Efficient Cryptographic Proofs from RAA Codes

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Succinct Non-Interactive Arguments (SNARGs)

[Kilian'92, Micali'94]

Prover: (x, w)

proof π

Verifier: x

Key bottleneck in practice

Targets:

- *Succinct* proof: $|\pi| \ll |w|$
- *Efficient* prover (linear time)
- *Efficient* verifier (polylog time)

Soundness against *poly-time* provers

$L = \{x : \exists w, \Gamma(x, w) = 1\}$

Recall: Interactive Oracle Proofs (IOPs)

[Reingold-Rothblum-Rothblum'16, BenSasson-Chiesa-Spooner'16]

Hybrid between (public-coin) interactive proofs and PCPs

Prover: (x, w)

Verifier: x

queries $\ll |w|$

proof π_1 →

← challenge r_1

proof π_2 →

← challenge r_2

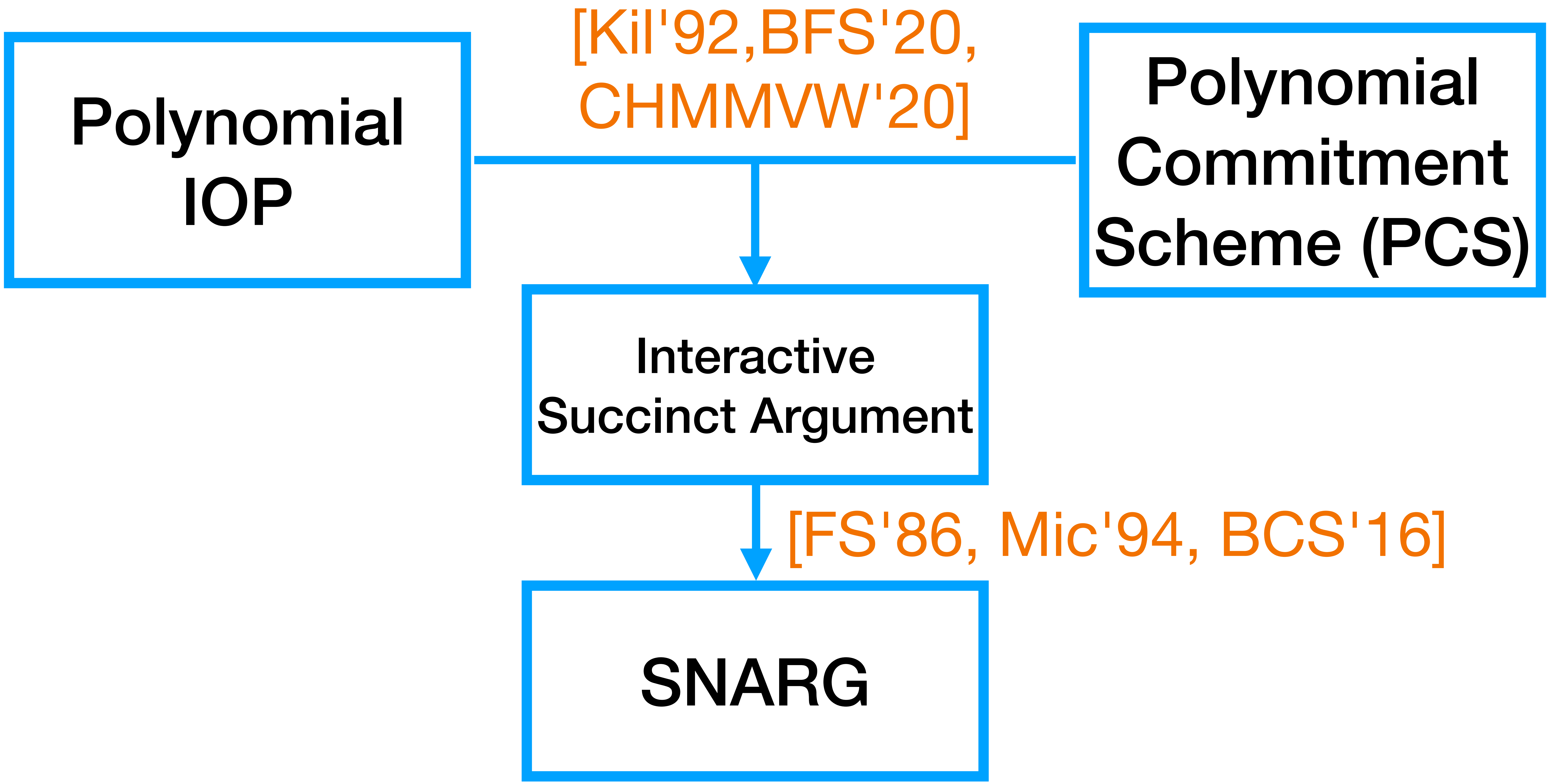
⋮

$x \overset{?}{\in} L$

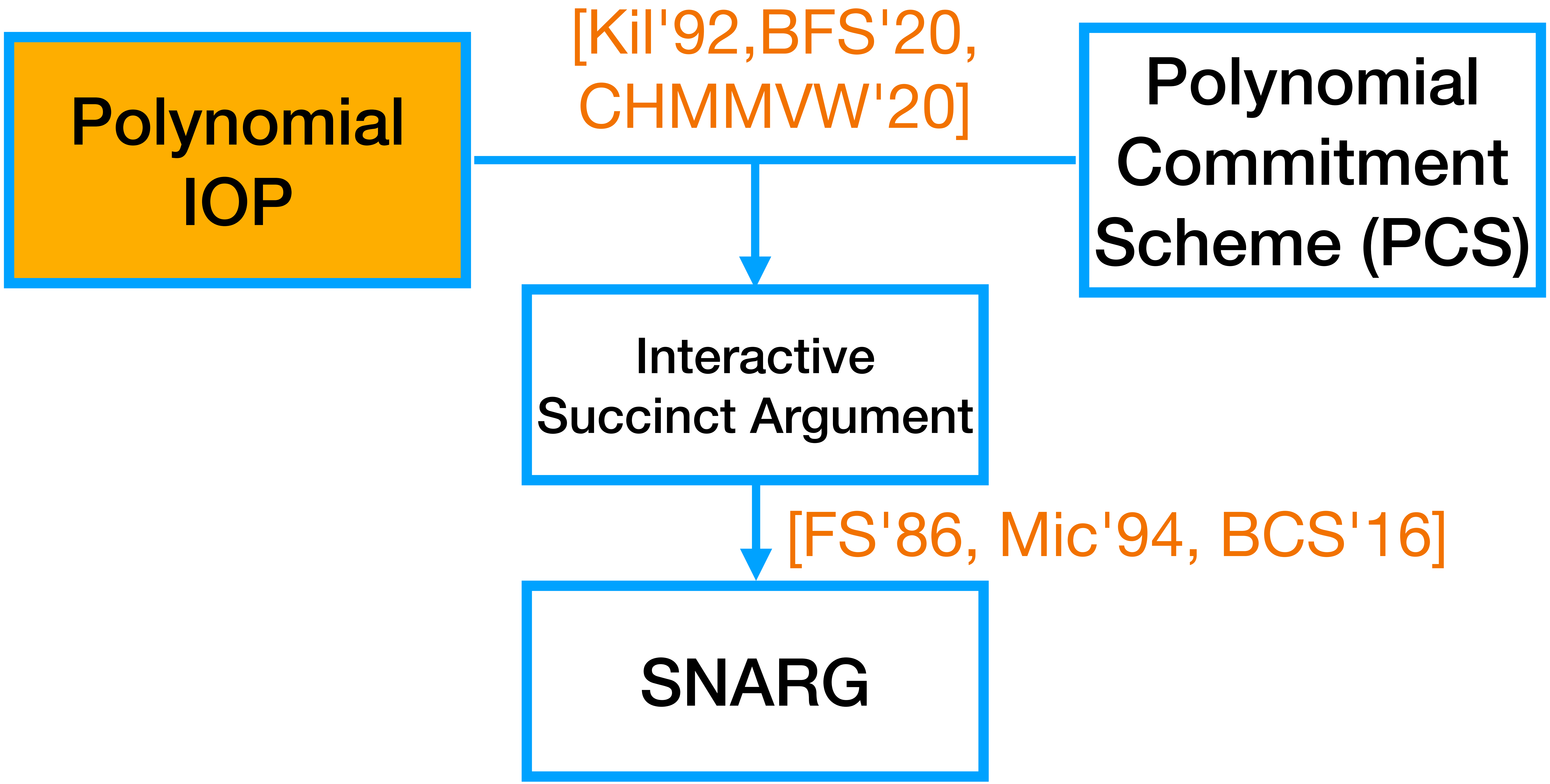
$L = \{x : \exists w, \Gamma(x, w) = 1\}$

Soundness against *arbitrary* provers

Common SNARG Construction Paradigm



Common SNARG Construction Paradigm



Polynomial IOPs

[RRR'16, BFS'20, CHMMVW'20]

Prover: (x, w)

