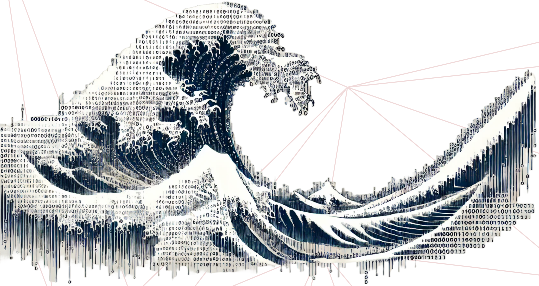

Zeyu Guo

zguotcs@gmail.com

Ohio State University

April 25, 2025

Random Gabidulin Codes Achieve List Decoding Capacity in the Rank Metric

joint work with Chaoping Xing (Shanghai Jiao Tong University), Chen Yuan (Shanghai Jiao Tong University), and Zihan Zhang (Ohio State University)

Introduction

Minimum Distance and Rate

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- Codes attaining the Singleton bound are called MDS codes.

Rank Codes: Matrix Formulation

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- The Singleton bound also holds with respect to the rank metric.

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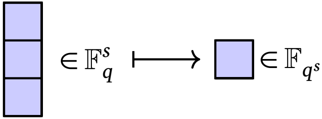

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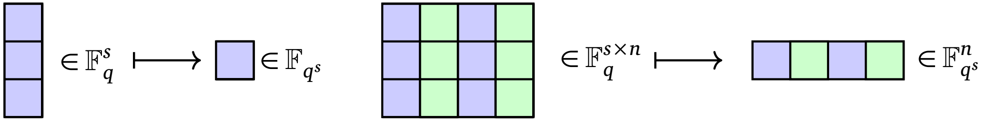

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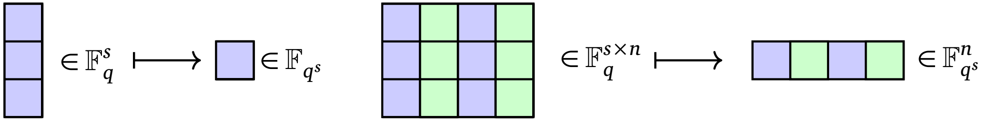

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Corollary

All MRD codes are MDS codes.

Definition (Linearized polynomials)

Let q be a prime power. We say $f(x)$ is q-linearized if it has the form

$$f(x) = a_d x^{q^d} + a_{d-1} x^{q^{d-1}} + \cdots + a_1 x^q + a_0 x.$$

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- The composition of two q-linearized polynomial is again a q-linearized polynomial.
- Composition is generally noncommutative: $(cX) \circ (X^q) = cX^q$ while $(X^q) \circ (cX) = c^q X^q$.

Definition (Gabidulin codes)

Given $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}_{q^s}$ that are linearly independent over $\mathbb{F}_q$, the corresponding $[n,k]$ Gabidulin code is

$$\mathsf{G}_{n,k}(\alpha_1, \dots, \alpha_n) := \left\{ \big(f(\alpha_1), \dots, f(\alpha_n)\big) \;\middle|\; \begin{array}{l} q\text{-linearized } f \in \mathbb{F}_{q^s}[x], \\ \deg_q(f) < k \end{array} \right\} \subseteq \mathbb{F}_{q^s}^n .$$

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Theorem

Gabidulin codes are MRD codes.

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- Connected with two-source rank condensers [Forbes–Guruswami '14], dimension expanders [Guruswami–Resch–Xing '18], and deterministic extractors [Guo–Volk–Jalan–Zuckerman '23].

