The Jacobi Factoring Circuit

Classically Hard Factoring in Sublinear Quantum Space and Depth

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Integer Factorisation

Given an n-bit integer $N < 2^n$, find its prime factorisation in $\mathsf{poly}(n)$ time.

Classical Factoring: A Crash Course

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For more: see Pomerance's survey "A Tale of Two Sieves"!

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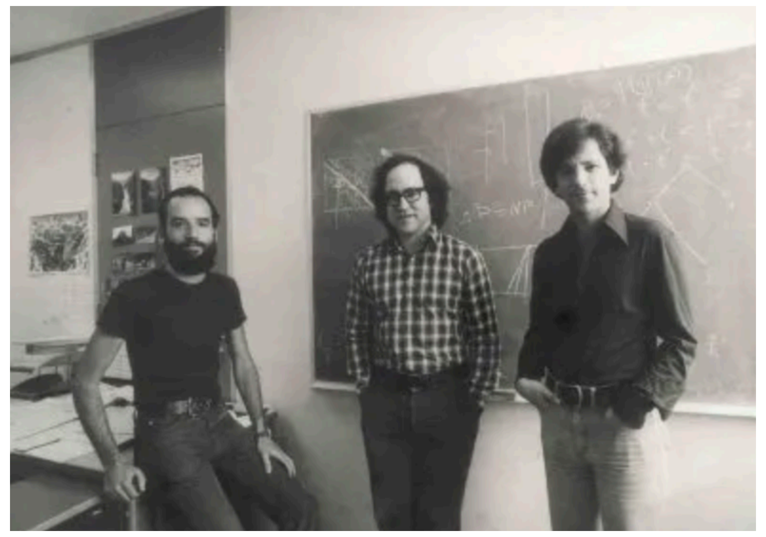
- **Quantum algorithms:** $\mathsf{poly}(n)$ **time!** (Shor 1994)

Implication 1: Breaking Cryptography

- RSA public-key cryptography:

pk $= N\ (= pq)$

sk $= (p, q)$



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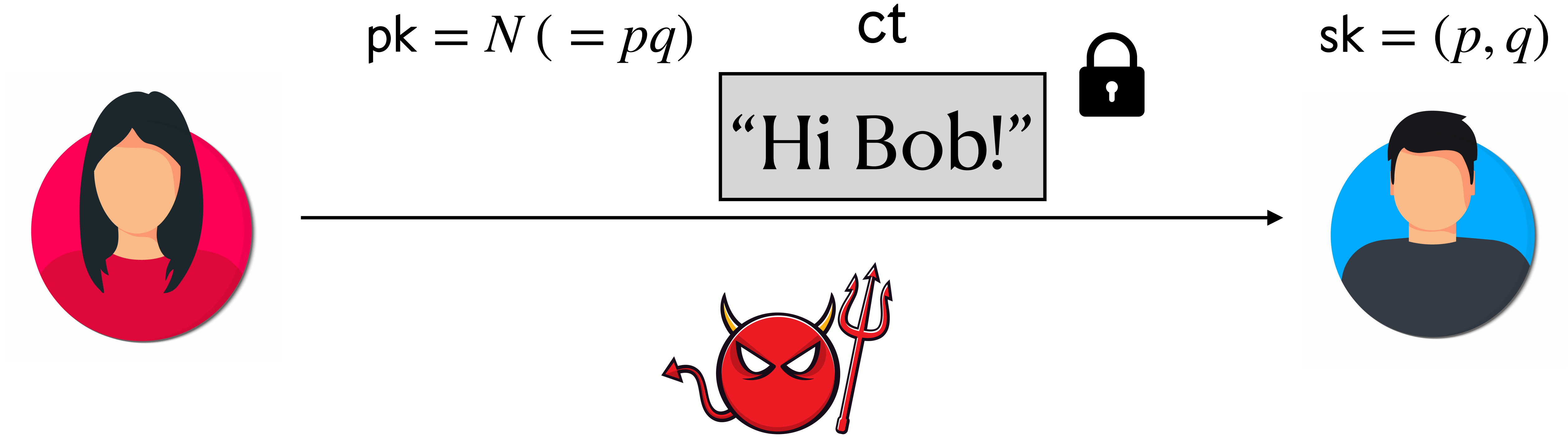

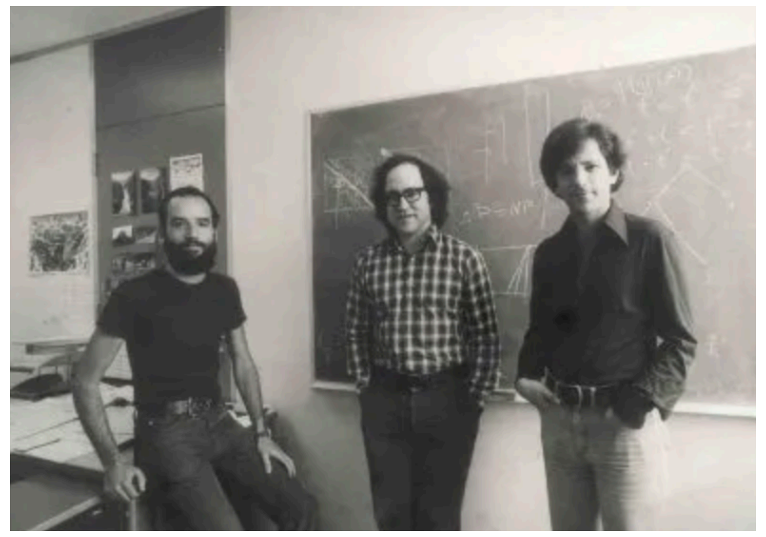

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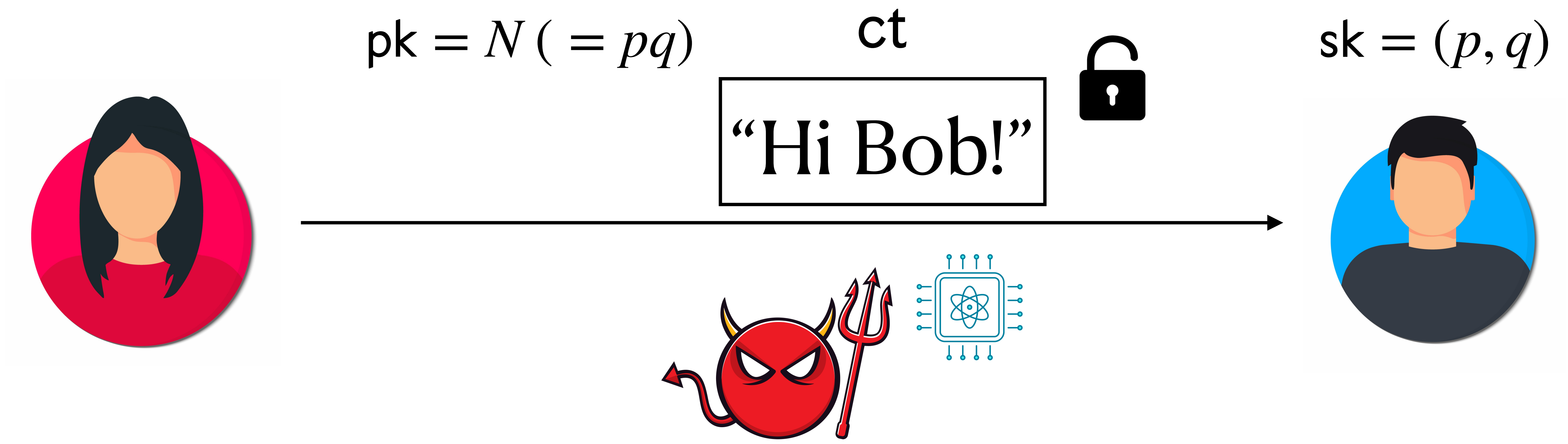

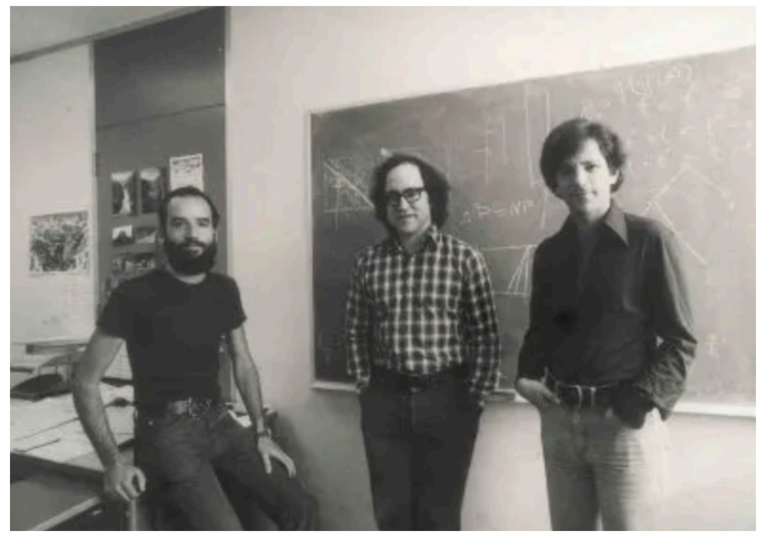

- **Completely broken if the eavesdropper has a large quantum computer (learn Bob's secret key by factoring $N = pq$)**