Verifier: x

(evaluations of) polynomial P_1 →

← challenge r_1

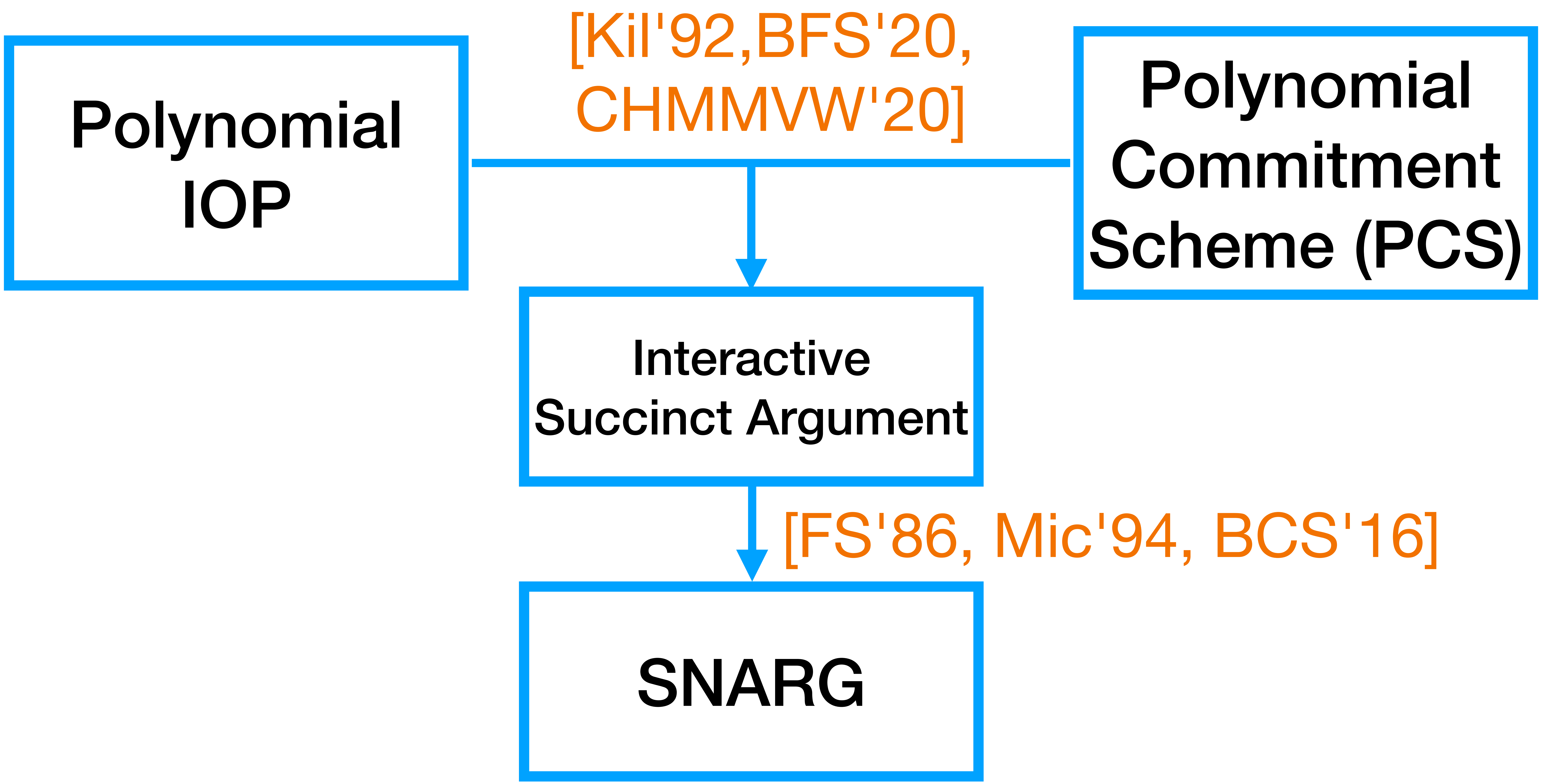
(evaluations of) polynomial P_2 →

← challenge r_2

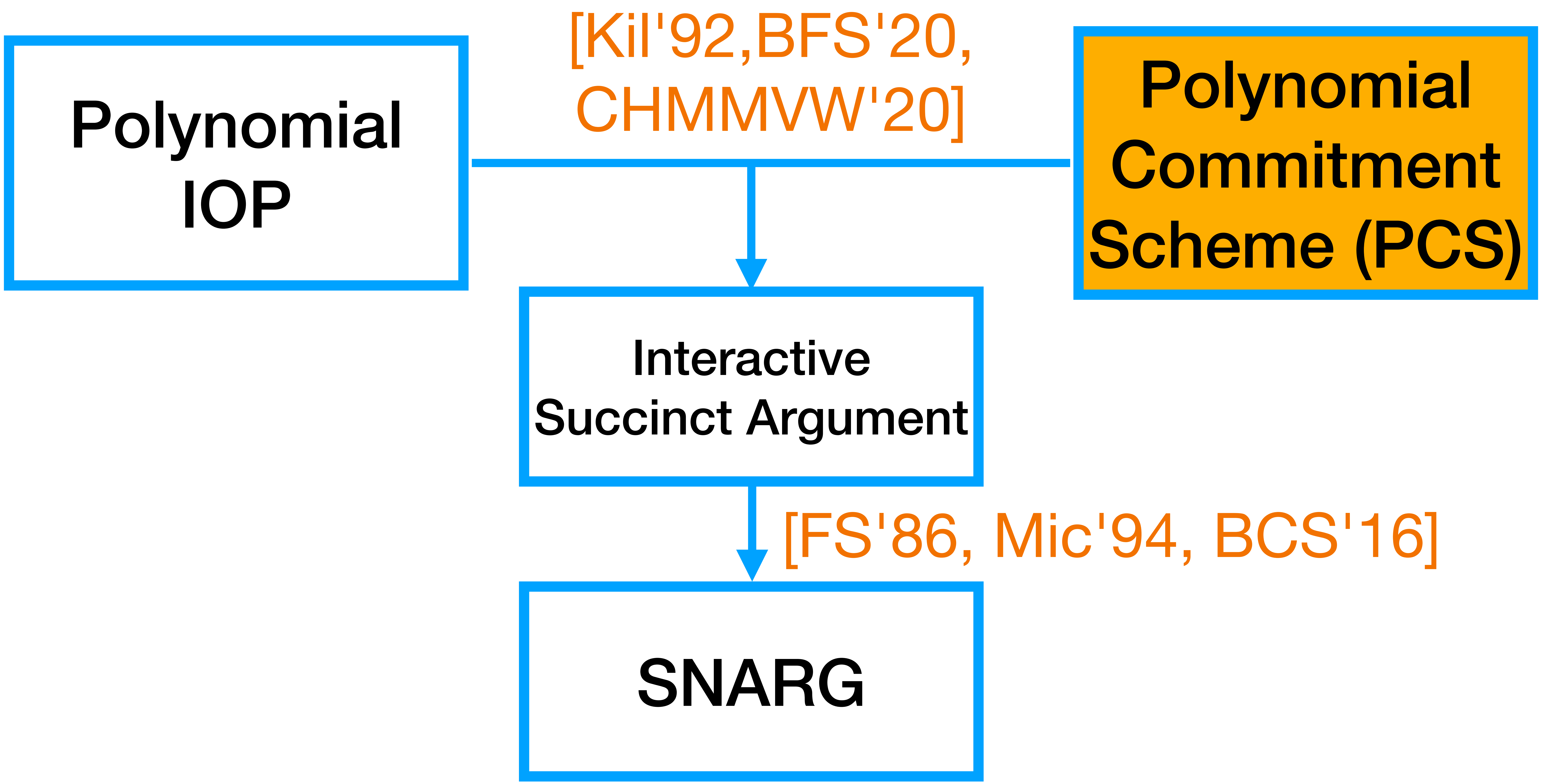
⋮

Queries:
what is $P_i(\alpha)$?

Common SNARG Construction Paradigm

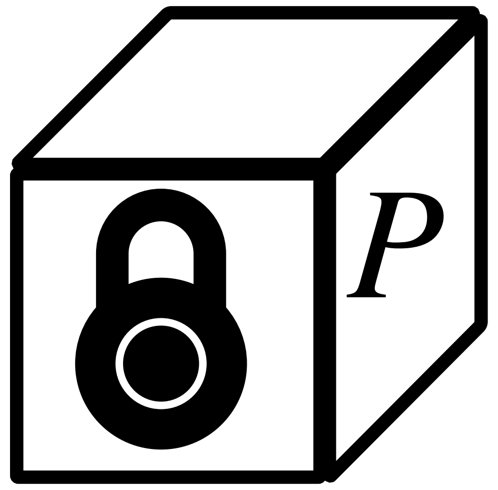


Common SNARG Construction Paradigm



Polynomial Commitment Scheme (PCS)

Key bottleneck in modern SNARGs



(short) commitment to P

I'm convinced $P(x) = y$

Later...

what is value of P on x?

it is y

Blaze: a new fast *Multilinear* PCS (MLPCS)

- Focus on committing to multilinear polynomials over characteristic 2 fields

- That is, $P : \mathbb{F}_{2^\lambda}^r \to \mathbb{F}_{2^\lambda}$ is of the form

$$P(X_1, \ldots, X_r) = \sum_{S \subseteq [r]} c_S \prod_{i \in S} X_i, \qquad c_S \in \mathbb{F}_{2^\lambda}$$

- Go hand-in-hand with highly efficient multilinear polynomial IOPs: Spartan [Set'20], Lasso/Jolt [STW'24, AST'24], Hyperplonk [CBBS'23] and generally GKR-based schemes such as Orion [XZS'22]

Why Characteristic 2?

Pros

- Addition is fast
- Eliminate "embedding overhead" [DP23]
- Friendly to arithmetization of logical operations

Cons

