Pseudorandom generators from polarizing random walks

Kaave Hosseini (UC San Diego)

Eshan Chattopadhyay (Cornell)

Pooya Hatami (UT Austin → Ohio State)

Shachar Lovett (UC San Diego)

Outline

Introduce Pseudorandom generators (PRGs)

New approach to construct PRGs

Open problems

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$\mathcal{F} = \{f : \mathcal{D} \longrightarrow \mathbb{R}\}$ family of functions : tests

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Usually dealing with functions $f: \{-1,1\}^n \to \{-1,1\}$

So we take $\mathcal{D} = \{-1,1\}^n$

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s *is called seed length*

Example

Example 1: Tests: $\mathbb{F}_2^n$ characters

$$\mathcal{F} = \{f(x) = \prod_{i \in S} x_i \quad : \quad S \subseteq [n]\}$$

X : ε-bias random variable

- PRGs with optimal seed length $O(\log(n/\varepsilon))$ are known.

Some known approaches to construct PRGs

- Use basic PRGs:

Viola[09]: sum of a d many ε-biased PRGs fools degree-d $\mathbb{F}_2$-polynomials.

Some known approaches to construct PRGs

- Pseudorandom restriction:

Ajtai-Wigderson85, Ajt93, CR96, AAI+01, GMR+12, IMP12, GMR13, TX13, GW14, HT17, ST18, …

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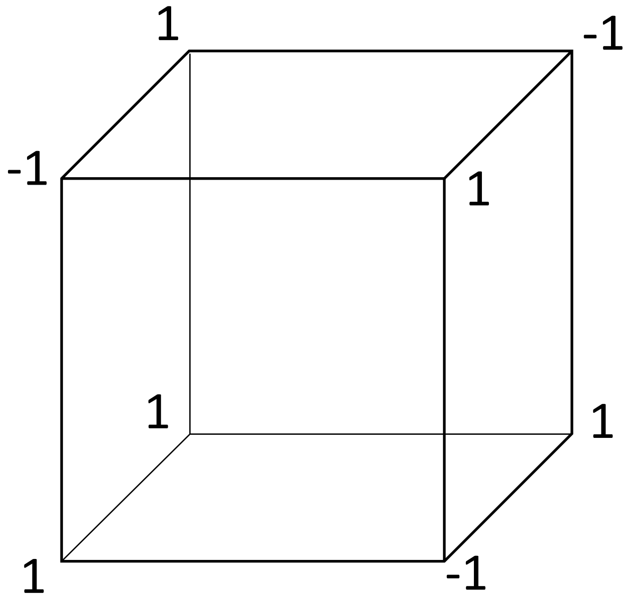
- Consequence of this work:

Generic method to do step 2 for arbitrary $\mathcal{F}$.

Fractional PRGs

$f: \{-1,1\}^n \to \{-1,1\}$



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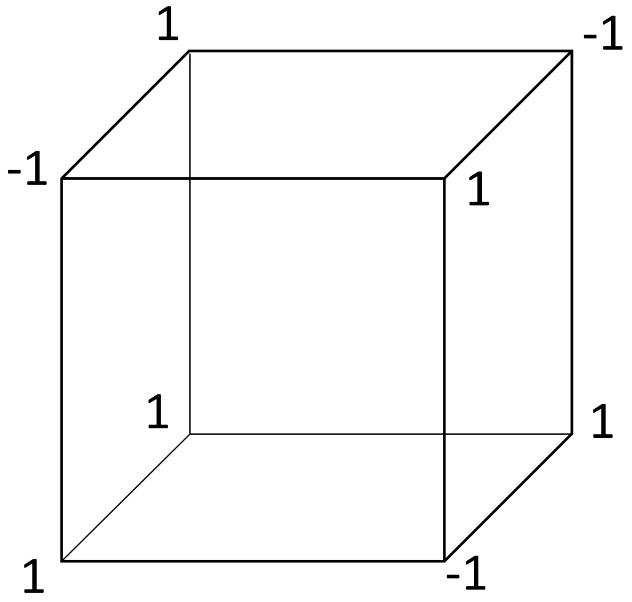
$f : \{-1,1\}^n \to \{-1,1\} \xrightarrow{\text{multi-linear extension}} f : \mathbb{R}^n \to \mathbb{R}$