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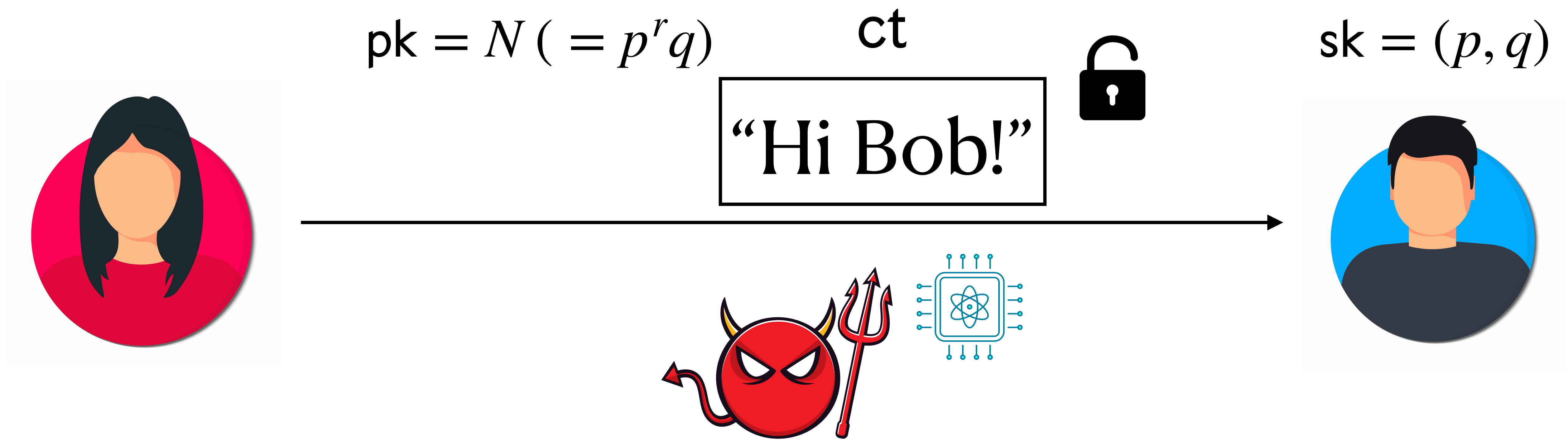
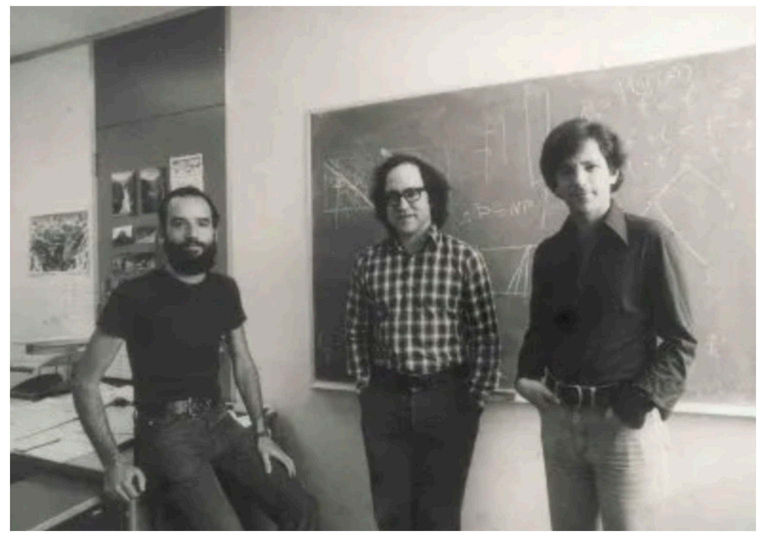
Goal: faster decryption than RSA





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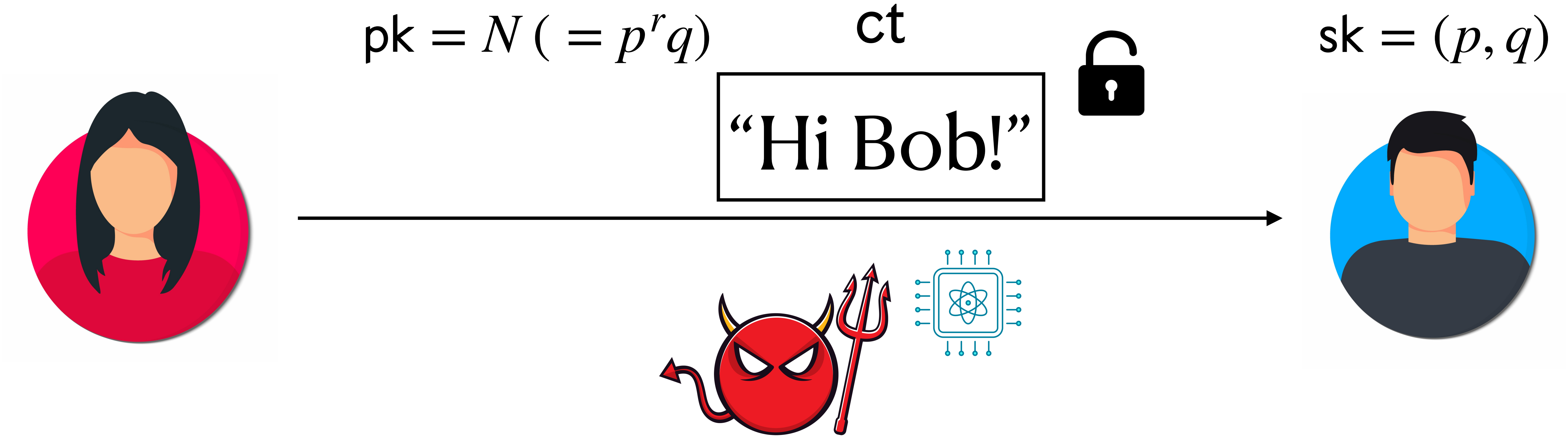

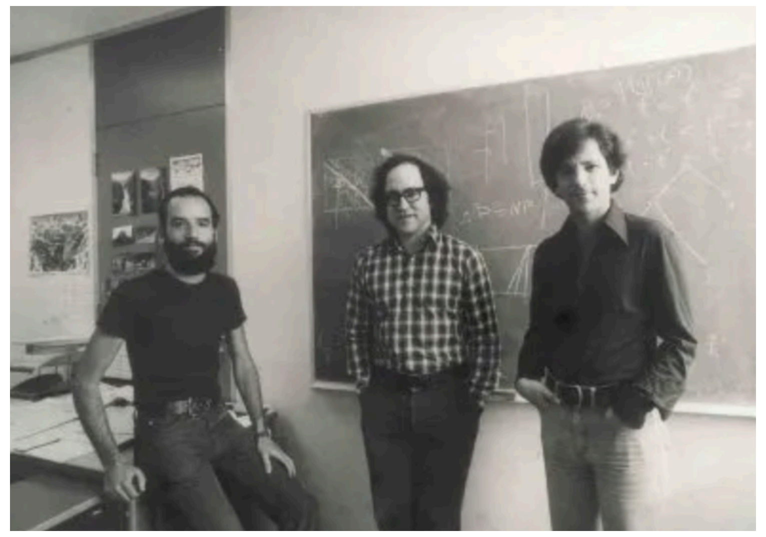

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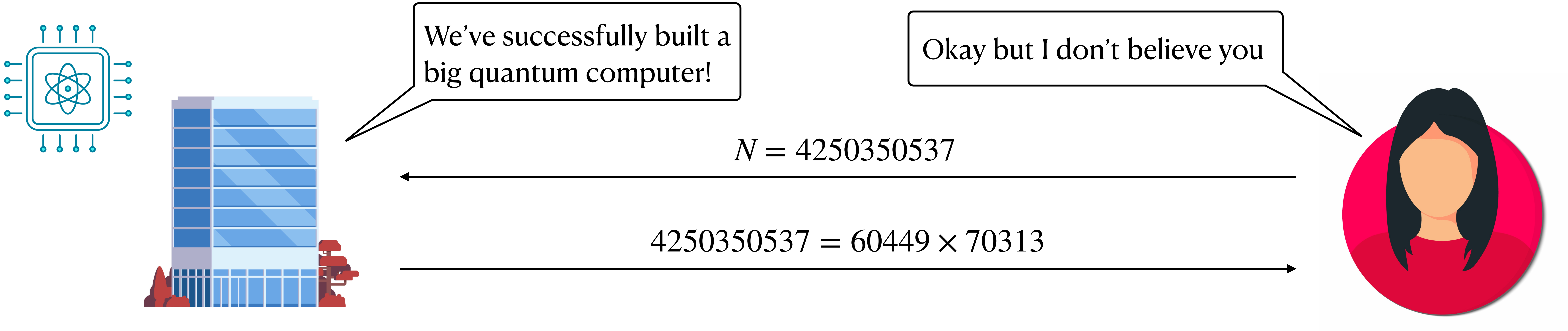
Coming up: even better quantum circuits for factoring $p^r q$ ($r > 1$)

