Crouzeix's Conjecture

Let A be a square matrix or a bounded linear operator on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle, \| \cdot \|)$. The *field of values* or *numerical range* of A is defined as

$$W(A) := \{\langle A\mathbf{q}, \mathbf{q} \rangle : \langle \mathbf{q}, \mathbf{q} \rangle = 1\}.$$

In 2004, Michel Crouzeix made the following simple conjecture: For any such A and any polynomial p,

$$\|p(A)\| \leq 2 \sup_{z \in W(A)} |p(z)|. \qquad (\textbf{Crouzeix's conjecture})$$

Here $\|p(A)\| := \sup_{\|\mathbf{x}\|=1} \|p(A)\mathbf{x}\|$.

The conjecture, while it is widely believed to be true, remains open today. In 2007, Crouzeix proved such an inequality with 2 replaced by 11.08, but he realized that a very different approach would be needed to get a constant close to 2. Then, in 2017, Crouzeix and Palencia found such a different approach and were able to replace the constant by $1 + \sqrt{2} \approx 2.414$. Except for a few classes of matrices (e.g., 2 by 2 matrices, matrices whose numerical range is a disk, weighted shift matrices, etc.), the constant 2 still has not been proved (or disproved).

I will go through the main pieces of the Crouzeix-Palencia proof. I will also discuss numerical studies and the insights that they can provide on this topic. Finally I will discuss some extensions, where $W(A)$ is replaced by, for instance, $W(A)$ with a disk removed, and the constant is adjusted appropriately. Such extensions can be useful, for example, in bounding the residual norm in the GMRES algorithm.