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→ Algebraic (typically rank-based) techniques

Rest of talk

1. Lower bounds for non-commutative ABPs. [N'91]
2. Lower bounds for homogeneous $\Sigma\Pi\Sigma$ formulas [NW'95]
3. Extensions [K'12, ...]
4. Some open questions.

Non-commutative polynomials

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monomials = strings over $\{x_1, \ldots, x_n\}$

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$$p(x,y) = xxxx + xyyx + yxxy + yyyy \in \mathbb{F}\langle x,y \rangle$$

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→ Still open.

Non-commutative complexity classes

$VF^{nc} \subseteq VBP^{nc} \subseteq VP^{nc}$

Thm [N'91]: $Per \notin VBP^{nc}$.

Any n.c. ABP for Per_n must have size $\geq 2^n$.

Lower bd for non-comm. ABPs

Depth redn.: A ABP size s deg. d $\Longrightarrow A = \sum_{i=1}^{s} \underset{\sim}{P_i} \cdot \underset{\sim}{Q_i}$

deg. $d/2$

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- Find explicit f s.t. f large

Nisan's complexity measure

$f \in \mathbb{F}\langle x_1, \ldots, x_n \rangle$ homog. deg. d.

$$f = \sum_{m:\, \deg(m) = d} c_f(m) \cdot m$$

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m_2

m_1

$\to$ monomial of deg. $d/2$

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$m_1 \longrightarrow c_f(m_1 \cdot m_2)$

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$\rightarrow f = \underbrace{P}_{\deg d/2} \cdot \underbrace{Q}$

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$$c_f(m_1 m_2) = c_P(m_1)\, c_Q(m_2)$$

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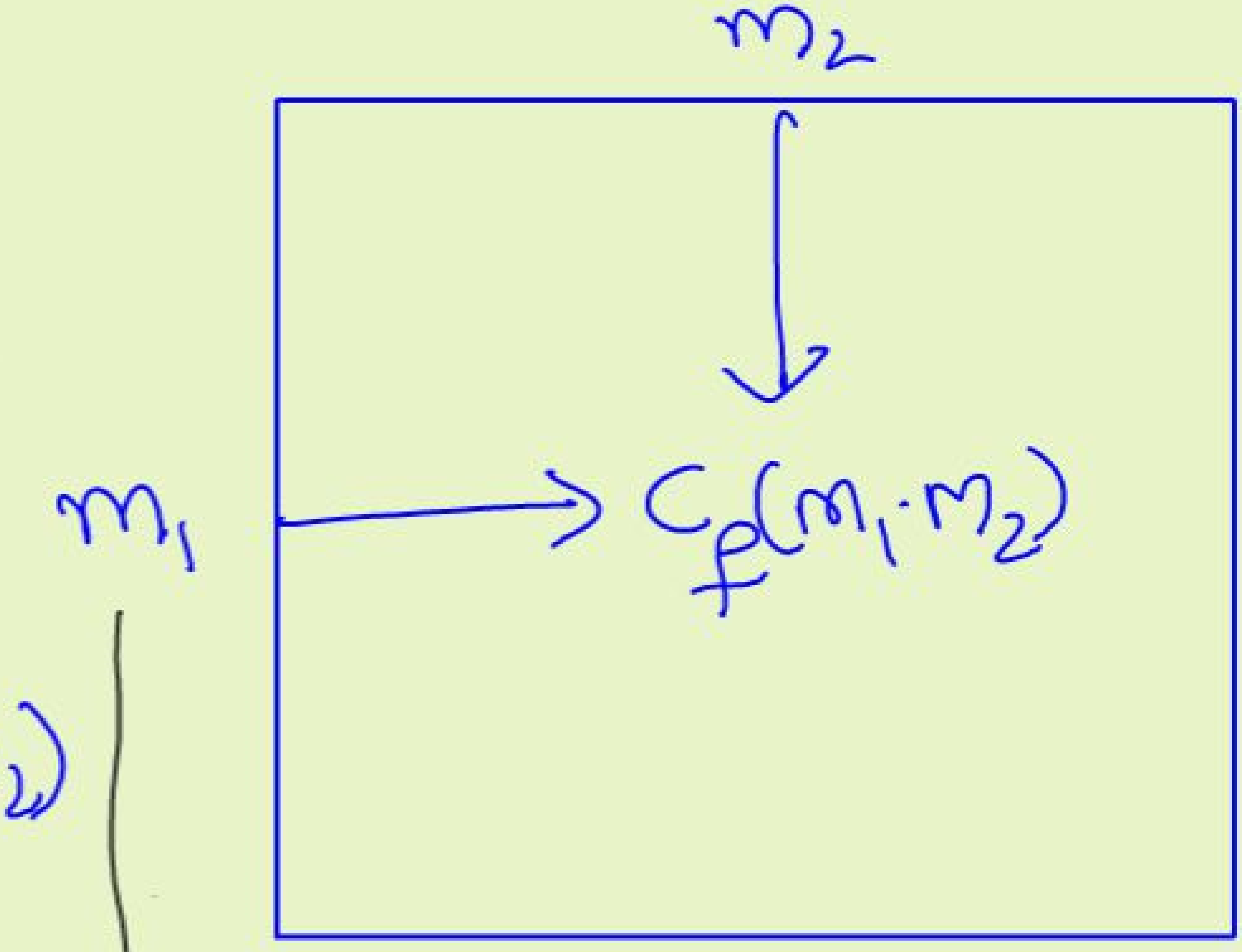
$$\Rightarrow rk(M(f)) \leq 1$$