Implication 2: Proofs of Quantumness



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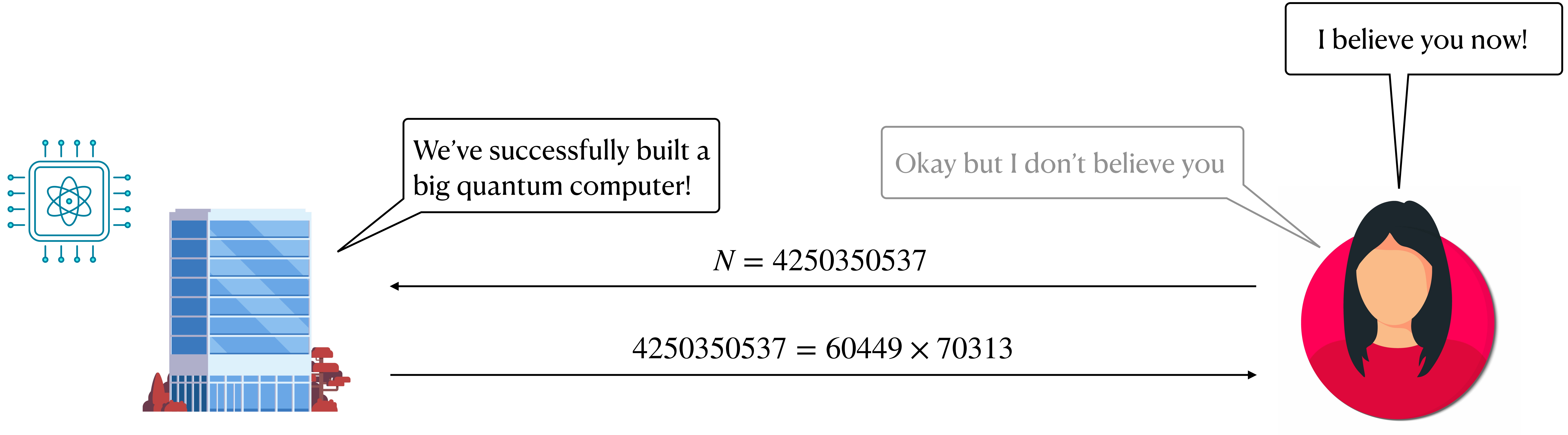

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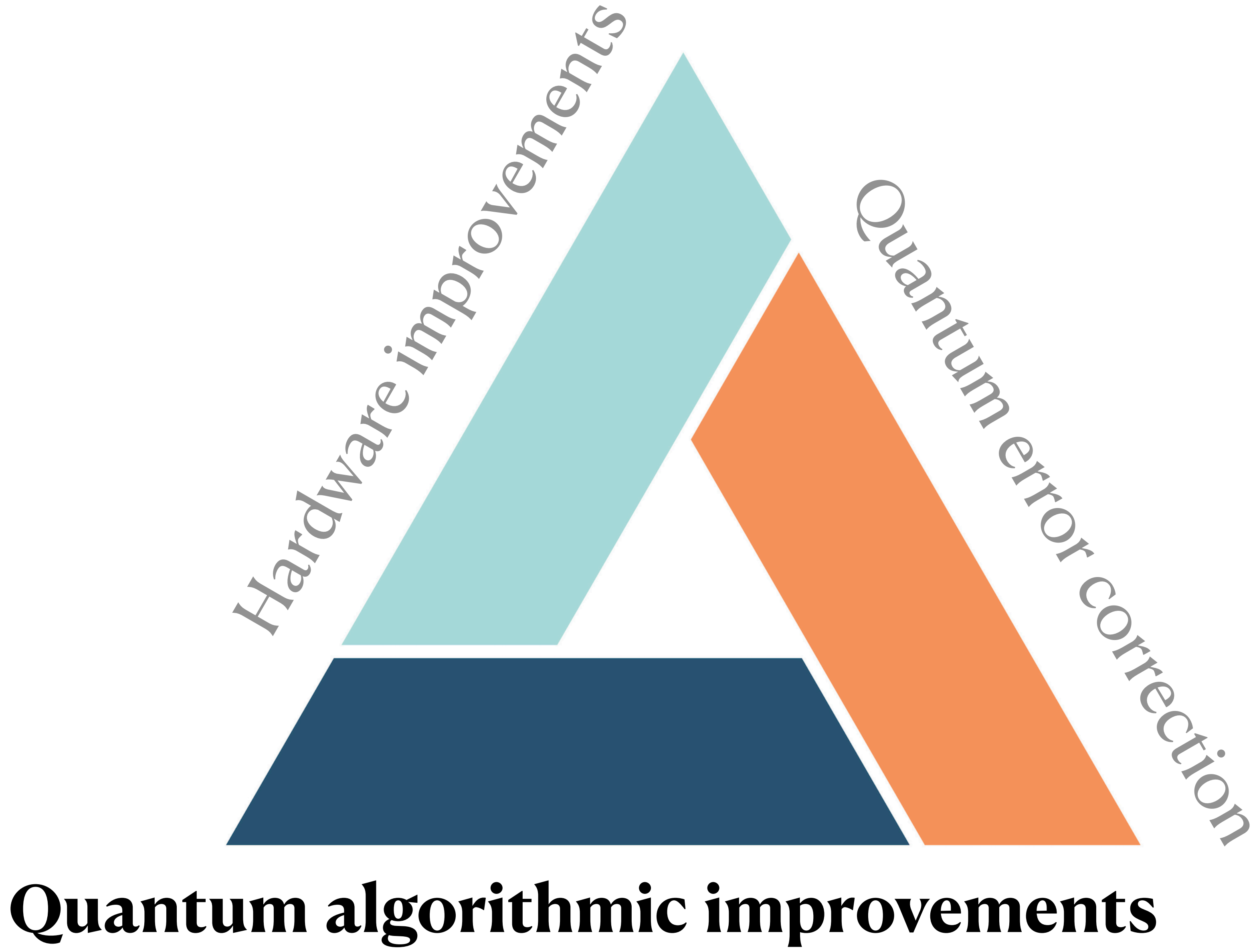

Proofs of Quantumness from Factoring

Authors	Types of inputs	Gates	Space	Depth
Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$	$\tilde{O}(n)$

n is the number of bits in the input N

$\tilde{O}(\cdot)$ hides constant and poly$(\log n)$ factors

All results in this talk are using fast integer multiplication (multiply n-bit integers in $\tilde{O}(n)$ time)

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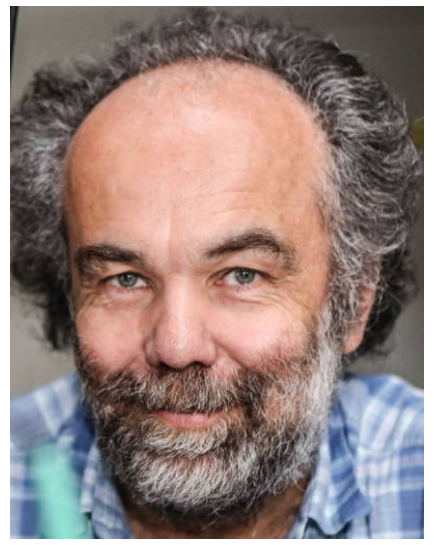

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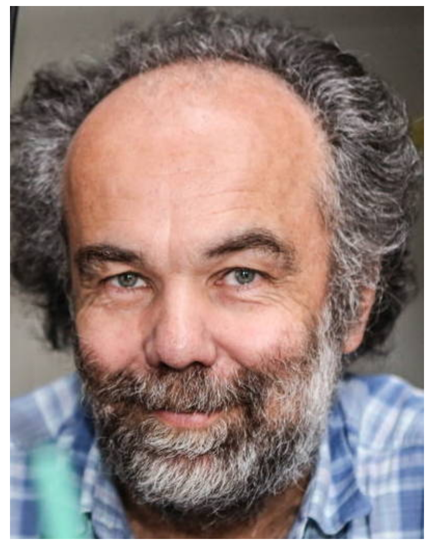
Any input: would break RSA cryptography, and suffice as a proof of quantumness