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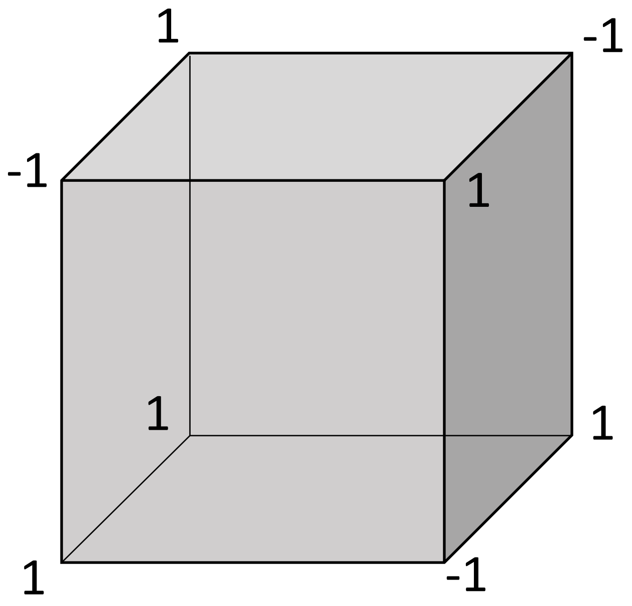
Only consider points in $[-1,1]^n$ so $f: [-1,1]^n \to [-1,1]$



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Equivalent definition of PRG:

$X \in \{-1,1\}^n$ ε-fools $\mathcal{F}$ if

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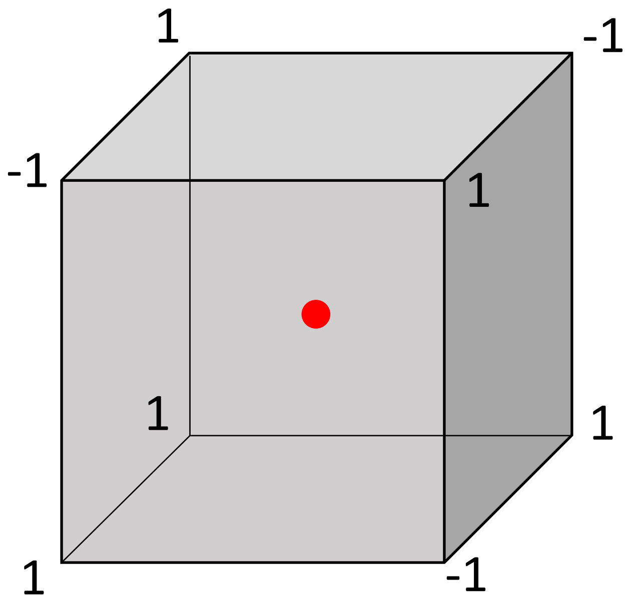

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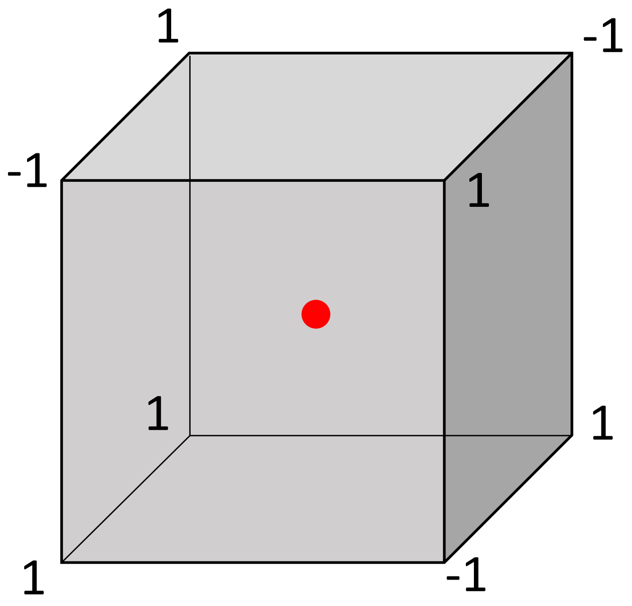
because $\mathbb{E}f(U_n) = f(\mathbb{E}U_n) = f(0)$



