cone = \{0\}.

No closures (same for all \textit{finite} group actions).
Example 2

\[ G = GL_n(C) \] acts on \[ V = M_n(C) \] by conjugation.

\[ M_A X = AXA^{-1}. \]

• **Invariants**: generated by \( \text{tr}(X^i) \).

• **Null cone**: nilpotent matrices.
Example 3

\[ G = SL_n(C) \times SL_n(C) \] acts on \[ V = M_n(C) \] by left-right multiplication.

\[ M_{(A,B)} X = AXB. \]

- **Invariants**: generated by \( \text{Det}(X) \).
- **Null cone**: Singular matrices.
Example 4: Matrix Scaling

\( ST_n \): group of \( n \times n \) diagonal matrices with determinant 1.
\( G = ST_n \times ST_n \) acts on \( V = M_n(C) \) by left-right multiplication.
\[
M_{(A,B)} X = AXB.
\]

- Invariants: generated by matchings \( X_{1,\sigma(1)}X_{2,\sigma(2)} \cdots X_{n,\sigma(n)} \).
- Null cone \( \leftrightarrow \) \( \text{Per}(X)=0 \)
- \( A_H \) is in null cone \( \leftrightarrow \) \( H \) has no perfect matching.
Example 5: Operator Scaling

\[ G = SL_n(C) \times SL_n(C) \] acts on \[ V = M_n(C)^{\oplus m} \]
by \textit{simultaneous left-right} multiplication.

\[ M_{(A,B)} (X_1, \ldots, X_m) = (AX_1B, \ldots, AX_mB). \]

- Invariants \([\text{DW 00, DZ 01, SdB 01}]\): generated by \( \text{Det}(\sum_i D_i \otimes X_i) \).
- Null cone \(\leftrightarrow\) Non-commutative singularity of symbolic matrices.
\(\leftrightarrow\) Non-commutative rational identity testing \(\leftrightarrow\) ...

\([\text{GGOW 16, IQS 16}]\): Deterministic polynomial time algorithms.

\([\text{DM 16}]\): Polynomial degree bounds on generators.
Example 6: Linear programming

$G = T_n$: (Abelian!) group of $n \times n$ diagonal matrices.

$V$: Laurent polynomials. $q \in V$ (poly w/some exponents negative).

$G$ acts on $V$ by scaling variables. $t \in T_n$, $t = \text{diag}(t_1, ..., t_n)$.

$$M_t q(x_1, ..., x_n) = q(t_1 x_1, ..., t_n x_n).$$

$q = \sum_{\alpha \in \Omega} c_\alpha x^\alpha$. $\text{supp}(q) = \{\alpha \in \Omega: c_\alpha \neq 0\}$.

Null cone $\leftrightarrow$ Linear Programming

$q \text{ not in null cone} \leftrightarrow 0 \in \text{conv}\{\text{supp}(q)\}$. (=Newton polytope $(q)$)

In non-Abelian groups, the null cone (membership) problem is a non-commutative analogue of Linear Programming.
GIT: computational perspective

What is complexity of null cone membership?

GIT puts it in $NP \cap coNP$ (morally).

• Hilbert-Mumford criterion:
  how to certify membership in null cone.
• Kempf-Ness theorem:
  how to certify non-membership in null cone.

Many mathematical characterizations have this flavor.
Begs for complexity theoretic quantification
(e.g proof complexity approach to Nullstellensatz, Positivstellensatz...)

Hilbert-Mumford

Group $G$ acts linearly on vector space $V$. How to certify $\nu \in N_G(V)$ (null cone)? Sequence of group elements $g_1, \ldots, g_k, \ldots$ such that $\lim_{k \to \infty} M_{g_k} \nu = 0$.

Compact description of the sequence? Given by one-parameter subgroups. [Hilbert 1893; Mumford 1965]: $\nu \in N_G(V)$ iff $\exists$ one-parameter subgroup $\lambda: \mathbb{C}^* \to G$ s.t. $\lim_{t \to 0} M_{\lambda(t)} \nu = 0$. 
One-parameter subgroups

One-parameter subgroup: Group homomorphism $\lambda: \mathbb{C}^* \to G$. Also this map is algebraic.

• $G = \mathbb{C}^*$: $\lambda(t) = t^a$, $a \in \mathbb{Z}$.

• $G = T_n = (\mathbb{C}^*)^\times n$: $\lambda(t) = \text{diag}(t^{a_1}, \ldots, t^{a_n})$, $a_i \in \mathbb{Z}$.