$N = P^2Q$: only suffices as a proof of quantumness



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- Con: quantum prover needs to store state between rounds




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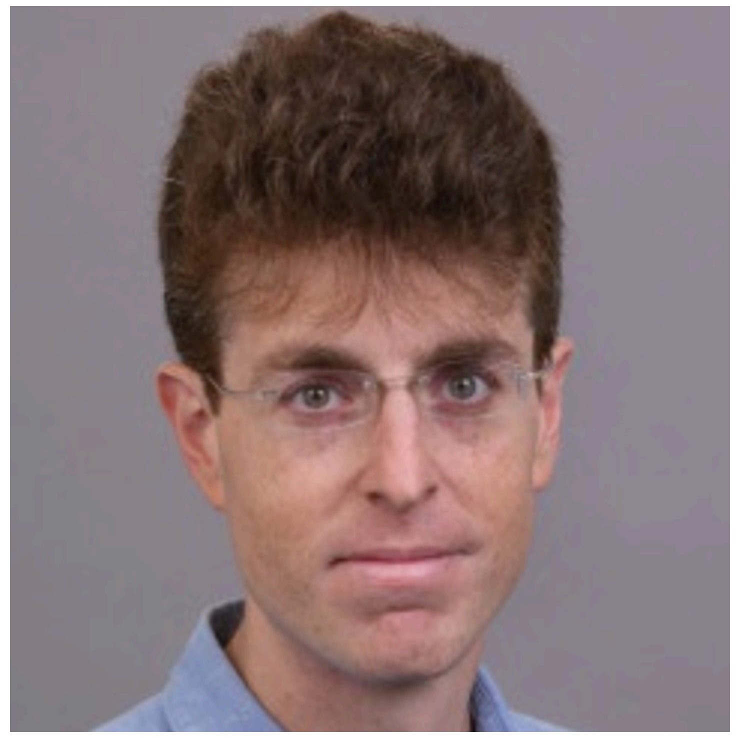
Regev (2023)	Any	$\tilde{O}(n^{1.5})$	$O(n^{1.5})$	$\tilde{O}(n^{0.5})$



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K**R**VV (2024)	$N = P^2Q\ (Q < 2^m)$	$\tilde{O}(n)$	$\tilde{O}(m)$	$\tilde{O}(n/m + m)$

This work: LPDS with space and depth proportional to $\log Q$ **rather than** $\log N$




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- Sweet spot: $m = \tilde{O}(n^{2/3}) \rightarrow$ gates $\tilde{O}(n)$, space $\tilde{O}(n^{2/3})$, depth $\tilde{O}(n^{2/3})$

Our Result

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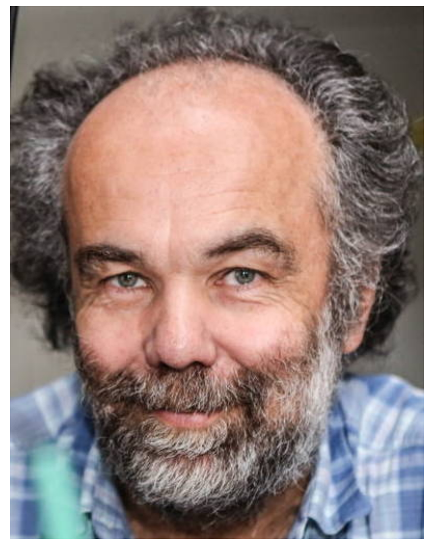
An algorithm that factors special-form integers (that are still classically as hard as RSA integers to factor) in <u>sublinear</u> space and depth!




Factoring P^2Q with LPDS12: A Sketch



Preliminary: Quantum Period Finding

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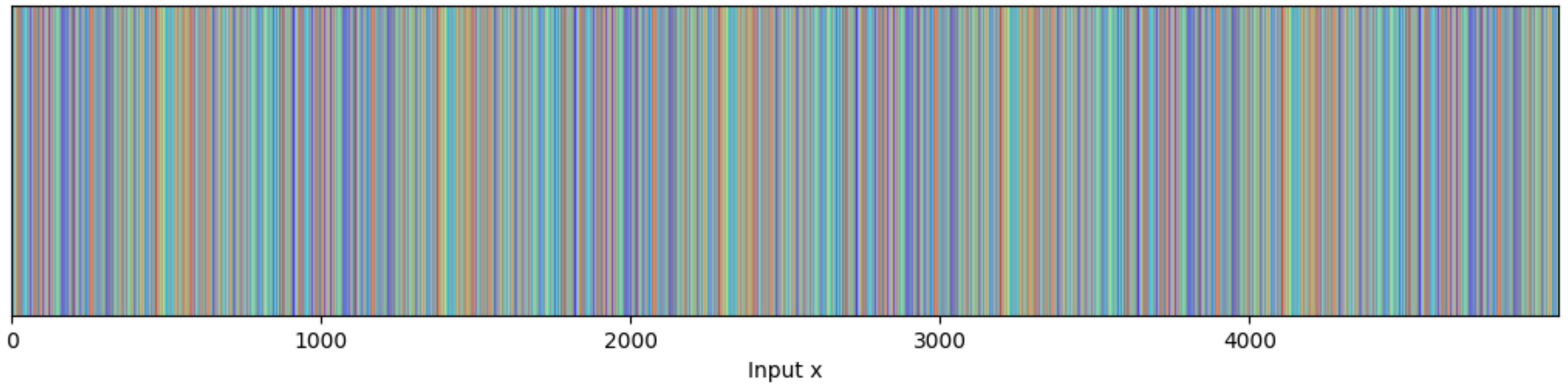

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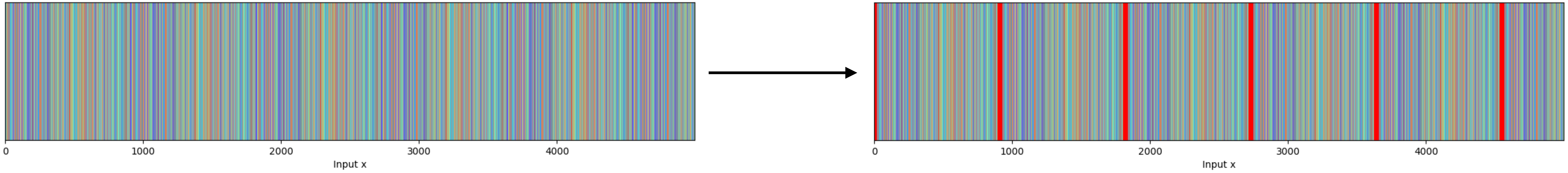
- Informal theorem statement: can quantumly recover a **uniformly random** multiple of $1/T$ (and hence T itself) using essentially only the gates/space needed to compute $f(x)$ for $|x| \leq \mathsf{poly}(T)$



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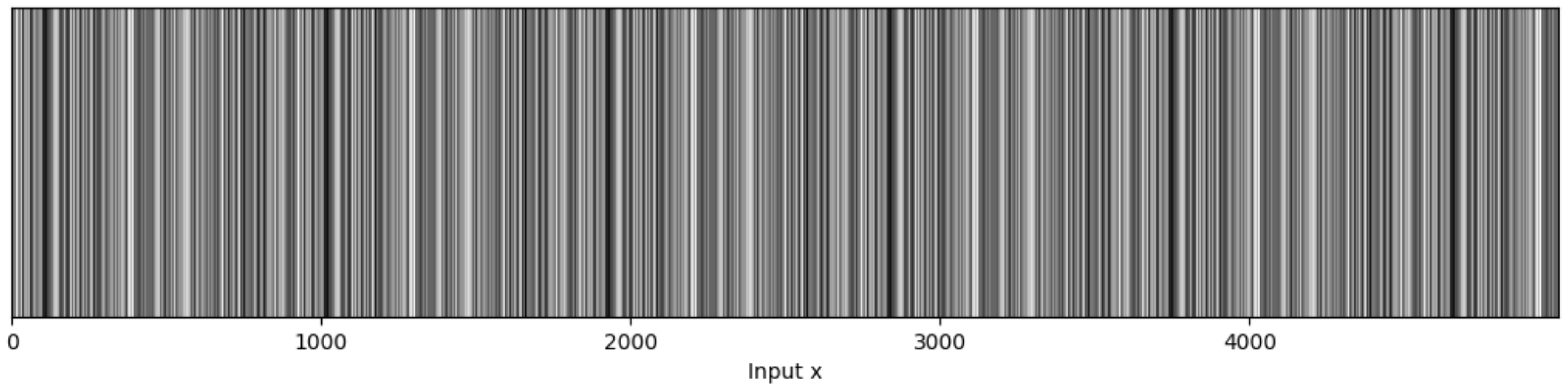