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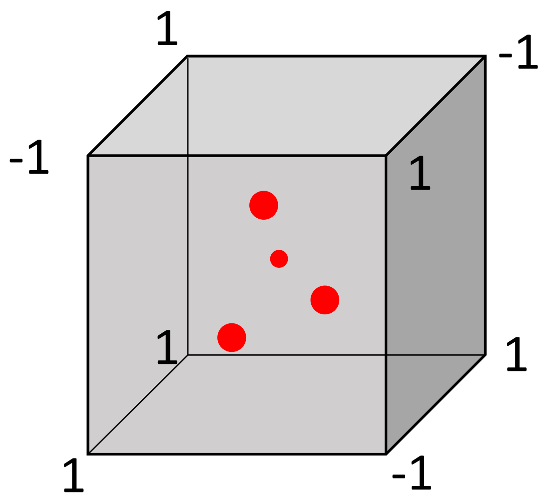

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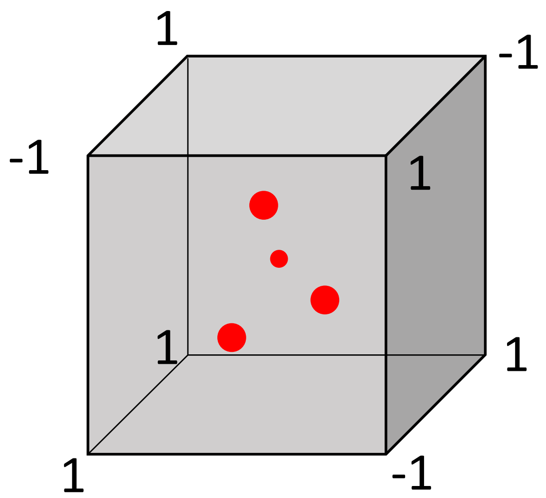

Trivial f-PRG: $X \equiv 0$; we will rule it out later.

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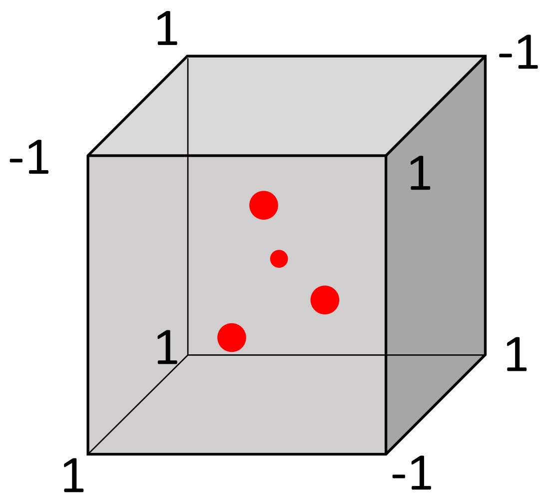

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Question. Are f-PRGs easier to construct than PRGs?

Can f-PRGs be used to construct PRGs?

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Recall: f-PRG is $X = (X_1, \cdots, X_n) \in [-1,1]^n$ where $|\mathbb{E}\, f(X) - f(0)| \leq \varepsilon$

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Need to enforce non-triviality: require $\mathbb{E}\, |X_i|^2 \geq p$ for all $i = 1, \ldots, n$

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- If X has seed length s then X' has seed length ts

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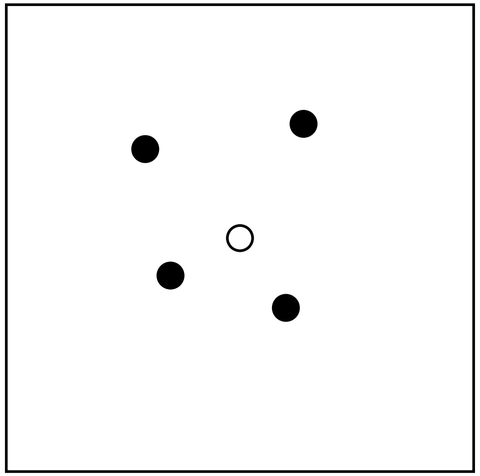
Goal: use the f-PRG to define a random walk