- Multiplication is a bit more complicated (but we won't be doing much!)
- Less friendly to integer arithmetic

Real reason: we use a code that we currently only know how to analyze over $\mathbb{F}_2$…

Blaze Asymptotics

- Cost of committing to $P : \mathbb{F}^r \to \mathbb{F}$ (described by $N = 2^r$ field elements)
- Commitment generation: $8N$ additions + 1 Merkle Hash
- Evaluation proof generation: $6N$ additions + $5N$ multiplications
- Proof length anc $\mathrm{N})^2\big)$

Prior works:

$O(N \log N)$, or $O(N)$

with unspecified constant

Techniques: Bird's Eye View

1. Building on code-switching technique [RonZewi-Ron'20], using *code interleaving* we build an MLPCS from

   MLPCS for "smaller" polynomials

   Leads to proof-size & verification time

   Error-correcting code

   Leads to proving time & committing time

2. Use Repeat-Accumulate-Accumulate (RAA) codes, which have *very* fast encoding and (usually) good distance

The Multi-Linear Polynomial Commitment Scheme (MLPCS)

Multilinear Extension

Will identify $u \in \mathbb{F}^k$ with a function $u : \{0,1\}^{\log k} \to \mathbb{F}$

- Given a function $f : \{0,1\}^r \to \mathbb{F}$, there exists a unique $\hat{f} : \mathbb{F}^r \to \mathbb{F}$ s.t.

