Undecidability of Linear Inequalities Between Graph Homomorphism Densities

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joint work with Sergey Norin

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Introduction
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Few techniques are very common (Induction, Cauchy-Schwarz, ...).

Discovery of rich algebraic structure underlying many of these techniques.

Neater proofs with no low-order terms.

Methods for applying these techniques in semi-automatic ways.
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Density of $H$ in $G$

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\frac{\text{number of copies of } H \text{ in } G}{\binom{|V(G)|}{|V(H)|}}.
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We can think of these densities as "moments" of the graph $G$. Many fundamental theorems in extremal graph theory can be expressed as algebraic inequalities between subgraph densities.
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Razborov’s flag algebras

A formal calculus capturing many standard arguments (induction, Cauchy-Schwarz,...) in the area.
Applications

Automatic methods for proving theorems (based on SDP):
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- **HH, Hladky, Kral, Norin, Razborov**: A conjecture of Erdős.
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- **Razborov**: Turán’s hypergraph problem under mild extra conditions.
- other conjectures of Erdös, crossing number of complete bipartite graphs, etc.
How far can we go?

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Question (Razborov)

Can every true algebraic inequality between subgraph densities be proved using a finite amount of manipulation with subgraph densities of finitely many graphs?
How far can we go?

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Question (Razborov)

*Can every true algebraic inequality between subgraph densities be proved using a finite amount of manipulation with subgraph densities of finitely many graphs?*

HH-Norine 2011

The answer is negative in a strong sense.
Formal definitions
Extremal graph theory

Studies the relations between the number of occurrences of different subgraphs in a graph $G$. 
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Equivalently one can study the relations between the “homomorphism densities”.
Homomorphism Density

Definition

- Map the vertices of $H$ to the vertices of $G$ independently at random.
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$$t_H(G) := \Pr[\text{edges go to edges}].$$
Definition

A map $f : H \to G$ is called a homomorphism if it maps edges to edges.
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$$t_H(G) = \Pr[f : H \to G \text{ is a homomorphism}].$$
Asymptotically $t_H(\cdot)$ and subgraph densities are equivalent.
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The functions $t_H$ have nice algebraic structures:

$$t_{H_1 \sqcup H_2}(G) = t_{H_1}(G)t_{H_2}(G).$$
Many fundamental theorems in extremal graph theory can be expressed as algebraic inequalities between homomorphism densities.
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Example (Goodman’s bound 1959)

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\[ a_1 t_{H_1}(G) + \ldots + a_m t_{H_m}(G) \geq 0. \]
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Example (Goodman’s bound 1959)

\[ t_{K_3}(G) - 2t_{K_2 \oplus K_2}(G) + t_{K_2}(G) \geq 0. \]
Algebra of Partially labeled graphs
Definition

A partially labeled graph is a graph in which some vertices are labeled by distinct natural numbers.

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Definition

Let \( H \) be partially labeled with labels \( L \). For \( \phi : L \to G \), define

\[ t_{H,\phi}(G) := \Pr[f : H \to G \text{ is a hom.} \mid f|_L = \phi] . \]
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Example

$$t_{H,\phi}(G) = \frac{3}{6} = \frac{1}{2}.$$
Example

Definition

Let $[H]$ be $H$ with no labels.
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$$E_{\phi} \left[ t_{H,\phi}(G) \right] = t_{[H]}(G)$$
Recall that:

\[ t_{H_1 \cup H_2}(G) = t_{H_1}(G)t_{H_2}(G). \]
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\[ t_{H_1 \sqcup H_2}(G) = t_{H_1}(G)t_{H_2}(G). \]