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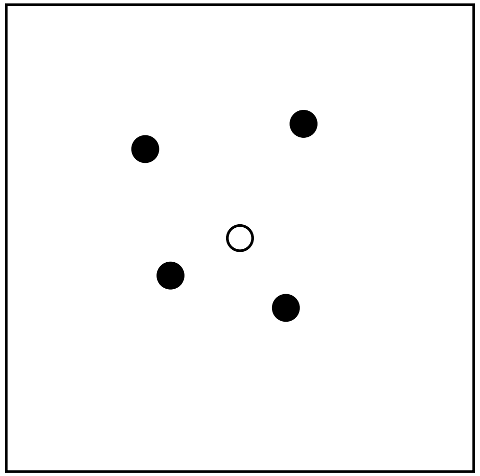

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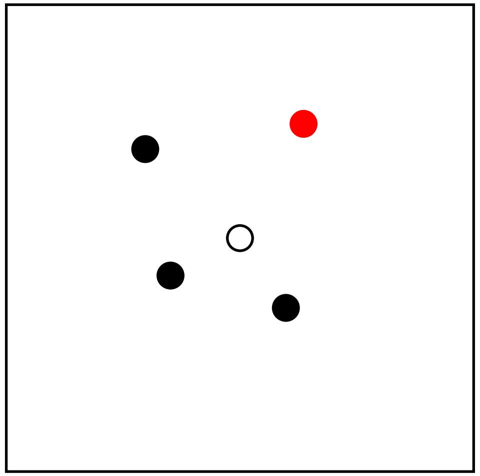

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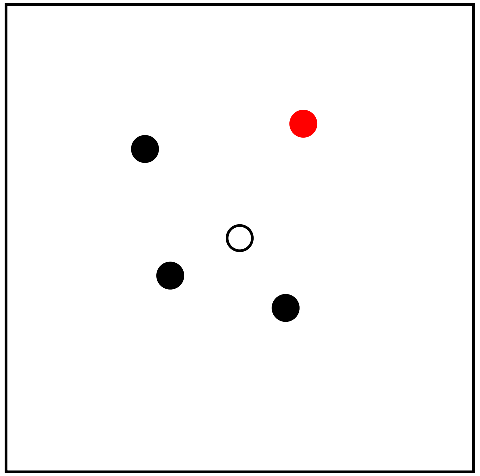

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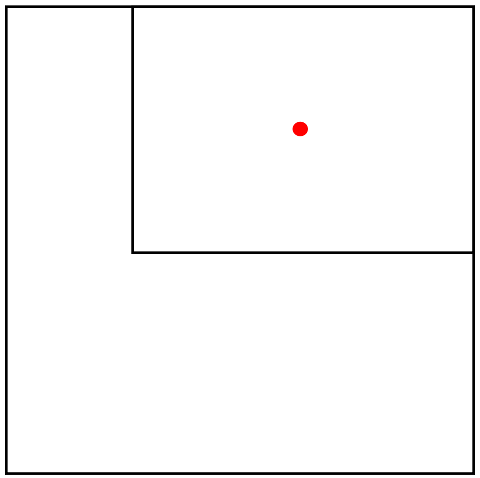

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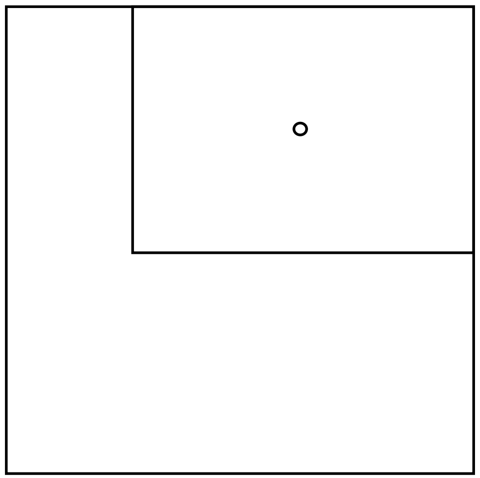