$\hat{f}$ is multilinear

$\hat{f}(x) = f(x)$ for all $x \in \{0,1\}^r$

- Specifically, take

$$\hat{f}(x) = \sum_{b \in \{0,1\}^r} f(b) \cdot \mathrm{eq}(b, x)$$

where $\mathrm{eq}(b, x) = \prod_{i=1}^{r} \left(b_i x_i + (1 - b_i)(1 - x_i)\right)$

MLPCS from Codes & IOPP

$(k = 2^r)$

Fix a code $\mathscr{C} : \mathbb{F}^k \to \mathbb{F}^n$

To get a multilinear polynomial commitment scheme (MLPCS),

suffices to design an interactive oracle proof of proximity (IOPP) for the language [FS'86, Kil'92, Mic'94, BCS'16]

$$\{\mathscr{C}(m) \in \mathbb{F}^n : m \in \mathbb{F}^k \text{ and } \hat{m}(z) = v\}$$

only need to reject if input is far (in Hamming distance) from language

Interleaved Codes

$t = \Theta(\log n)$

Identify $M \in \mathbb{F}^{t \times k}$, $M \in \mathbb{F}^{tk}$ and $M : \{0,1\}^{\log t + \log k} \to \mathbb{F}$

Given $\mathscr{C} : \mathbb{F}^k \to \mathbb{F}^n$ and integer t

Construct new code $\mathscr{C}^t : \mathbb{F}^{tk} \to \mathbb{F}^{tn}$ by interleaving [Ligero, BCGGHJ'17]

Assuming IOPP for $\{\mathscr{C}(m) \in \mathbb{F}^n : m \in \mathbb{F}^k$ and $\hat{m}(z) = v\}$ exists,

we construct IOPP for $\{\mathscr{C}^t(M) : M \in \mathbb{F}^{t \times k}, \hat{M}(z) = v\}$ $(z \in \mathbb{F}^{\log(tk)})$

$M \in \mathbb{F}^{t \times k}, v \in \mathbb{F}$
$z \in \mathbb{F}^{\log(t)+\log(k)}$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix}$$

$v \in \mathbb{F},$
$z \in \mathbb{F}^{\log(t)+\log(k)}$

Idea: decompose to claim on *rows*

Need to prove
$\hat{M}(z) = v$

$$\hat{M}(z) = \sum_{b \in \{0,1\}^{\log t + \log k}} \mathrm{eq}(b, z) \cdot M(b)$$
$$= \sum_{b_1 \in \{0,1\}^{\log t},\ b_2 \in \{0,1\}^{\log k}} \mathrm{eq}(b_1, z_1) \cdot \mathrm{eq}(b_2, z_2) \cdot M(b_1, b_2)$$
$$= \sum_{b_1 \in \{0,1\}^{\log t}} \mathrm{eq}(b_1, z_1) \cdot \hat{M}_{b_1}(z_2)$$

$z = (z_1, z_2) \in \mathbb{F}^{\log t} \times \mathbb{F}^{\log k}$

Multilinear Evaluation with Interleaving

1. Prover sends function $u : \{0,1\}^{\log t} \to \mathbb{F}$ defined as

$$u(x) = \hat{M}_x(z_2)$$

Send random linear combination of $m_1, \ldots, m_t$

2. Verifier checks that

$$\hat{u}(z_1) = \sum_{b_1 \in \{0,1\}^{\log t}} \mathrm{eq}(b_1, z_1) \cdot u(b_1) = v$$

If Verifier check passes but $\hat{M}(z) \neq v$, then Prover sent incorrect u

How to enforce correctness?

$M \in \mathbb{F}^{t \times k}, v \in \mathbb{F}$
$z \in \mathbb{F}^{\log(t)+\log(k)}$

$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_t \end{bmatrix}$

$v \in \mathbb{F},$
$z \in \mathbb{F}^{\log(t)+\log(k)}$

$\xrightarrow{u}$

$\xleftarrow{r}$

$\xrightarrow{c_0}$

Compute $u(x) = \hat{M}_x(z_2)$

Check $\hat{u}(z_1) \stackrel{?}{=} v$

Sample $r \in \mathbb{F}^t$

$c_0 := \Sigma_i\ r_i c_i$, which is $\mathscr{C}$-encoding of $m_0 := \Sigma_i\ r_i m_i$

Run IOPP for $\mathscr{C}$ with $m_0, c_0, z_2, v_0 := \Sigma_i\ r_i u_i$

Check consistency of c_0 and C for $\Omega(\lambda/\delta)$ $i \in [n]$

Proximity Gaps

What if $C \in \mathbb{F}^{t \times n}$ is far from $\mathscr{C}^t$?