List Decodability

Combinatorial List Decodability

Definition (Combinatorial list decodability)

For $\rho \in [0,1]$ and $L \geq 1$, a code $C \subseteq \mathbb{F}_q^n$ is (ρ, L) list decodable if for all $y \in \mathbb{F}_q^n$ and $L+1$ distinct codewords $c_0, c_1, \ldots, c_L \in C$,

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Definition (Average-radius combinatorial list decodability)

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Combinatorial List Decodability of RS Codes in Hamming Metric

Theorem (Johnson Bound)

Any $[n,k]$ *code* C *with distance* $d(C)$ *is* $\left(1-\sqrt{1-\frac{d(C)}{n}}, qnd(C)\right)$ *list decodable in the Hamming metric.*

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Corollary

Any $[n,k]$ *RS code* C *of rate* R *over* $\mathbb{F}_q$ *is* $\left(1-\sqrt{R}, L\right)$ *list decodable in the Hamming metric, where* $L = qnd(C)$.

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List decoding Gabidulin codes in the **rank metric** is more challenging:

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What about a random Gabidulin code $\mathsf{G}_{n,k}(\alpha_1, \ldots, \alpha_n)$?

Generalized Singleton Bound

Theorem (Generalized Singleton Bound [Shangguan–Tamo '20])

If a linear code C of rate R is (ρ, L) list decodable in the Hamming metric, then

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We show that random Gabidulin codes (over sufficiently large fields) achieve the generalized Singleton bound, even in the rank metric.

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  proves the main result of [Brakensiek–Gopi–Makam '23].

Our Results

Combinatorial List Decodability of Random Gabidulin Codes

Theorem (Guo–Xing–Yuan–Zhang '24)

For any $L \geq 1$, a random Gabidulin code of rate R over a sufficiently large field is, w.h.p, $\left(\frac{L}{L+1}(1-R), L\right)$ average-radius list decodable in the rank metric.

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To prove the theorem, we develop a theory of "higher-order MRD codes" for rank codes, analogous to the theory of higher-order MDS codes developed by Brakensiek, Gopi, and Makam.

Higher Order MRD Codes

- We introduce an analogous theory by introducing three notions of "higher order MRD codes": $\mathrm{MRD}(L)$, $\mathrm{GKP}(L)$, and $\mathrm{LD\text{-}MRD}(L)$.

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Corollary ([Guo–Xing–Yuan–Zhang '24])

For all $L \geq 1$, *random Gabidulin codes of rate* R *over large enough fields are w.h.p.* $\left(\frac{L}{L+1}(1-R), L\right)$ *average-radius list decodable in the rank metric.*

An Interesting Formula for Generic Intersections

Generic Intersection Formula

Definition

For symbolic (generic) matrix $W = \left(X_{ij}\right) \in \mathbb{F}_q(X_{11}, \cdots, X_{kn})^{k \times n}$ and a set $A \subseteq [n]$, define the subspace

$$W_A := \operatorname{span}_{\mathbb{F}_q(X_{11},\cdots,X_{kn})}\{i\text{-th column vector of } W : i \in A\} \subseteq \mathbb{F}_q(X_{11}, \cdots, X_{kn})^k.$$

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Theorem (Generic Intersection Formula [Brakensiek–Gopi–Makam '23])

Given $A_1, \ldots, A_\ell \subseteq [n]$ of size at most k, for a $k \times n$ generic matrix W,

$$\dim\left(W_{A_1} \cap \cdots \cap W_{A_\ell}\right) = \max_{P_1 \sqcup P_2 \sqcup \cdots \sqcup P_s = [\ell]} \left(\sum_{i \in [s]} \left| \bigcap_{j \in P_i} A_j \right| - (s-1)k \right).$$

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Related Works

Reducing the Alphabet Size

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- Analogous to [Guo–Zhang '23] and [Alrabiah–Guruswami–Li '23].

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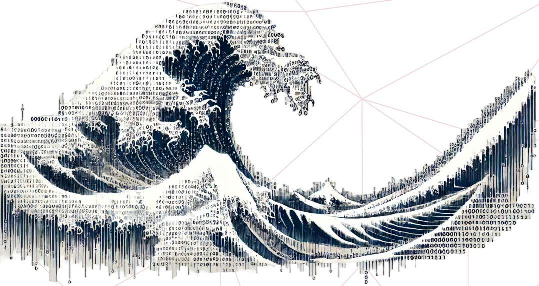

The End