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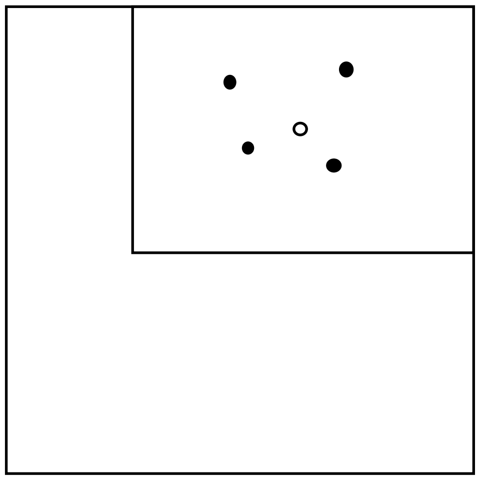

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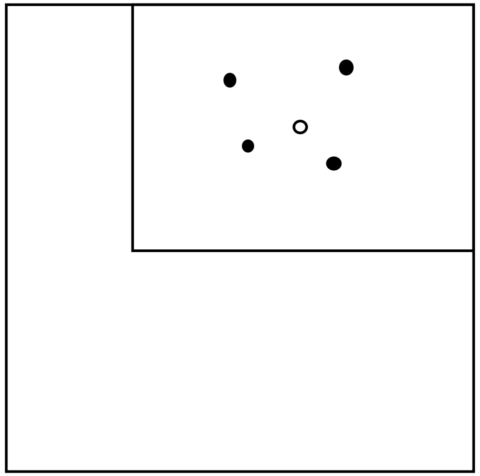

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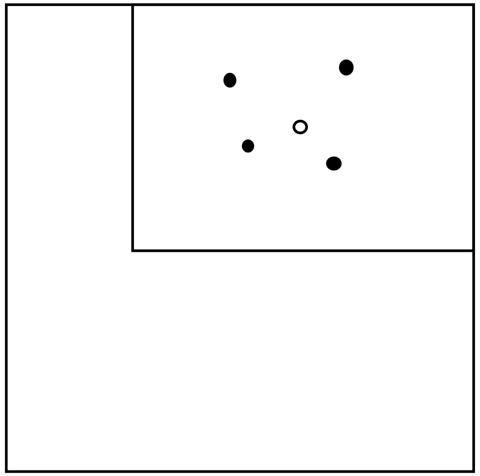

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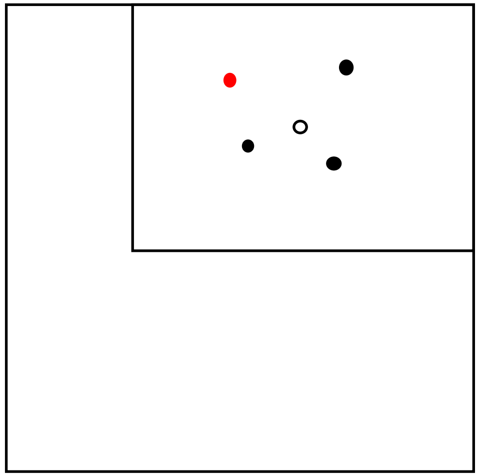

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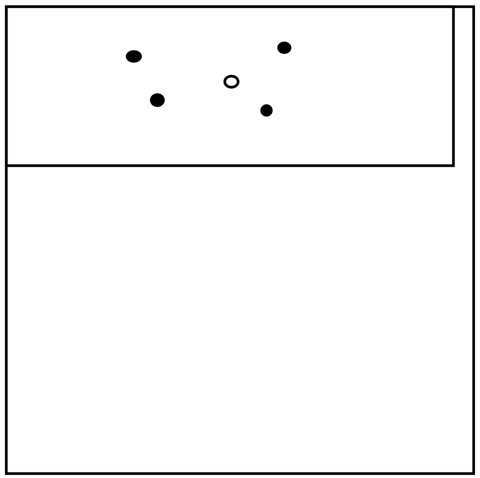

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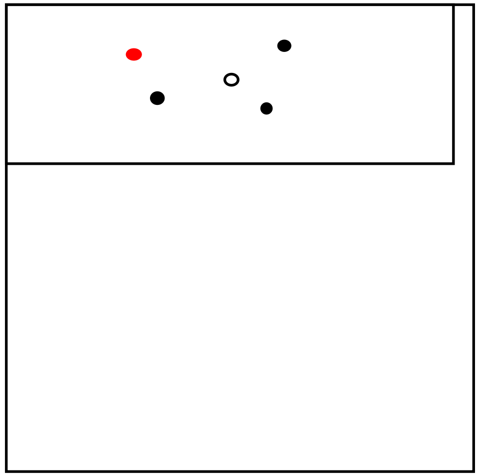