Can use elegant results on *proximity gaps* for codes [RVW'13, BKS'18] to guarantee that whp, $c_0 = \Sigma_i\ r_i c_i$ is far from $\mathscr{C}$

Theorem: ([BKS'18]) Suppose $\mathscr{C} \leq \mathbb{F}^n$ has min. distance δ, let $U \subseteq \mathbb{F}^n$ be an affine space and suppose $\exists u \in U$ for which $\Delta(u, \mathscr{C}) > \tau$. Then if $\varepsilon > 0$ is s.t. $\tau - \varepsilon < \delta/3$,

$$\mathbb{P}_{u \in U}[\Delta(u, \mathscr{C}) < \tau - \varepsilon] \leq (\varepsilon \mathbb{F})^{-1}$$

Summary

Prover time dominated by computation of t $\mathscr{C}$-encodings

So: pick code with *blazing fast* encoding!

Additionally: for soundness need good distance

can use "off-the-shelf" constructions

Actually, $\mathscr{C}$ better already have an IOPP for multilinear evaluation…

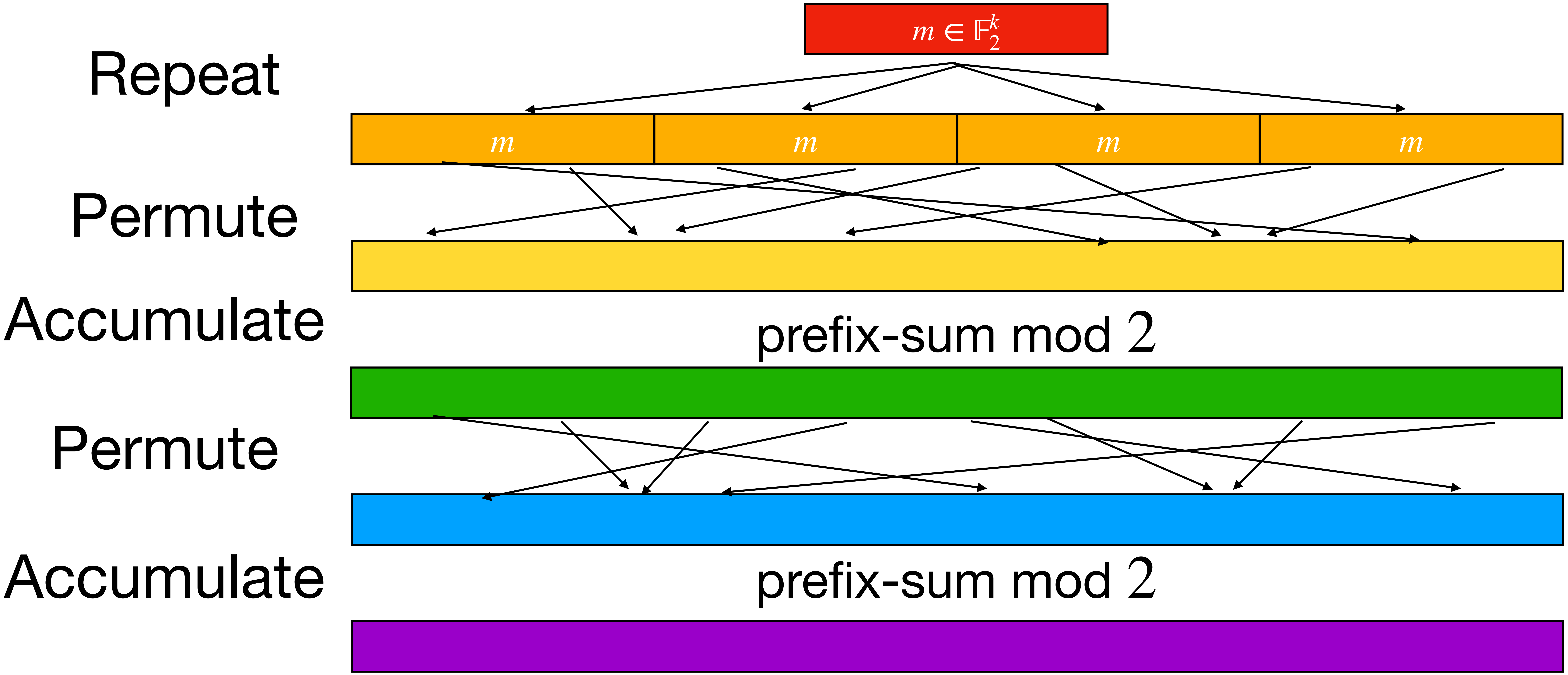
but we actually provide tailor-made one for our choice of $\mathscr{C}$

Repeat-Accumulate-Accumulate Codes [Divsalar-Jin-McEliece'98]