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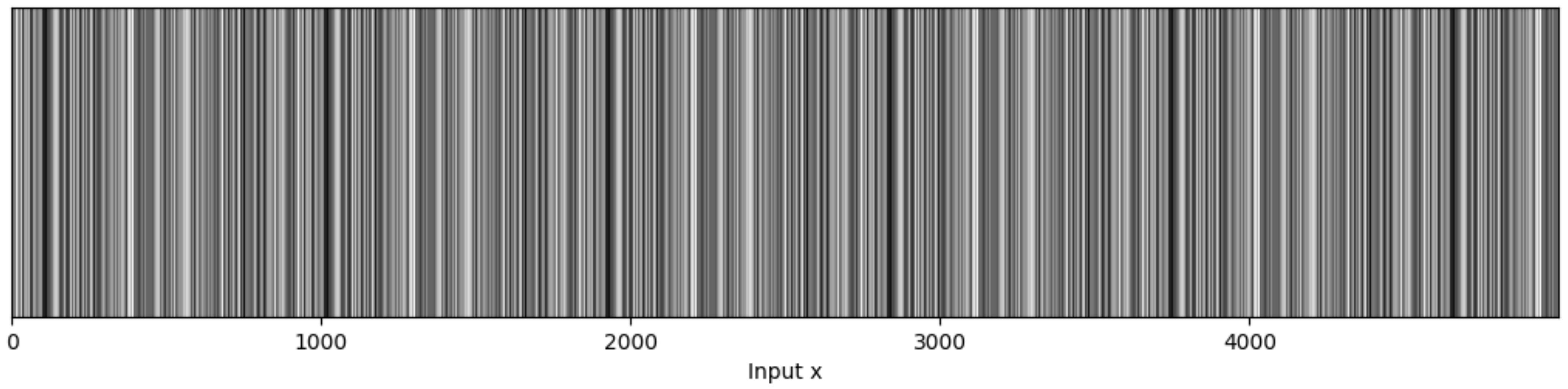

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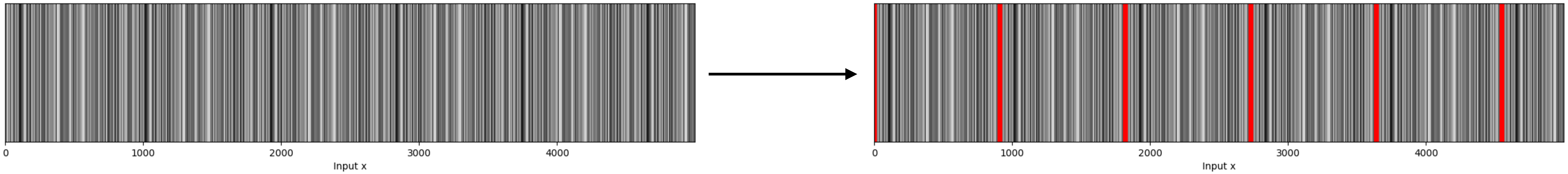
- Informal theorem statement: for "reasonable" f, this is still sufficient to recover T



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Hales-Hallgren '98, May-Schlieper '22

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- Even better, we can recover T from one sample if $\hat{f}$ concentrates on values a such that $\gcd(a, T) = 1$

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- Legendre symbol essentially indicates whether this is the case:

$$\left(\frac{a}{p}\right) = \begin{cases} 1, \text{ if } a \text{ is a nonzero quadratic residue modulo } p\text{; and} \\ -1, \text{ if } a \text{ is not a quadratic residue modulo } p\text{; and} \\ 0, \text{ if } a \text{ divisible by } p\,. \end{cases}$$

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  - Could have $\left(\frac{a}{N}\right) = 1$ without a being a quadratic residue modulo N

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- Useful property: $a \equiv b \pmod{N} \Rightarrow \left(\frac{a}{N}\right) = \left(\frac{b}{N}\right)$
- Theorem (from Euclid to Schönhage 1971): can compute $\left(\frac{a}{N}\right)$ efficiently without knowing the factorisation of N — in fact, in time $\tilde{O}(\log N)$

Algorithms Computing the Jacobi Symbol

ft. Euclid, 2000 years ago

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Jacobi Properties

- Periodicity: $\left(\frac{a}{b}\right) = \left(\frac{a \bmod b}{b}\right)$
- Reciprocity:* $\left(\frac{a}{b}\right) = (-1)^{f(a,b)}\left(\frac{b}{a}\right)$

  for a very simple f

* modulo minor technical caveats; requires a, b odd

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$$f(a,b) = \begin{cases} 0, \text{ if } a \equiv 1 \pmod 4 \text{ or } b \equiv 1 \pmod 4 \\ 1, \text{ if } a \equiv b \equiv 3 \pmod 4 \end{cases}$$

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<u>Greatest Common Divisor (GCD) Properties</u>

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Extended Euclidean algorithm solves both these problems!

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- Extended Euclidean recursion: if $a < b$, swap a, b. Else, update $a \leftarrow a \bmod b$.
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Acta Informatica 1, 139—144 (1971)

Schnelle Berechnung von Kettenbruchentwicklungen

A. Schönhage

Eingegangen am 16. September 1970

Summary. A method, given by D. E. Knuth for the computation of the greatest common divisor of two integers u, v and of the continued fraction for u/v is modified in such a way that only $O(n(\lg n)^2(\lg\lg n))$ elementary steps are used for $u, v < 2^n$.

Zusammenfassung. Ein von D. E. Knuth angegebenes Verfahren, für ganze Zahlen u, v den größten gemeinsamen Teiler und den Kettenbruch für u/v zu berechnen, wird so modifiziert, daß für n-stellige Zahlen nur $O(n(\lg n)^2(\lg\lg n))$ elementare Schritte gebraucht werden.

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A Unified Approach to HGCD Algorithms for polynomials and integers

Klaus Thull and Chee K. Yap*

Freie Universität Berlin
Fachbereich Mathematik
Arnimallee 2-6
D-1000 Berlin 33
West Germany

March, 1990

Abstract

We present a unified framework for the asymptotically fast Half-GCD (HGCD) algorithms, based on properties of the norm. Two other benefits of our approach are (a) a simplified correctness proof of the polynomial HGCD algorithm and (b) the first explicit integer HGCD algorithm. The integer HGCD algorithm turns out to be rather intricate.

Keywords: Integer GCD, Euclidean algorithm, Polynomial GCD, Half GCD algorithm, efficient algorithm.

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ON SCHÖNHAGE'S ALGORITHM AND SUBQUADRATIC INTEGER GCD COMPUTATION

NIELS MÖLLER

Abstract. We describe a new subquadratic left-to-right GCD algorithm, inspired by Schönhage's algorithm for reduction of binary quadratic forms, and compare it to the first subquadratic GCD algorithm discovered by Knuth and Schönhage, and to the binary recursive GCD algorithm of Stehlé and Zimmermann. The new GCD algorithm runs slightly faster than earlier algorithms, and it is much simpler to implement. The key idea is to use a stop condition for HGCD that is based not on the size of the remainders, but on the size of the next difference. This subtle change is sufficient to eliminate the back-up steps that are necessary in all previous subquadratic left-to-right GCD algorithms. The subquadratic GCD algorithms all have the same asymptotic running time, $O(n(\log n)^2 \log\log n)$.

Factoring from Jacobi Symbol Periodicity

- For RSA integers ($N = PQ$): product of two periodic functions with smaller periods but itself only has period N

$$\left(\frac{a}{N}\right) = \left(\frac{a}{P}\right)\left(\frac{a}{Q}\right)$$

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- What about $N = P^2Q$?

$$\left(\frac{a}{N}\right) = \left(\frac{a}{P}\right)^2\left(\frac{a}{Q}\right) = \left(\frac{a}{Q}\right)$$, which is periodic* with period Q!