• $G = ST_n$: $\lambda(t) = \text{diag}(t^{a_1}, \ldots, t^{a_n})$, $a_i \in \mathbb{Z}$, $\sum_i a_i = 0$.

• $G = GL_n$: $\lambda(t) = S \ \text{diag}(t^{a_1}, \ldots, t^{a_n})S^{-1}$, $S \in GL_n$, $a_i \in \mathbb{Z}$. (Abelian, up to a basis change $S$)
Example: Matrix Scaling & Perfect Matching

\[ G = ST_n \times ST_n \quad (ST_n: n \times n \text{ diagonal matrices with } \det 1) \]
acts on \( V = M_n \quad (X \text{ an } n \times n \text{ matrix}) \)
\[
M_{(A,B)} X = AXB.
\]

\( X \) in null cone \( \iff \exists a_1, ..., a_n, b_1, ..., b_n \in \mathbb{Z}: \)
\[
\sum_i a_i = \sum_j b_j = 0
\]
s.t. \( a_i + b_j > 0 \quad \forall (i,j) \in \text{supp}(X). \)
\( \iff \text{Supp}(X) \) has no perfect matching (Hall’s theorem)
\[
\text{Supp}(X) = \{(i,j) \in [n] \times [n]: X_{i,j} \neq 0\} \quad \text{(adjacency matrix of } X)\]

1-parameter subgroups: \( \lambda(t) = \left( (t^{a_1}, ..., t^{a_n}), (t^{b_1}, ..., t^{b_n}) \right) \)
\[
a_i, b_j \in \mathbb{Z}: \quad \sum_i a_i = \sum_j b_j = 0.
\]
\( \lambda(t) \) sends \( X \) to 0 \( \iff a_i + b_j > 0 \quad \forall (i,j) \in \text{supp}(X) \)
Kempf-Ness

Group $G$ acts linearly on vector space $V$. How to certify $v$ is not in null cone?

**Algebraic:** Exhibit *invariant* polynomial $p$ s.t. $p(v) \neq 0$. Typically doubly exponential time...

*Invariants hard* to find, high degree, high complexity etc.

**Analytic:** Kempf-Ness provides a more efficient way.
An optimization perspective (+ duality!)

Finding *minimal norm* elements in orbit-closures!

Group $G$ acts linearly on vector space $V$.

$$\text{cap}(v) = \inf_{g \in G} \|M_g v\|_2^2.$$  

$\text{cap}(v) = 0 \iff v \in \text{Null cone}$

$\text{cap}(v) > 0 \iff v \notin \text{Null cone}$

$\iff \mu_G(v^*) = 0$  

$\mu_G$ moment map (gradient)

$\iff v$ can be "scaled"

Minimizing $\mu_G$ is a dual optimization problem.
Moment map

Group \( G \) acts linearly on vector space \( V \).

\textit{Moment map} \( \mu_G(v) \): gradient of \( \| M_g v \|^2 \) at \( g = id \).

How much \textit{norm} of \( v \) decreases by \textit{infinitesimal action} near \( id \).

\( \mu_G(v) \): a linear function (like the familiar gradient), on a linear space called the Lie algebra of the group \( G \).

\( \mu_G \) can be defined in more general contexts.

Moment \( \rightarrow \textit{momentum} \).

Fundamental in \textit{symplectic geometry} and \textit{physics}.

Minimizing \( \mu_G(v) = 0 \) (finding \( \mu_G(v^*) = 0 \)) is a scaling problem!
Example 1: Matrix Scaling

\[ G = ST_n \times ST_n \] acts on \( V = M_n \). \hspace{1cm} \( M_{(A,B)} X = AXB. \)

Consider only \( w: \sum_j w(j) = 0 \)

\[ A(s) = \text{diag} \exp(s \ q_1), \quad B(s) = \text{diag} \exp(s \ q_2) \]

Directional derivative: action of \( (A(s), B(s)) \) on \( X, s \approx 0 \).

\[ \mu_G(X) = (p_1, p_2), \quad \sum_i p_1(i) = \sum_j p_2(j) = 0 \quad \text{s.t.} \]

\[ \langle p_1, q_1 \rangle + \langle p_2, q_2 \rangle = \frac{d}{ds} \left[ \| M_{(A,B)} X \|_F^2 \right]_{s=0} \]

\[ = \langle r_X, q_1 \rangle + \langle c_X, q_2 \rangle \]

\[ = \langle r_X - \alpha \mathbf{1}, q_1 \rangle + \langle c_X - \alpha \mathbf{1}, q_2 \rangle \]

\[ \mu_G(X) = (r_X - \alpha \mathbf{1}, c_X - \alpha \mathbf{1}), \quad (\alpha = \langle r_X, \mathbf{1} \rangle = \langle c_X, \mathbf{1} \rangle) \]

\( r_X, c_X \) vectors of row and column \( \ell_2^2 \) norms of \( X \).