RAA Encoding

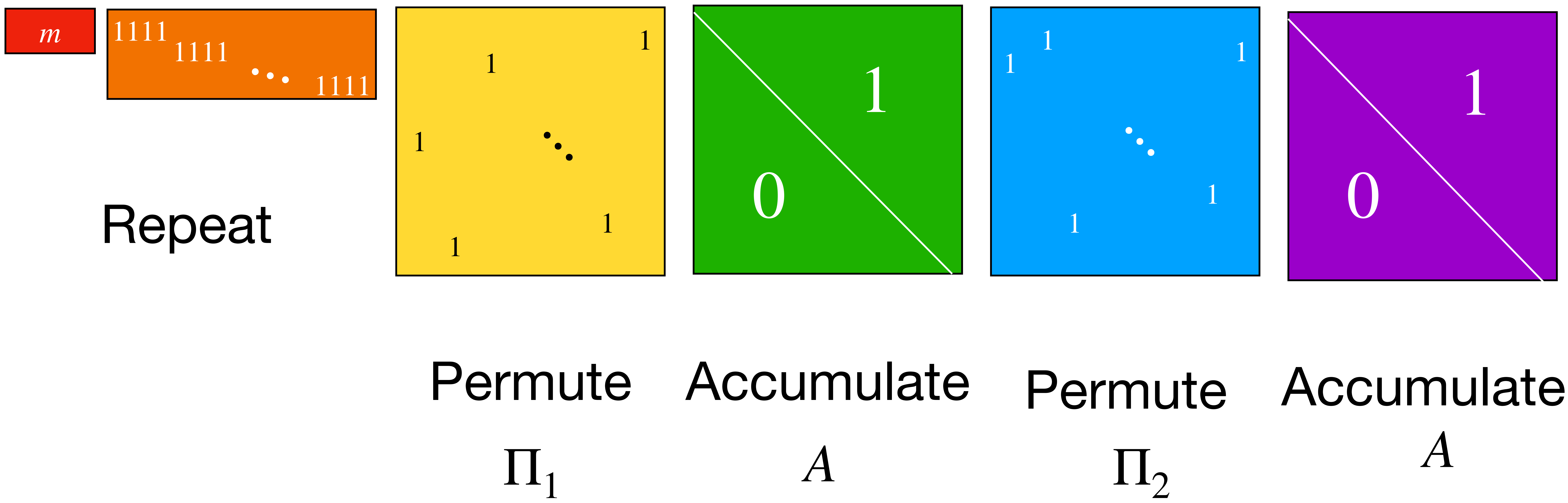
working over $\mathbb{F}_2$...

(rate $1/4$ case)



RAA Generator Matrix

(rate $1/4$ case)



Analysis

Over random choice of 2 permutations, we show whp

a rate $1/4$ RAA code has min. distance ≥ 0.19

GV bound for rate $1/4$: ~ 0.21

This builds off prior works [PS'03, BMS'07, KZKC'07, RF'09]

Use *Input-Output-Weight-Enumerator Function*

$$N(a,b) := |\{x \in \mathbb{F}_2^n : \mathrm{wt}(x) = a \text{ and } \mathrm{wt}(xA) = b\}|$$

$$= \binom{b-1}{\lceil a/2 \rceil - 1} \cdot \binom{n-b}{\lfloor a/2 \rfloor}$$

$$p_{a \to b} := \mathop{\mathbb{P}}_{\substack{x \in \mathbb{F}_2^n, \\ \mathrm{wt}(x) = a}} \left[\mathrm{wt}(xA) = b\right] = \frac{N(a,b)}{\binom{n}{a}}$$

Analysis

Let $X =$ number of codewords of weight $\leq d$

By Markov: $\mathbb{P}[\mathrm{dist}(\mathscr{C}) \leq d] = \mathbb{P}[X \geq 1] \leq \mathbb{E}[X]$

number of messages

prob. of going from *repeated* message weight to intermediate weight

$$\mathbb{E}[X] = \sum_{a=1}^{n/4} \sum_{b=1}^{n} \sum_{w=1}^{d} \binom{n/4}{a} \cdot p_{4a \to b} \cdot p_{b \to w}$$

choices for message weight

choices for intermediate weight

choices for codeword weight

prob. of going from intermediate weight to codeword weight

Analysis

$$\mathbb{E}[X] = \sum_{a=1}^{n/4} \sum_{b=1}^{n} \sum_{w=1}^{d} \binom{n/4}{a} \cdot p_{4a\to b} \cdot p_{b \to w}$$

Break up sum based on b

$b \le h = \omega(\log n)$:

use $\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$;

final bound $1/n^\varepsilon$

$b \ge h = \omega(\log n)$:

use $\binom{a}{b} \le 2^{aH(b/a)}$;

final bound $2^{-\Omega(h)}$

Tests

From crypto perspective: failure probability $1/n^{\varepsilon}$ quite large…

also, $1/n^{O(1)}$ unavoidable (consider prob. of wei
$O(1) \to O(1) \to O(1)$; all terms $1/\mathrm{poly}(n)$)