* modulo minor technical caveats; could have $\left(\frac{a}{P}\right) = 0$ for a tiny fraction of inputs a

Quantumly Factoring $N = P^2Q$

Li, Peng, Du, Suter (2012)

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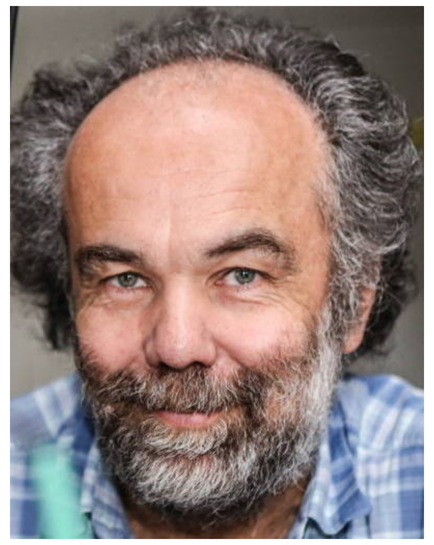
- So quantum period finding → recover Q (and hence P)



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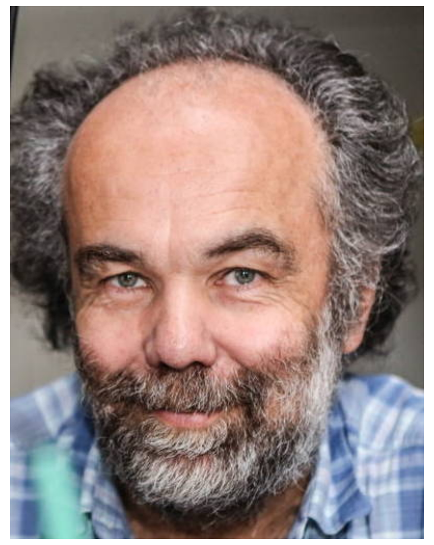

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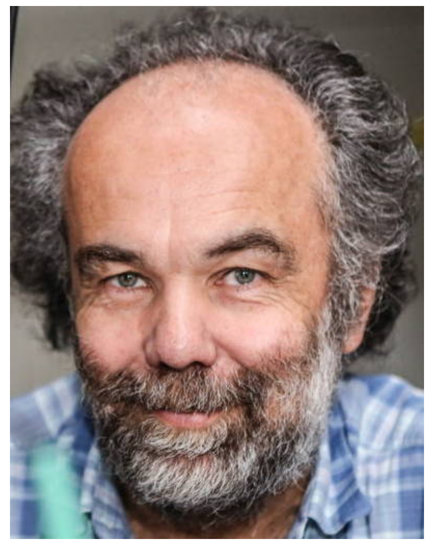
- Space and depth (if naively implemented): also $\tilde{O}(\log N)$



Is the Jacobi Function "Reasonable"?

- Generalised quantum period finding: we recover $\frac{a}{Q}$ with probability

$$\left|\hat{f}(a)\right|^2 = \frac{1}{Q}\left|\sum_{x=0}^{Q-1}\left(\frac{x}{Q}\right)\cdot\exp\left(\frac{2\pi i a x}{Q}\right)\right|^2$$

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The Jacobi function isn't just "reasonably good" for general quantum period finding, it's actually magically well-suited to it — even more so than the periodic function used by Shor to factor!

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Our Contribution: Pushing Space and Depth Down to $\tilde{O}(\log Q)$

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- **<u>Hope 1:</u>** when factoring P^2Q with Jacobi: the period is just $Q \rightarrow O(\log Q)$ qubits could suffice!

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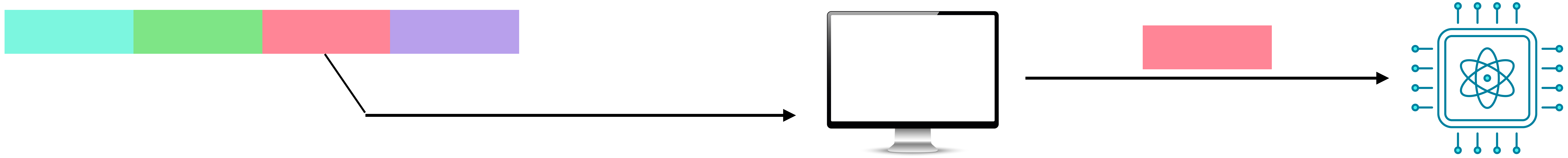

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The "only" bottleneck: computing $|a\rangle \mapsto |a\rangle|N \bmod a\rangle$

Our Result, Distilled

- <u>Theorem (K**R**VV24):</u> for quantum a and classically known N, we can compute

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in $\tilde{O}(\log N)$ gates (near-linear) and $\tilde{O}(\log a)$ qubits (enough qubits to write down a)

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Open question: other applications of these results?

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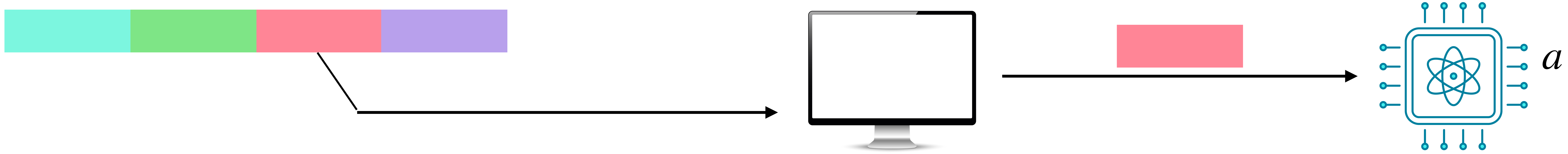
- **Corollary 2: we can factor $N = P^2Q$ in $\tilde{O}(\log N)$ gates and $\tilde{O}(\log Q)$ qubits**
  - **Just need the above theorem for $a \leq \mathsf{poly}(Q)$**

Computing $N \bmod a$ with Quantum Streaming

Notation: N has n bits, a has $m = O(\log Q)$ bits

Bits of N, split into chunks of size $O(\log Q)$

Quantum computer with $\tilde{O}(\log Q)$ qubits



Classical computer sending instructions to the quantum computer

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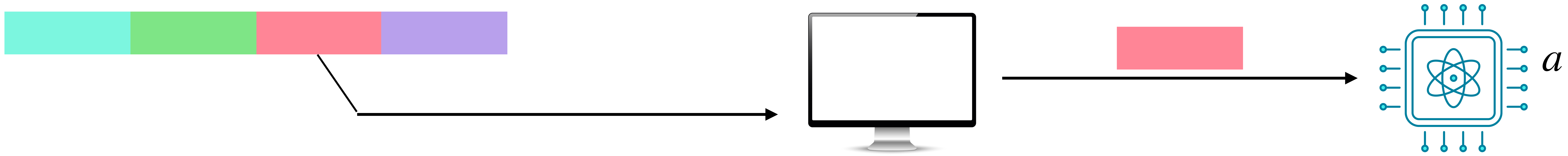

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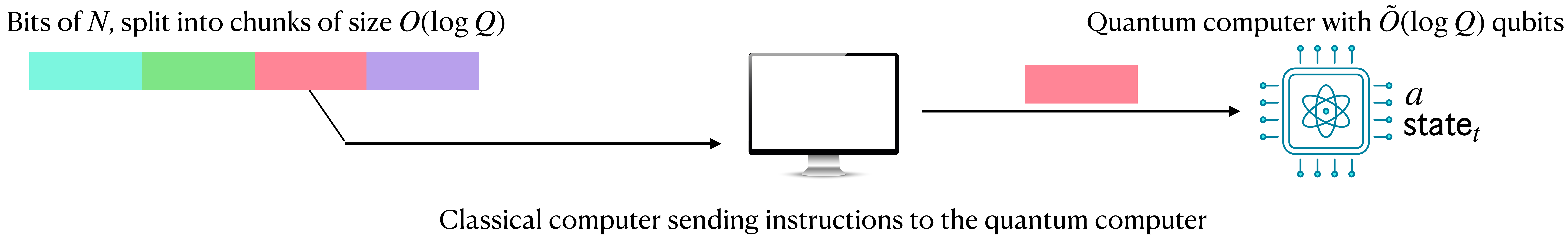

Computing N mod a with Quantum Streaming

- Proceeds in $t_{\max} = O(n/m)$ time steps (one for each m-bit chunk of N)
  - After time step t: quantum computer has some state_t
- Desiderata:
  - Correctness: N mod a is efficiently recoverable from $\mathsf{state}_{t_{\max}}$

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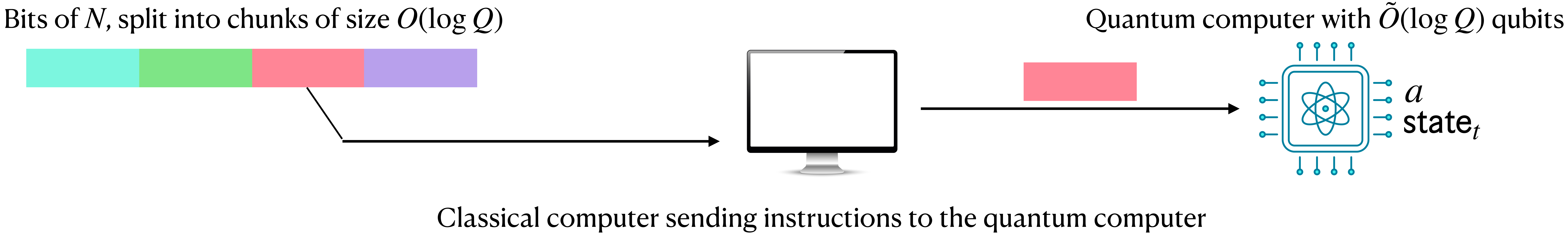