This motivates us to define \( H_1 \times H_2 := H_1 \sqcup H_2 \).
Definition

The product $H_1 \cdot H_2$ of partially labeled graphs $H_1$ and $H_2$: take their disjoint union, and then identify vertices with the same label. If multiple edges arise, only one copy is kept.
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Example

```
\begin{align*}
\begin{tikzpicture}[scale=0.8]
  \node[red] (1) at (0,0) {1};
  \node[red] (2) at (1,0) {2};
  \node[red] (3) at (0,1) {3};
  \node[red] (4) at (1,1) {4};
  \draw (1) -- (2);
  \draw (1) -- (3);
  \draw (1) -- (4);
  \draw (2) -- (3);
  \draw (2) -- (4);
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\end{tikzpicture}
\end{align*}
\times \begin{align*}
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```

Hamed Hatami (McGill University)
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Let \( \phi : L_1 \cup L_2 \to G \).
We have \( t_{H_1,\phi}(G)t_{H_2,\phi}(G) = t_{H_1 \times H_2,\phi}(G) \).

**Example**

\[ \begin{align*}
H_1 & \times H_2 \\
G & \\
\phi & \\
1 & 2 & \times & 1 & 2
\end{align*} \]
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- Let $H_1, \ldots, H_k$ be partially labeled graphs with the set of labels $L$.
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\[
0 \leq \left( \sum b_i t_{H_i,\phi}(G) \right)^2 = \sum b_i b_j t_{H_i,\phi}(G) t_{H_j,\phi}(G) = \sum b_i b_j t_{H_i \times H_j,\phi}(G)
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Reflection positivity

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Let \( H_1, H_2, \ldots \) be all partially labeled graphs. For every \( G \):
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Let \( H_1, H_2, \ldots \) be all partially labeled graphs. For every \( G \):
- **Condition I:** \( t_{K_1}(G) = 1 \).
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Let \( H_1, H_2, \ldots \) be all partially labeled graphs. For every \( G \):

- **Condition I:** \( t_{K_1}(G) = 1 \).
- **Condition II:** \( t_{H \cup K_1}(G) = t_H(G) \) for all graph \( H \).
Let $H_1, H_2, \ldots$ be all partially labeled graphs. For every $G$:

- **Condition I:** $t_{K_1}(G) = 1$.
- **Condition II:** $t_{H \sqcup K_1}(G) = t_H(G)$ for all graph $H$.
- **Condition III:** The infinite matrix whose $ij$-th entry is $t_{[H_i \times H_j]}(G)$ is positive semi-definite.
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**Theorem (Freedman, Lovász, Shrijver 2007)**

These conditions describe the closure of the set

\[ \{(t_{F_1}(G), t_{F_2}(G), \ldots) : G \in [0, 1]^N \} \]
Quantum Graphs
A quantum graph is a formal linear combination of graphs:

\[ a_1 H_1 + \ldots + a_k H_k. \]
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A quantum graph \( a_1 H_1 + \ldots + a_k H_k \) is called positive, if for all \( G \),

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\[ K_3 - 2(K_2 \sqcup K_2) + K_2 \geq 0. \]
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We want to understand the set of all positive quantum graphs.
A partially labeled quantum graph is a formal linear combination of partially labeled graphs:

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Partially labeled quantum graphs form an algebra:

\[
(a_1 H_1 + \ldots + a_k H_k) \cdot (b_1 L_1 + \ldots + b_\ell L_\ell) = \sum a_i b_j H_i \cdot L_j.
\]
Unlabeling operator

\[ \cdot : \text{partially labeled quantum graph} \mapsto \text{quantum graph} \]
Unlabeling operator

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\cdot : \text{partially labeled quantum graph} \rightarrow \text{quantum graph}
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Recall

\[
\left( \sum b_i H_i \right)^2 = \sum b_i b_j [H_i \times H_j] \geq 0
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Equivalently

For every partially labeled quantum graph \( g \) we have \( [g^2] \geq 0 \).
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For every partially labeled quantum graph \( g \) we have \( [g^2] \geq 0 \).

Corollary

Always

\[
\left[ g_1^2 + \ldots + g_k^2 \right] \geq 0.
\]
Question (Lovász’s 17th Problem, Lovász-Szegedy, Razborov)

Is it true that every $f \geq 0$ is of the form

$$f = \left[ g_1^2 + g_2^2 + \ldots + g_k^2 \right]$$
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Observation (Lovasz-Szegedy and Razborov)

If \( f \geq 0 \) and \( \epsilon > 0 \), there exists a positive integer \( k \) and quantum labeled graphs \( g_1, g_2, \ldots, g_k \) such that