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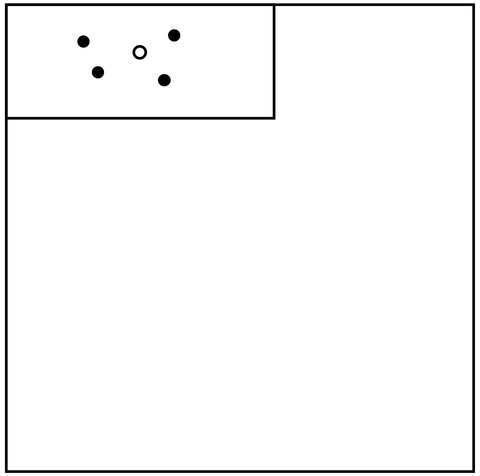

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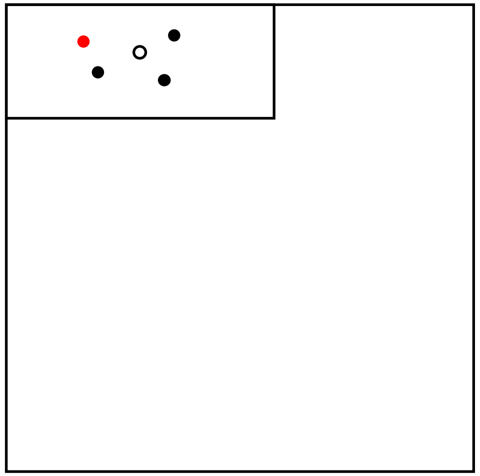

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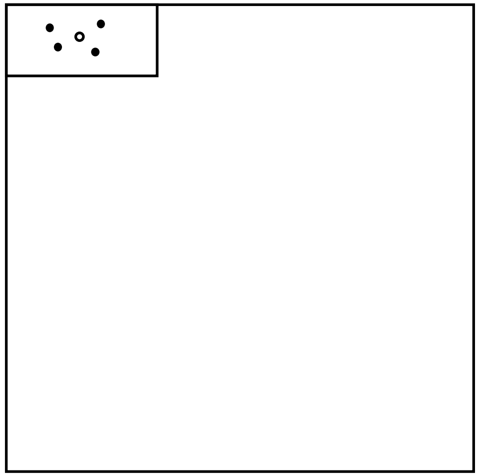

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Proof of main theorem: fast convergence

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Round to sign$\{Y_t\}$ once the random walk is close enough to the boundary

Construction of fractional PRGs

$$f: \{-1,1\}^n \to \{-1,1\}$$

Fourier coefficients: $\hat{f}(S) = \mathbb{E}\, f(x) \prod_{i \in S} x_i$, $S \subseteq [n]$

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f has bounded Fourier growth if

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$\mathrm{c} = n$ is a trivial bound.

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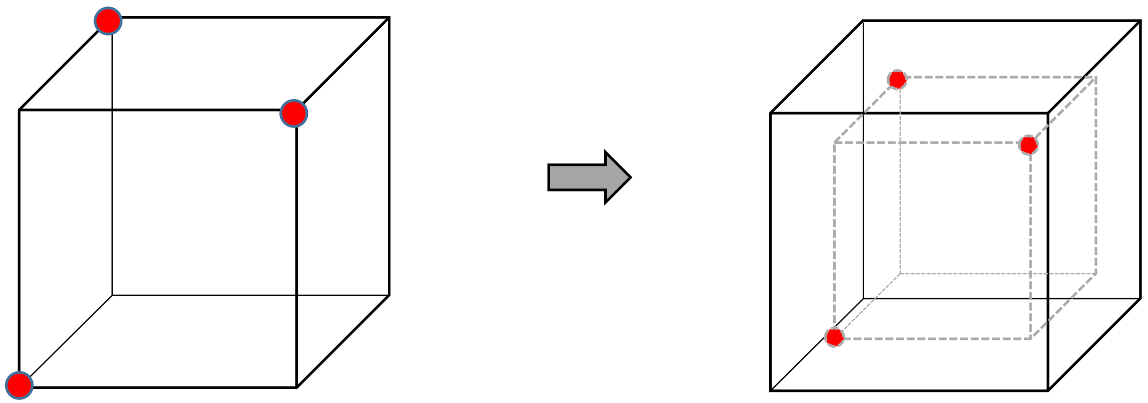