Computing N mod a with Quantum Streaming

- Proceeds in $t_{\max} = O(n/m)$ time steps (one for each m-bit chunk of N)
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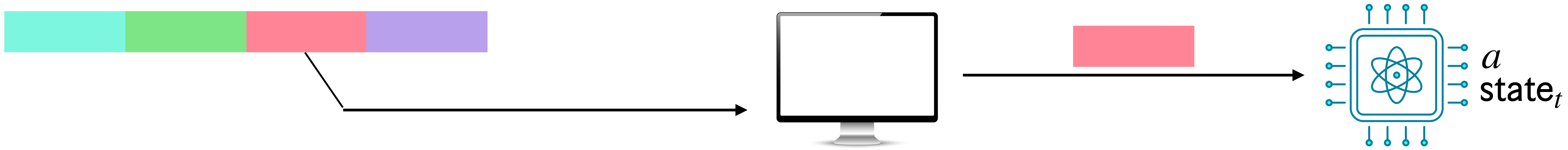
Quantum computer with $\tilde{O}(\log Q)$ qubits



Classical computer sending instructions to the quantum computer

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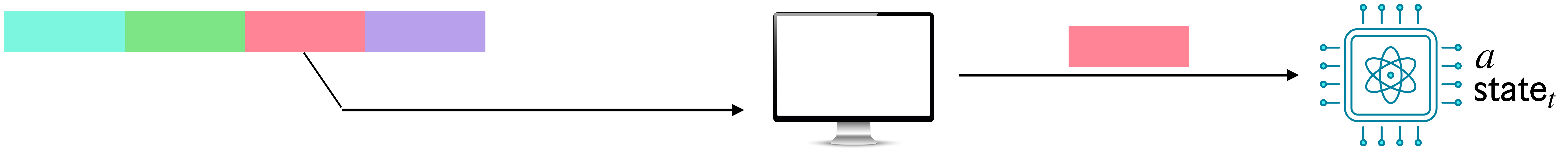

Classical computer sending instructions to the quantum computer

Our Construction, Simplified

- At time $t = 0, \ldots, n - m$, let N_t be a multiple of a such that $N \equiv N_t \pmod{2^t}$
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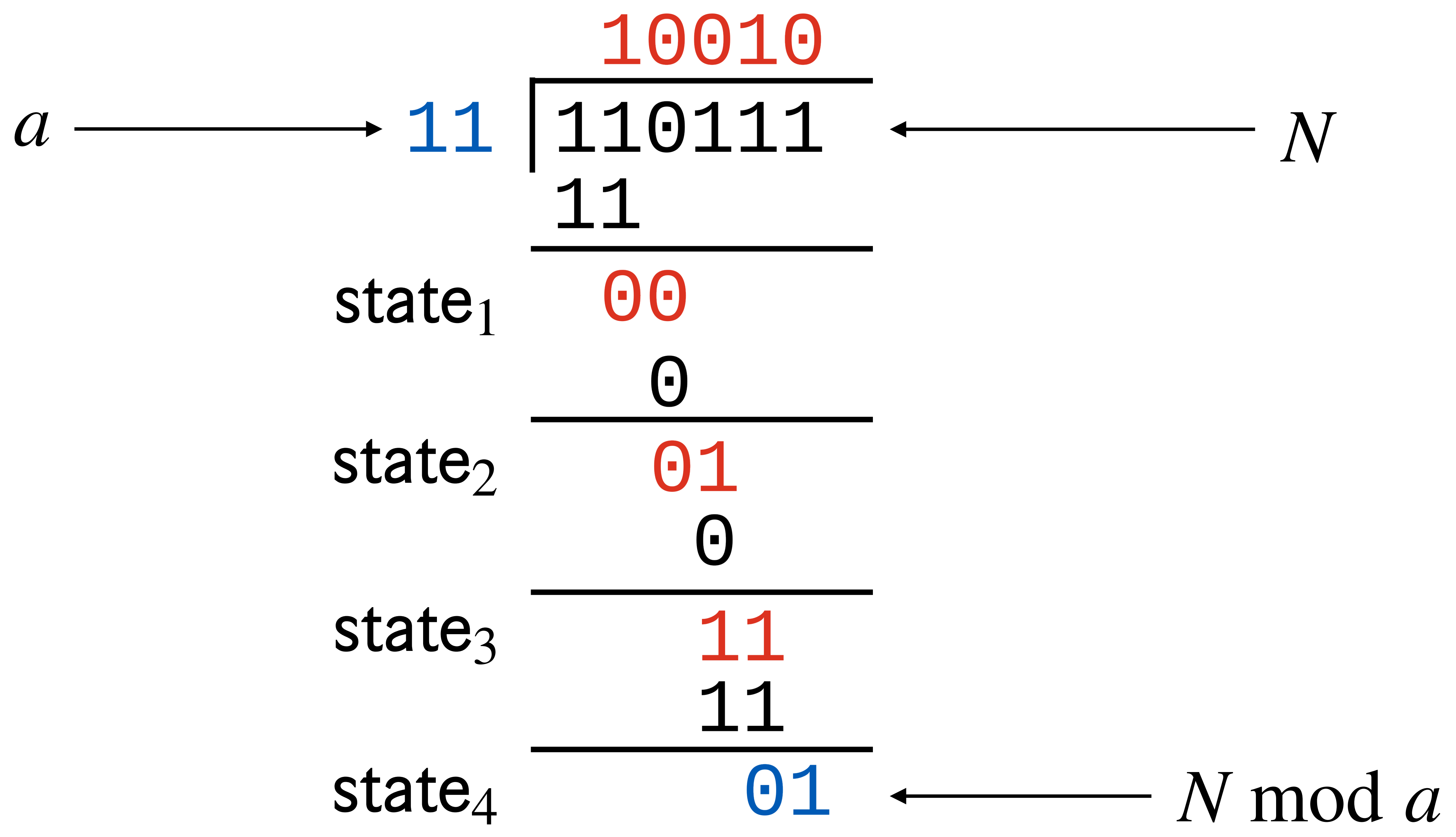
- It turns out that the final state state_{n-m} suffices to reconstruct $N \bmod a$

Our Construction, In Detail

A Natural Attempt: Long Division



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Excess state that we cannot clean up

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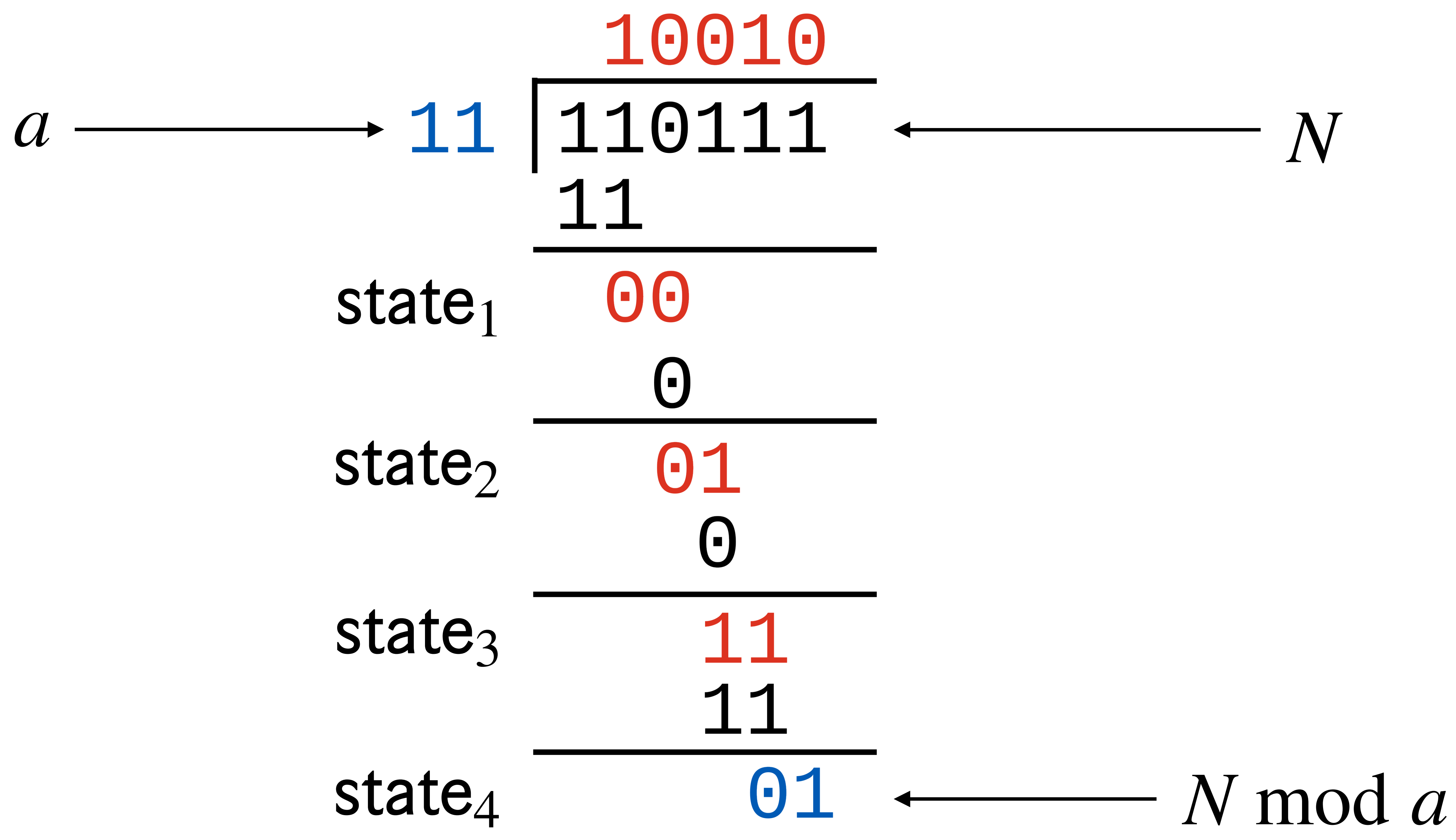