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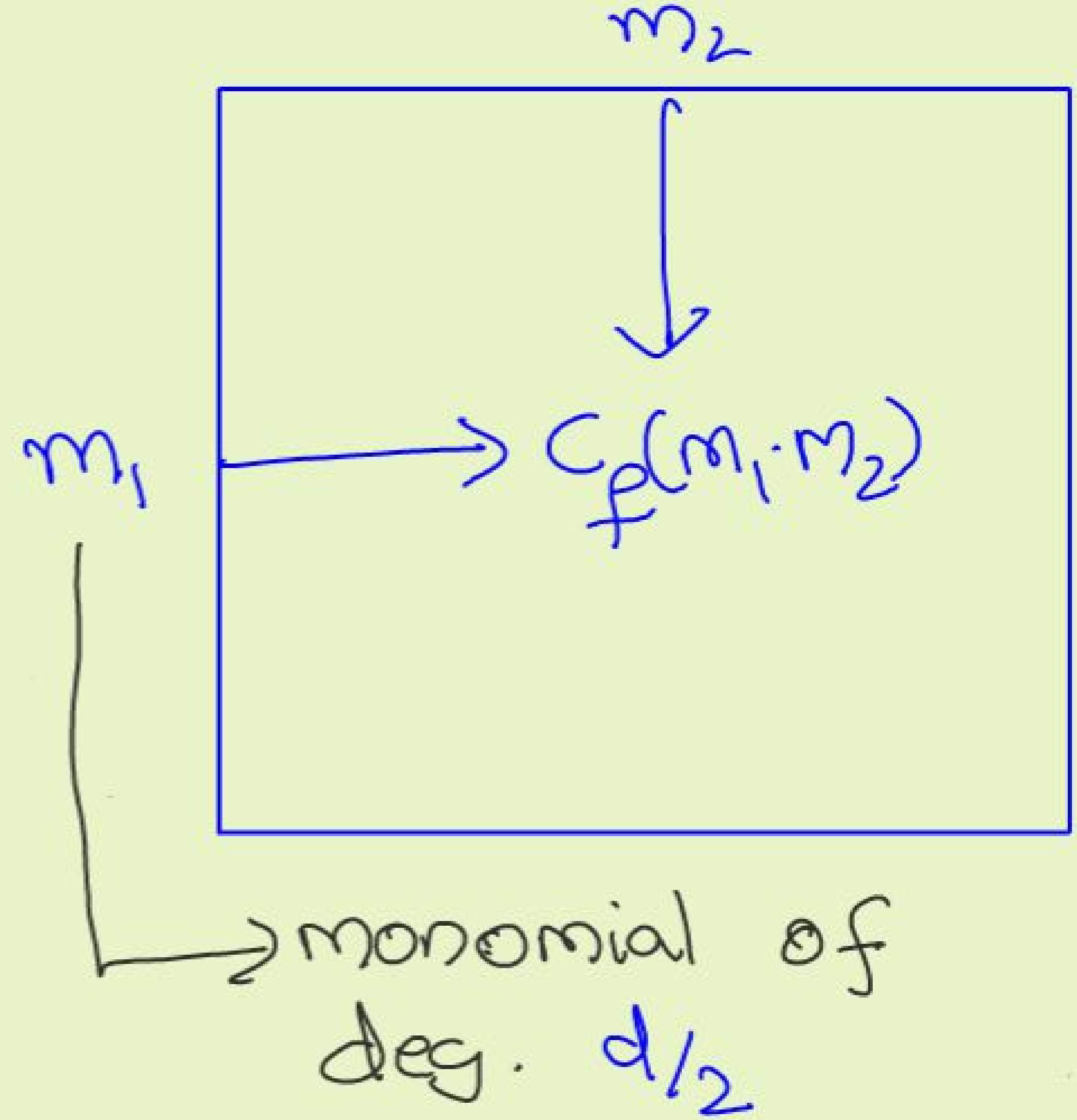