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Tal'17: $c = \log^d s$

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Questions

- One way to view our construction is as follows

X_1
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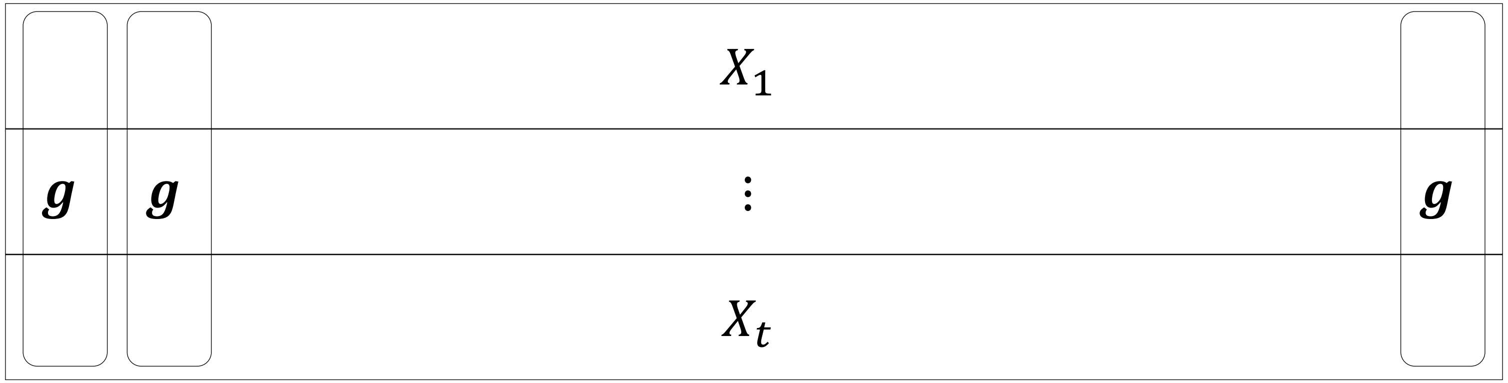

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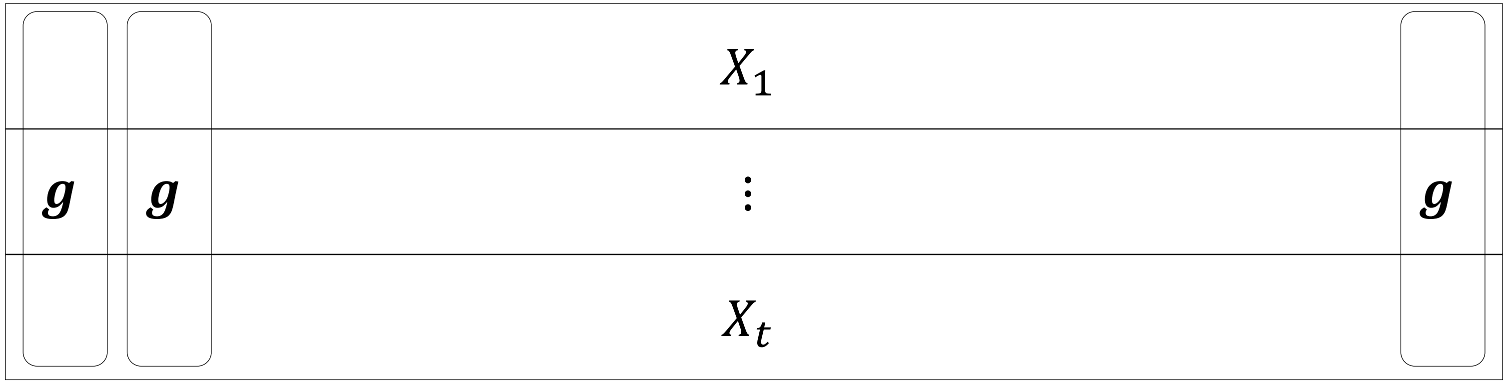

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- Can we construct hitting sets this way?

Thank you!