Actually, just check first round accumulation

To boost failure probability: check encoding of low-weight messages

Conditioned on test passing, failure probability decreases substantially

but with poly-time test, failure probability still $\geq 1/\mathrm{poly}(n)$…

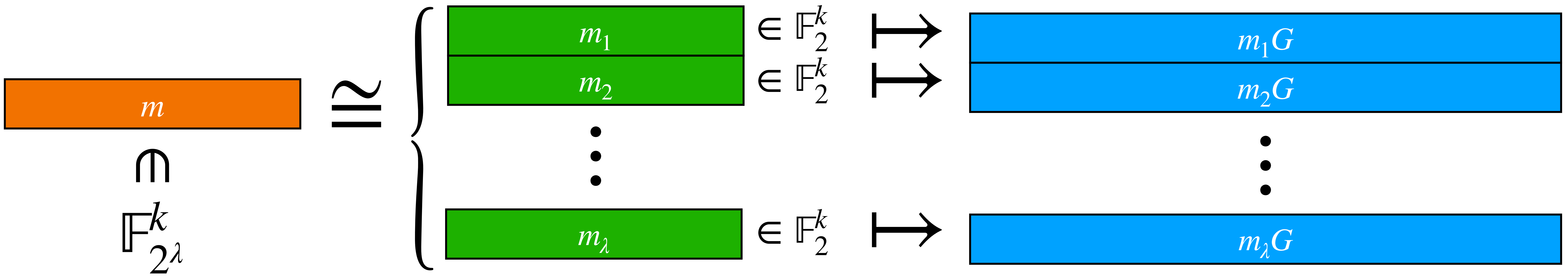
RAA over Extension Fields

Thus far: analyzed RAA codes over $\mathbb{F}_2$

But: for soundness, need $|\mathbb{F}| = 2^\lambda, \lambda \approx 128$

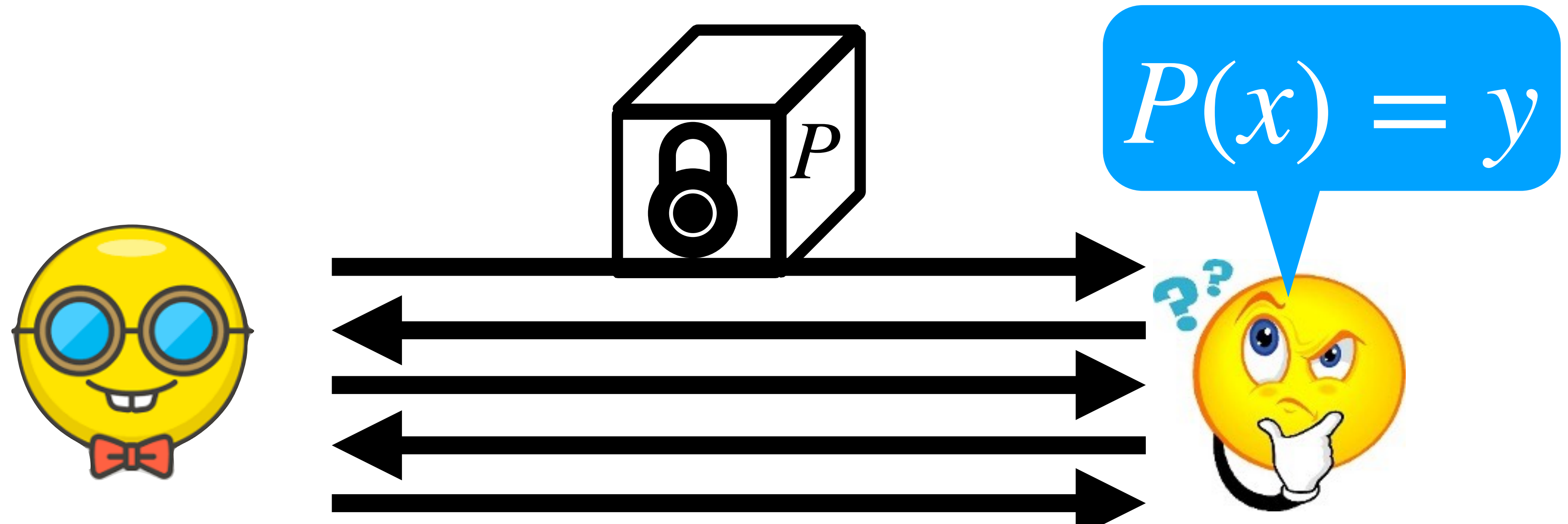
Can use same generator matrix G to define code over $\mathbb{F}_{2^\lambda}$

Can implement by "bit-slicing"



Conclusion

Recap



1. Building on code-switching technique [RonZewi-Ron'20], using *code interleaving* we build an MLPCS from

"Smaller" MLPCS

Leads to proof-size & verification time

$\Longleftarrow$ IOPP for $\{\mathscr{C}(m) \in \mathbb{F}^n : m \in \mathbb{F}^k$ and $\hat{m}(z) = v\}$

Error-correcting code

Leads to proving time & committing time

2. Use Repeat-Accumulate-Accumulate (RAA) codes, which have *very* fast encoding and are (usually) near GV-bound

Open Problems

- Analyze RA* codes directly over larger alphabets? [BFKTWZ'24] Can we approach the Singleton bound?
- (More) explicit constructions? Like [Applebaum-Kachlon'20], design test that leads to negligible $n^{-\omega(1)}$ failure probability?
- Could puncturing be useful?
- Concrete bounds for smaller n? (currently need $n \approx 2^{20}$)
- Improved proximity gaps? Maybe tailor-made for RAA codes?

Thank you!