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-\epsilon \leq f - \left[ g_1^2 + g_2^2 + \ldots + g_k^2 \right] \leq \epsilon.
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Theorem (HH and Norin)

The answer to the above question is negative.
positive polynomials
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**Theorem (Hilbert 1888)**

*There exist 3-variable positive homogenous polynomials which are not sums of squares of polynomials.*
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Theorem (Hilbert 1888)

There exist 3-variable positive homogenous polynomials which are not sums of squares of polynomials.

Example (Motzkin’s polynomial)

\[ x^4 y^2 + y^4 z^2 + z^4 x^2 - 6x^2 y^2 z^2 \geq 0. \]
Extending to quantum graphs
Theorem (HH and Norin)

There are positive quantum graphs $f$ which are not sums of squares. That is, always $f \neq \left[ g_1^2 + \ldots + g_k^2 \right]$. 

Theorem (HH and Norin)

There are positive quantum graphs \( f \) which are not sums of squares. That is, always \( f \neq [g_1^2 + \ldots + g_k^2] \).

- The proof is based on converting \( x^4 y^2 + y^4 z^2 + z^4 x^2 - 6x^2 y^2 z^2 \) to a quantum graph.
Theorem (Artin 1927, Solution to Hilbert’s 17th Problem)

Every positive polynomial is of the form

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Corollary

The problem of checking the positivity of a polynomial is decidable.
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Corollary

The problem of checking the positivity of a polynomial is decidable.

- Co-recursively enumerable: Try to find a point that makes \( p \) negative.
- recursively enumerable: Try to write \( p = \sum (p_i/q_i)^2 \).
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**Theorem (HH and Norin)**

The following problem is undecidable.

**QUESTION:** Does the inequality \(a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0\) hold for every graph \(G\)?
Proof
Theorem (HH and Norin)

The following problem is undecidable.

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The following problem is undecidable.

**QUESTION:** Does the inequality \( a_1 t_{H_1}(G) + \ldots + a_k t_{H_k}(G) \geq 0 \) hold for every graph \( G \)?

Equivalently

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Instead I will prove the following theorem:

Theorem

The following problem is undecidable.

- **INSTANCE:** A polynomial $p(x_1, \ldots, x_k, y_1, \ldots, y_k)$.
- **QUESTION:** Does the inequality $p(t_{K_2}(G_1), \ldots, t_{K_2}(G_k), t_{K_3}(G_1), \ldots, t_{K_3}(G_k)) \geq 0$ hold for every $G_1, \ldots, G_k$?
Matiyasevich 1970 Solution to Hilbert’s 10th problem: Checking the positivity of \( p \in \mathbb{R}[x_1, \ldots, x_k] \) on \( \{1 - \frac{1}{n} : n \in \mathbb{Z}\}^k \) is undecidable.
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- Bollobás, Razborov: Goodman’s bound is achieved only when $t_{K_2}(G) \in \{1 - \frac{1}{n} : n \in \mathbb{Z}\}$. 

![Graph showing the relationship between $t(K_2; G)$ and $t(K_3; G)$ with annotations for Kruskal-Katona, Goodman, and Razborov.]
Let $S$ be the grey area and $g(x) = 2x^2 - x$. (Goodman: $t_{K_3}(G) \geq 2t_{K_2}(G)^2 - t_{K_2}(G)$.)

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**Lemma**

Let $p \in \mathbb{R}[x_1, \ldots, x_k]$. Define $q(x_1, \ldots, x_k, y_1, \ldots, y_k)$ as

$$q := p \prod_{i=1}^{k} (1 - x_i)^6 + C_p \times \left( \sum_{i=1}^{k} y_i - g(x_i) \right).$$

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- $q < 0$ for some $x_i = t_{K_2}(G_i)$ and $y_i = t_{K_3}(G_i)$. *(reduction)*
Where do we go from here?
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Question: What about unions of cliques?
Let $R$ denote the closure of $\{ (t_{H_1}(G), t_{H_2}(G), \ldots) : G \} \subset [0, 1]^\mathbb{N}$. 
Graphons: The points in $R$ (graph limits) can be represented by symmetric measurable $W : [0,1]^2 \rightarrow [0,1]$. 

Finitely Forcible: A point in $R$ is finitely forcible if a finite number of coordinates uniquely determine it.

Lovász's Conjecture: Every feasible inequality $a_1 t_{H_1}(W) + \ldots + a_k t_{H_k}(W) < 0$ has a finitely forcible solution $W$.

Lovász-Szegedy's Conjecture: Finitely forcible graphons have simple structures (finite dimensional).

There are finitely forcible $W$'s such that \{ $W(x, \cdot)$ : $x \in [0,1]$ \} with the $L^1$ distance contains a subset homeomorphic to $[0,1]$. 

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**[Glebov, Klimošová, Král 2013+]**

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