Scaling = Minimizing \( \mu_G(X) = \) DS \( G \)-scaling \( Y \) of the matrix \( X \).
Example 2: Scaling polynomials

$T_n$: (Abelian!) group of $n \times n$ diagonal matrices.

$V$: Laurent polynomials (with negative exponents).

$G$ acts on $V$ by scaling variables. $t \in T_n$, $t = \text{diag}(t_1, \ldots, t_n)$.

$$M_t q(x_1, \ldots, x_n) = q(t_1x_1, \ldots, t_nx_n).$$

$T(s) = \text{diag } \exp(sw), \ w \in \mathbb{R}^n$

Directional derivative: action of $T(s)$ on $q$, $s \approx 0$.

$$\mu_G(q) = u, \ u \in \mathbb{R}^n \text{ s.t.}$$

$$\langle u, w \rangle = \frac{d}{ds} \left[ \| M_{T(s)}q \|_2^2 \right]_{s=0} = \langle \text{grad } \hat{q}(1), w \rangle$$

$$\mu_G(q) = \text{grad } \hat{q}(1) \quad \text{(the usual gradient)}$$

$$q = \sum_{\alpha \in \Omega} c_\alpha x^\alpha \quad \hat{q} = \sum_{\alpha \in \Omega} |c_\alpha|^2 x^\alpha$$

Scaling = Minimizing $\mu_G(q)$ = finding extrema of $\hat{q}$
Kempf-Ness

Group $G$ acts linearly on vector space $V$.

[Kempf, Ness 79]: $v$ not in null cone iff there exist a non-zero $w$ in orbit-closure of $v$ s.t. $\mu_G(w) = 0$.

$w$ certifies $v$ not in null cone.

Easy direction.

• $v$ not in null cone. Take $w$ vector of minimal norm in the orbit-closure of $v$. $w$ non-zero.

• $w$ minimal norm in its orbit. $\Rightarrow$ Norm does not decrease by infinitesimal action around $id$. $\Rightarrow \mu_G(w) = 0$.

• global minimum $\Rightarrow$ local minimum.
Kempf-Ness

Hard direction: *local* minimum $\Rightarrow$ *global* minimum. Some "*convexity*".

- *Commutative* group actions – *Euclidean convexity*. (change of variables) [exercise].

- *Non-commutative* group actions: *geodesic convexity*. 
Example: Matrix Scaling

\[ G = ST_n \times ST_n \] acts on \( V = M_n \).
\[ M_{(A,B)} X = AXB. \]

[Hilbert-Mumford]: \( X \) in null cone iff bipartite graph defined by \( \text{supp}(X) \) does not have a perfect matching.

[Kempf-Ness]: \( X \) not in null cone \( \iff \)
\( \iff \) non-zero \( Y \) in orbit-closure s.t. \( \mu_G(Y) = 0 \) \( \iff \)
\( \iff \) \( X \) is scalable to "Doubly Stochastic"

Matrix scaling theorem [Rothblum, Schneider 89].
Moment polytopes
Moment polytopes

Group $G$ acts linearly on vector space $V$.

$$\Delta = \{\text{all gradients}\} = \{ \mu_G(w) : w \in V \}$$
$$\Delta_v = \{\text{all gradients in the orbit closure of } v\} = \{ \mu_G(w) : w \in \overline{O_v} \}$$

[Atiyah, Hilbert, Mumford]: All “such” are convex polytopes
( $\mu_G$ needs to be normalized, standardized)

Uniform Scaling: Given $v$, does $0 \in \Delta_v$? (null cone problem)
Non-uniform Scaling: Given $v \in V$, $r$, does $r \in \Delta_v$?

