Global dynamics of Kuramoto's equation.

Dan Burns[1]
Gregorio Malajovich[2]
Charles Pugh[3]
Indika Rajapakse[1]
Steve Smale[2]
[1] *University of Michigan*
[2] *Universidade Federal do Rio de Janeiro*
[3] *University of California at Berkeley*

Smale@95, a Conference in Honor of Steve Smale
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Kuramoto's equations, simplified

Differential equation

$$\frac{\partial}{\partial t}\theta_1(t) = \omega_1 - \sum_{k=1}^{m} a_{1k} \sin(\theta_1(t) - \theta_k(t))$$

$$\vdots$$

$$\frac{\partial}{\partial t}\theta_m(t) = \omega_m - \sum_{k=1}^{m} a_{mk} \sin(\theta_m(t) - \theta_k(t))$$

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Kuramoto's equations, simplified

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$$\begin{aligned}
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A potential vector field

$$
\frac{\partial}{\partial t}\Theta(t) = -\nabla V(\Theta(t)),
$$

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V(\Theta) = \frac{1}{2}\sum_{l,k=1}^{m} (1 - \cos(\theta_l - \theta_k))
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Group actions by Isometry

- $(\mathbb{Z}^m, +)$: $V(\Theta) = V(\Theta + 2\pi\mathbf{k})$
- $\mathbb{R}$: $V(\Theta) = V(\Theta + r\mathbf{1})$
- Notation: K is the quotient of $\mathbb{R}^m$ by the two actions above, $[\Theta]$ equivalence class of Θ.
- S_m: $V(\Theta) = V(\sigma(\Theta))$

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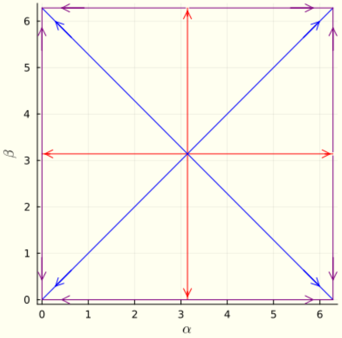
Low potential regime $0 < V < \frac{m^2-1}{2}$

Perfect Morse

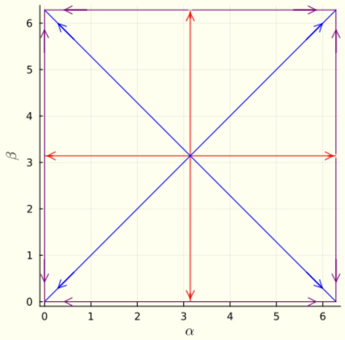


Fixed points

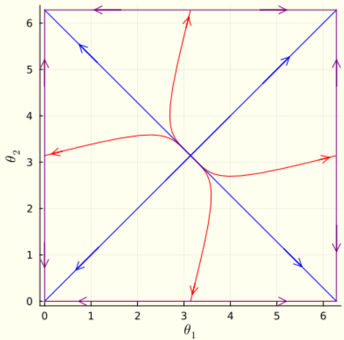
- There is a unique sink $[0, \dots, 0]$ at $V = 0$.
- For $1 \leqslant k \leqslant \lfloor (m^2 - 1)/2 \rfloor$, there are $\binom{m}{k}$ saddles of index k at $[0^{\times m-k}, \pi^{\times k}]$ and permutations.
- Let S be an index k saddle. Then $W^u(S)$ is a copy of $\mathbb{T}^k$ and the restriction $-\nabla V_{|W^u(S)}$ is topologically conjugated to 'Perfect Morse' $\dot{\theta}_i = -\sin(\theta_i)$ in $(S^1)^k$.

Low potential regime $0 < V < \frac{m^2-1}{2}$

Perfect Morse



Template for index 2 saddle, $m = 5$



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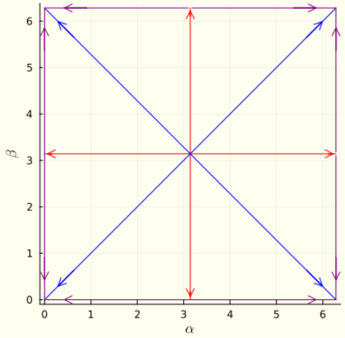
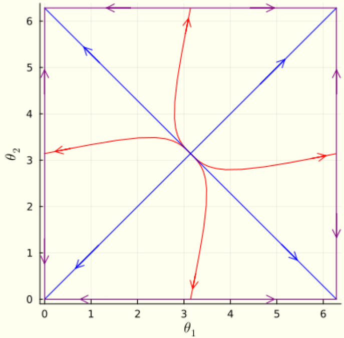


Template for index 2 saddle, $m = 6$



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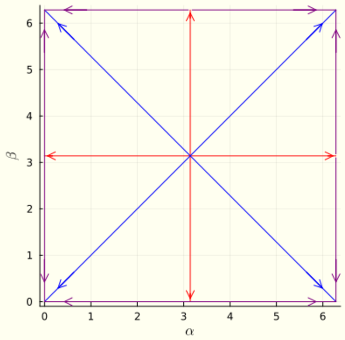
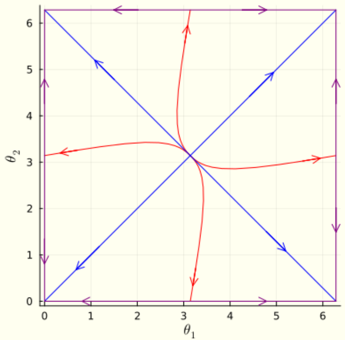