Nisan's lbd for non-comm. ABPs

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Rank arg.: $rk(M(A)) \leq \sum_i rk(M(P_i Q_i)) \leq s.$

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$f(x,y) = $ deg d $\Rightarrow rk(M(f)) = 2^{d/2}$

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Back to the commutative setting

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$\deg \leq \sqrt{d}$

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Lower bounds for homog. $\Sigma\Pi\Sigma\Pi$ formulas?

$\Sigma\Pi\Sigma$ and $\Sigma\Lambda\Sigma$ formulas

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$$F = \sum_{i=1}^{s} \underbrace{L_{i,1}}_{} \cdots \underbrace{L_{i,d}}_{\deg \leq 1}$$

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$\Sigma\Pi\Sigma$ and $\Sigma\Lambda\Sigma$ formulas

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Thm [S'01]: Any polynomial has a $\Sigma\Lambda\Sigma$ formula (of possibly exponential size). (Need $\mathrm{char}(\mathbb{F}) = 0$.)

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NW95: Rank of Partial Derivative matrix

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Also: S1851

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$$\tilde{f}(x_1, \ldots, x_4, y_1, \ldots, y_4)$$

$$= x_1x_2x_3x_4 + x_1y_2y_3x_4$$

$$+ y_1x_2x_3y_4 + y_1y_2y_3y_4 \in \mathbb{F}[x_1, \ldots, x_4, y_1, \ldots, y_4]$$

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$$\frac{\partial^2 \tilde{f}}{\partial x_1 \partial x_2} = 1\cdot x_3x_4 + 0\cdot x_3y_4 + 0\cdot y_3x_4 + 0\cdot y_3y_4$$

N-W Partial derivative measure

$f \in \mathbb{F}[x_1, \ldots, x_n]$

$$\partial^k f = \left\{ \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} : i_1 < i_2 < \ldots < i_k \right\}$$

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$= \dim(\mathrm{span}(\partial^k f))$

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Eg: $f_0 = x_1 x_2 \cdots x_n$

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$$\boxed{\mu_k(f_0) = \binom{n}{k}}$$

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Pf: $\dfrac{\partial^k l_i^d}{\partial x_{i_1} \cdots \partial x_{i_k}} = \alpha \cdot l_i^{d-k}$

all rows same up to scalar multiplication

Lower bounds for $\Sigma\Lambda\Sigma$ formulas

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$\boxed{\text{all rows same up to scalar multiplication}}$

$\Longrightarrow \mu_k(F) \leq s.$

Lower bounds for $\Sigma\Lambda\Sigma$ formulas

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F computes $f_0 \implies s \geq \binom{n}{n/2}$

(Nearly tight. Ubd.: $s \le 2^{n-1}$.)