We have algorithms!
Polyhedral combinatorics!!
Non-uniform matrix scaling

\((r, c)\): probability distributions over \(\{1, \ldots, n\}\).
Non-negative \(n \times n\) matrix \(X\).
Scaling of \(X\) with \textit{row sums} \(r_1, \ldots, r_n\)
and \textit{column sums} \(c_1, \ldots, c_n\)?

\[ \Delta_X = \{(r, c): r = Y1, c = Y^t1\}. \]

[...; Rothblum, Schneider 89]: \(\Delta_X\) \textit{convex polytope}!
Membership: Linear programming

\[ \Delta_X = \{(r, c): \exists Z, \text{supp}(Z) \subseteq \text{supp}(X), \ Z \text{ marginals } (r, c)\}. \]

\textit{Commutative group} actions: \textit{classical marginal} problems.
Also related to \textit{maximum entropy} distributions.
Quantum marginals

*Pure* quantum state $|\psi\rangle_{S_1,\ldots,S_d}$ ($d$ quantum systems): $\psi$ is a $d$-tensor

Underlying group action: Products of $GL$’s on $d$ —*tensors*. (``local’’ basis changes in each system)

Characterize marginals $\rho_{S_1}, \ldots, \rho_{S_d}$ (marginal states on systems)?

Only the *spectra* of $\rho_{S_i}$ matter (local rotations for free).

• Collection of such spectra $\Delta_\psi$ *convex polytope*!
• Follows from theory of *moment polytopes*.
• [BFGOWW 18]: Membership via *non-uniform tensor scaling*. 
More examples of moment polytopes

Schur-Horn: A $n \times n$ symmetric matrix.

$$\Delta_A = \{ \text{diag}(B) : B \text{ similar to } A \} \subseteq \mathbb{R}^n$$

Horn:

$$\Delta = \{ (\lambda_A, \lambda_B, \lambda_C) : A + B = C \} \subseteq \mathbb{R}^{3n}$$

Brascamp-Lieb: Feasibility of analytic inequalities

Newton: $q = \sum_{\alpha \in \Omega} c_\alpha x^\alpha \in C[x_1, \ldots, x_n]$, homogeneous polynomial

$$\Delta_q = \text{conv}\{ \alpha : \alpha \in \Omega \} \subseteq \mathbb{R}^n$$

Edmonds: $M, M'$ matroids on $[n]$ (over the Reals).

$$\Delta_{M,M'} = \text{conv}\{ 1_S : S \text{ basis for } M, M' \} \subseteq \mathbb{R}^n$$
Algorithms: membership in moment polytopes

Group $G$ acts linearly on vector space $V$. 
$v \in V \leftrightarrow \Delta_v \subseteq R^n$ moment polytope.

Non-uniform scaling: Given $v \in V$, $r \in R^n$, $\varepsilon > 0$ does $r \in \Delta_v$
or $\varepsilon$–far from $\Delta_v$

For general settings we have efficient:

- Alternating minimization: convergence poly($1/\varepsilon$)
- Geodesic optimization: convergence polylog($1/\varepsilon$)
Conclusions & Open problems
Summary: Invariant Theory + ToC

Lots of similar type questions, notions, results

- Algorithms are important, sought and discovered
- Has both an algebraic and analytic nature
- Quantitative, with many asymptotic notions
- Studies families of objects
- Needs comp theory structure, reductions, completeness
- Symmetry is becoming more central in ToC
Summary: Consequences

New efficient algorithmic techniques, solving classes of:
- non-convex optimization problems
- systems of quadratic equations
- linear programs of exponential size

Applicable (or potentially applicable) in:
- Derandomization (PIT)
- Analysis (Brascamp-Lieb inequalities)
- Non-commutative algebra (word problem)
- Quantum information theory (distillation, marginals, SLOCC)
- Representation theory (asymptotic Kronecker coefficients)
- Operator theory (Paulsen problem)
- Combinatorial optimization (moment polytopes)
Open problems

- PIT in $P$ ?
- Is PIT a null cone problem?

- Polynomial time algorithms for
  1. Null cone membership.
  2. Moment polytopes membership, separation, optimization.

- Extend algorithmic theory to group actions on algebraic varieties, Riemannian/symplectic manifolds
Learn more?

EATCS survey [Garg, Oliveira]
My CCC’17 tutorial:
http://www.computationalcomplexity.org/Archive/2017/tutorial.php
STOC 2018 tutorial:
https://staff.fnwi.uva.nl/m.walter/focs2018scaling/
A week of tutorials:
https://www.math.ias.edu/ocit2018

**Mathematics and Computation**
New book on my website