Template for index 2 saddle, $m = 7$



Low potential regime $0 < V < \frac{m^2-1}{2}$

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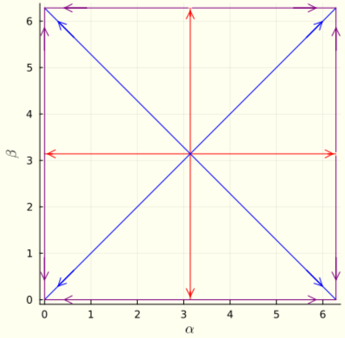
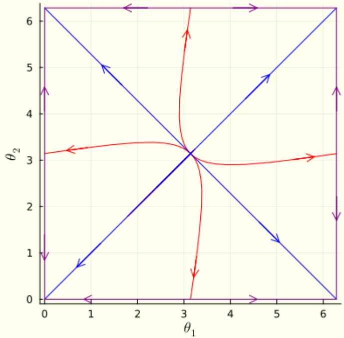


Template for index 2 saddle, $m = 8$



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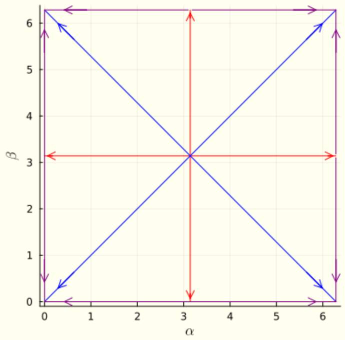
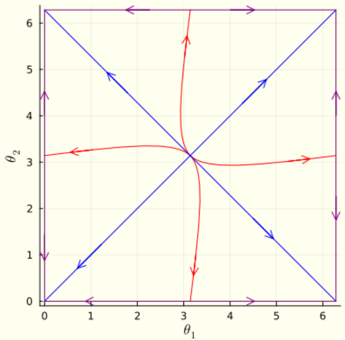


Template for index 2 saddle, $m = 9$



Low potential regime $0 < V < \frac{m^2-1}{2}$

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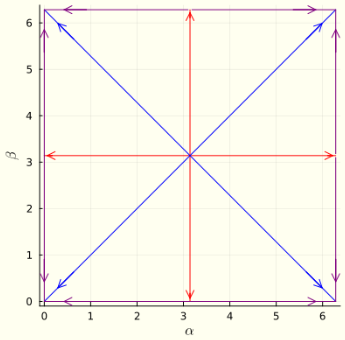
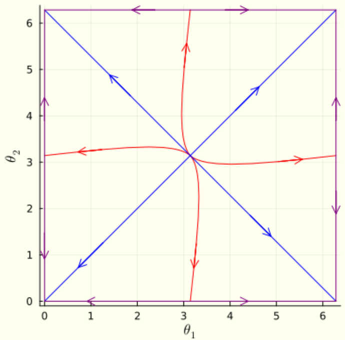


Template for index 2 saddle, $m = 10$



Low potential regime $0 < V < \frac{m^2-1}{2}$

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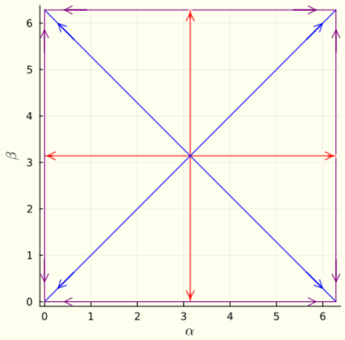
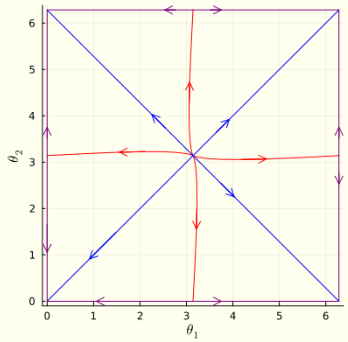


Template for index 2 saddle, $m = 20$



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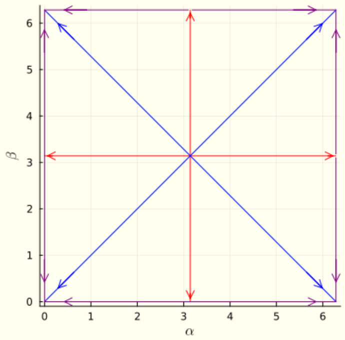
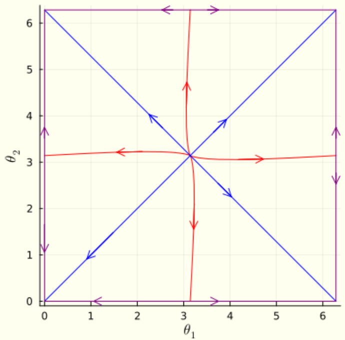