[Fischer's identity]

Lower bounds for homog. $\Sigma\Pi\Sigma$ formulas

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$$F = \sum_{i=1}^{s} L_{i,1} \cdots L_{i,d}$$

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$|B| = d-k$

$k = |A|$

$\boxed{\mathbb{1}[A \cap B = \phi]}$

"Disjointness matrix"

Lower bounds for homog. $\Sigma\Pi\Sigma$ formulas

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$$\mu_k(L_{i,1} \cdots L_{i,d}) \le \binom{d}{k}$$ [no. dep. on n]

$$f = \sum_{S \in \binom{[n]}{d}} \prod_{j \in S} x_j$$

$$\mu_k(f) = \binom{n}{k}$$ [if $k \le d/2$]

$|B| = d-k$

$k = |A|$

$\mathbb{1}[A \cap B = \phi]$

"Disjointness matrix"

Extensions of the Partial Deriv. method

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→ Extension: Shifted Partial Derivatives [K'12]

Shifted Partial Derivative method

$f \in \mathbb{F}[x_1, \ldots, x_n]$

$$\partial^k f = \left\{ \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} : i_1, \ldots, i_k \right\}$$

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$$\mu_{k,l}(f) = \dim(\mathrm{span}((\partial^k f)_{\leq l}))$$

Shifted Partial Derivative method

$$(\partial^k f)_{\leq \ell} = \left\{ \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \cdot m \;\middle|\; \begin{array}{l} i_1, \ldots, i_k \\ m : \text{mon. of deg} \leq \ell \end{array} \right\}$$

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$$\underbrace{\langle \partial^k f \rangle}$$

Ideal gen. by

$\partial^k f$

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$\dim((\partial^k f)_{\leq \ell})$ measures "size of ideal."

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$\rightarrow F = \sum_{i=1}^{s} \underbrace{P_i}_{\deg k}^{d/k}$ computing $x_1 x_2 \cdots x_n$

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→ F homog. $\Sigma\Pi\Sigma\Pi$ computing $\mathrm{IMM}_{n,d}$

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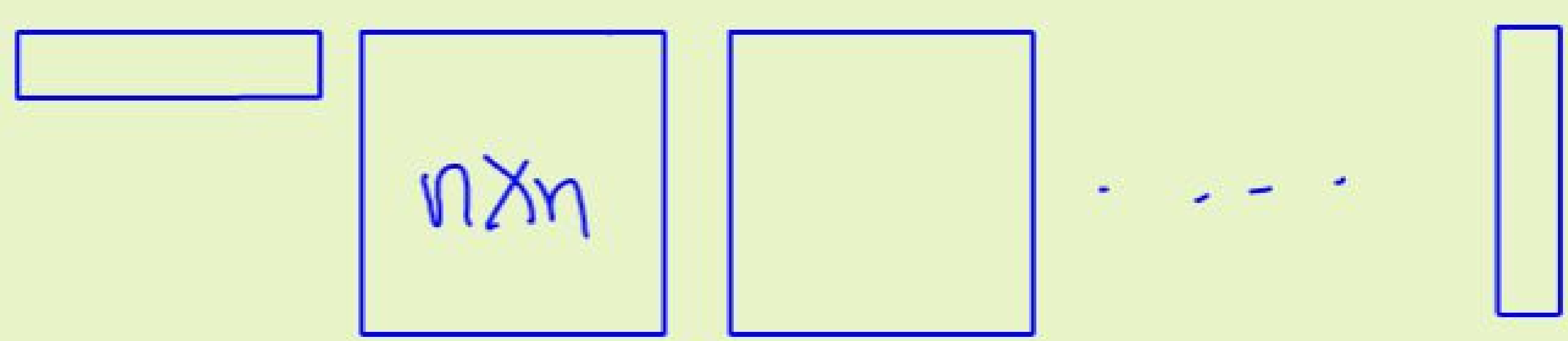
$IMM_{n,d} =$



OPEN QUESTIONS

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→ CILM '18 : Slightly non-trivial ckt. lower bds ⇒ strong ckt. lower bds.

OPEN QUESTIONS

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→ Non-comm. algebraic analogue of separating NL & NC^1

OPEN QUESTIONS

(3) Lower bounds for inhomog. $\Sigma\Pi\Sigma$ formulas

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→ Best lbd: $n^{3-o(1)}$ [KST'16]

OPEN QUESTIONS

4 Can our techniques prove superpoly.
(lbds)

OPEN QUESTIONS

(4) Can our techniques prove superpoly. lbds)

→ EGOW'18 : Limitations of rank-based techniques.

OPEN QUESTIONS

(4) Can our techniques prove superpoly. lbds?

→ EGOW'18: Limitations of rank-based techniques.

→ FSV'17, GrKSS'17, BNL'18: Algebraic "natural proofs" barriers

THANK YOU!