Template for index 2 saddle, $m = 40$



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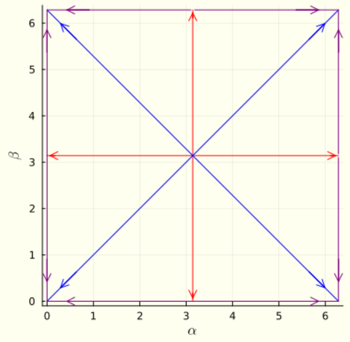
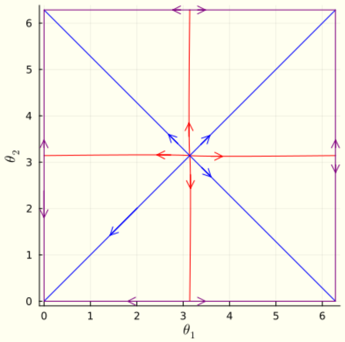


Template for index 2 saddle, $m = 100$



Mid and high potential regimes $\frac{m^2-1}{2} \leqslant V \leqslant \frac{m^2}{2}$

The source.

- $V^{\max} = \{[\Theta] : V(\Theta) = m^2/2\}$
- $[\Theta]$ fixed iff $[\Theta] \in V^{\max}$ iff $\sum_{k=1}^{m} e^{i\theta_k} = 0$. There are no other fixed points with potential $\geqslant \frac{m^2-1}{2}$.
- Subsets of $V^{\max}$ given by equalities of θ_i's are geodesic in $V^{\max}$. When they intersect transversally, they are orthogonal.
- From now on $\theta_1 = 0$ and permutations fix θ_1.

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The case $m = 4$

- $V^{\max}$ is the union of 6 lines $[0, \alpha, \pi, -\alpha]$, $0 \leqslant \alpha$, $0 \leqslant \alpha \leqslant \pi$, and permutations.
- They meet orthogonally in 3 vertices, namely $[0, 0, \pi, \pi]$ and permutations.

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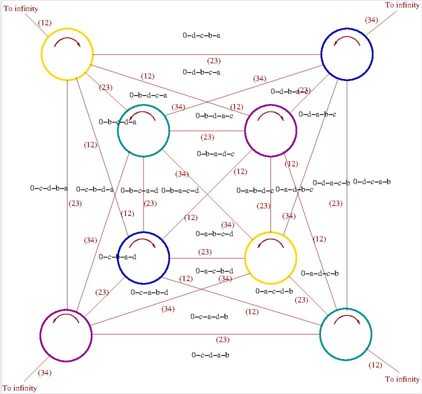
The case $m = 5$

- $V^{\max}$ union of $F = 24$ pentagonal cells, viz. $[0 = \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5]$ and permutations.
- There are $E = \frac{24 \times 5}{2} = 60$ edges.
- Because of right angles, $V = \frac{24 \times 5}{4} = 30$
- $\chi = F - E + V = 24 - 60 + 30 = -6$
- $\chi = 2 - 2g$ so $V^{\max}$ compact orientable surface of genus 4.



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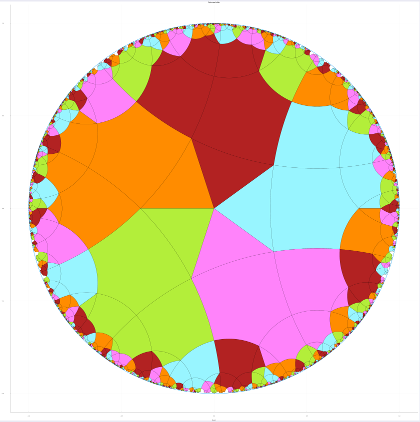

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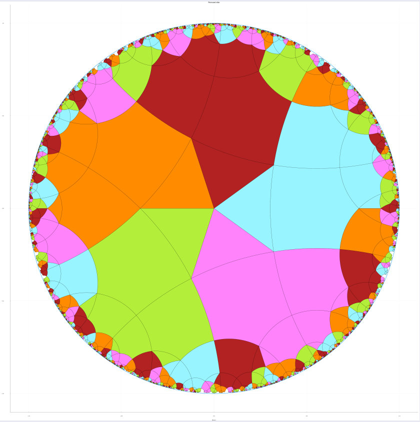
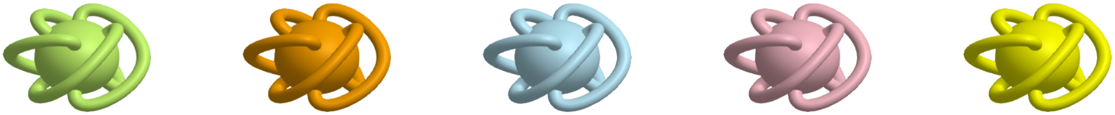
The general case

- When m is even, $V^{\max}$ is smooth.
- When m is odd, $[0, \ldots, 0, \pi, \ldots, \pi]$ and permutations are isolated singularities.
- The smooth part of $V^{\max}$ is normally hyperbolic.
- Homology of $V^{\max}$ is known.





Happy Birthday Steve!