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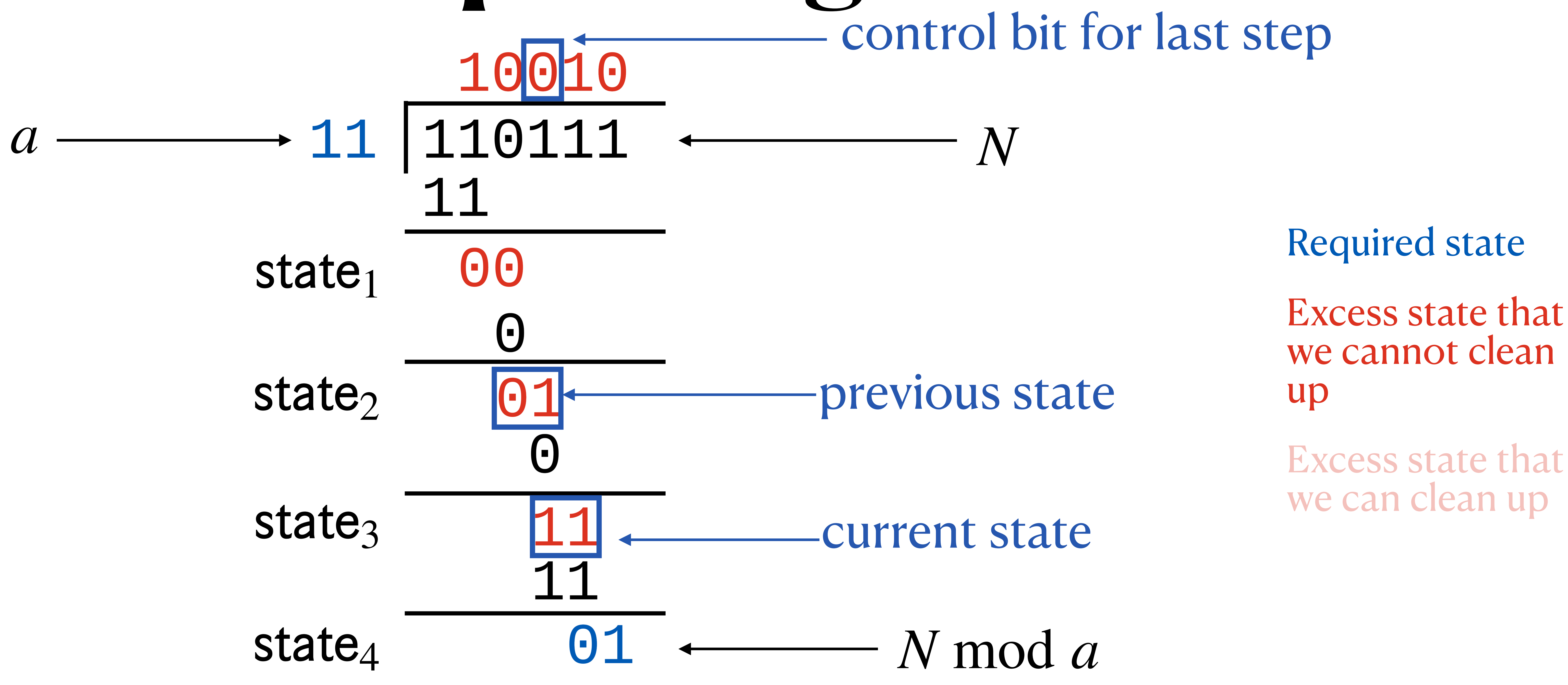
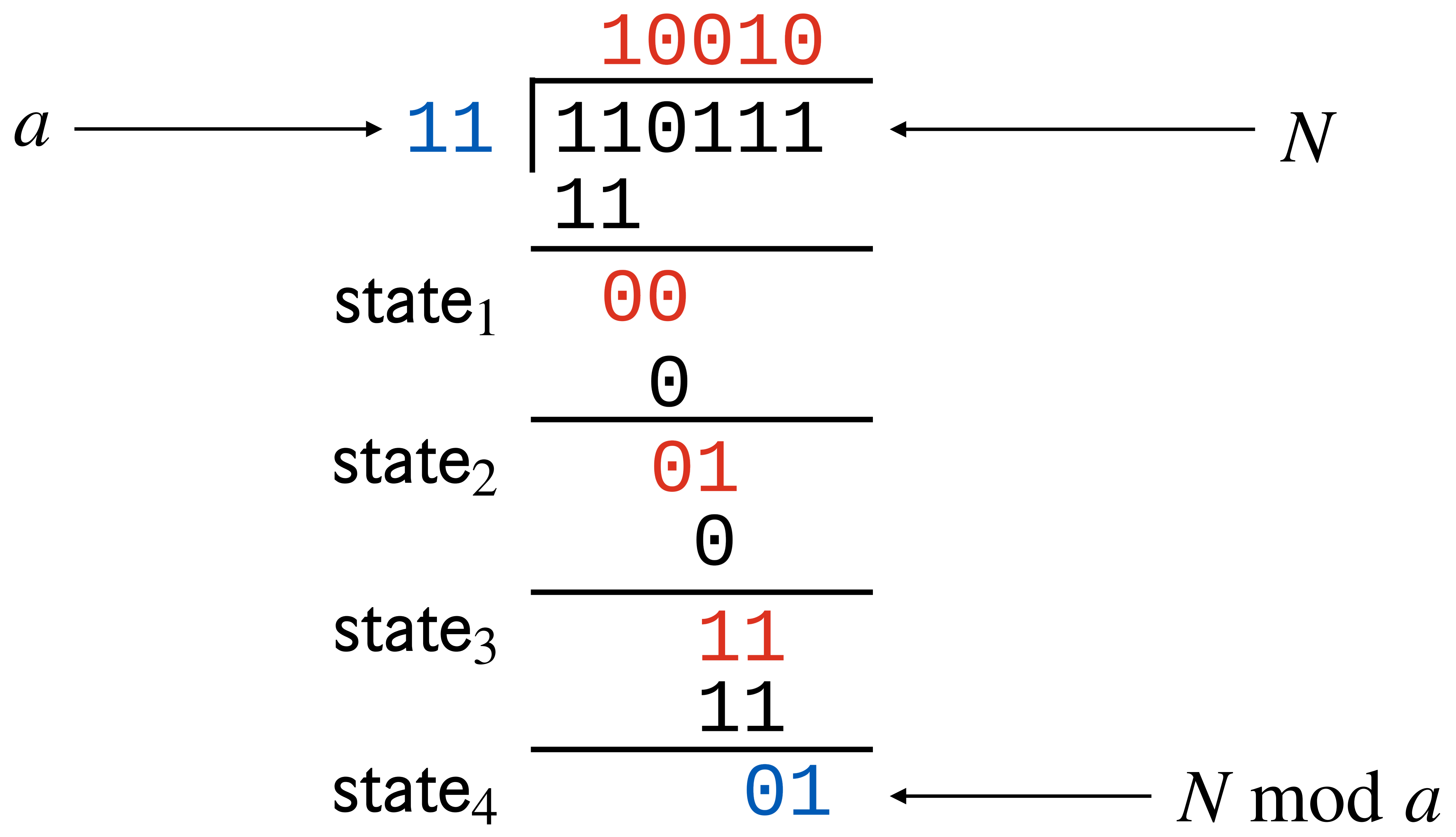


- Pro: only need to look at $O(m)$ bits of N at a time, and each state_t is compact
- Con: no reversibility → end up using $O(n)$ qubits anyway

Long Division: A Bird's Eye View



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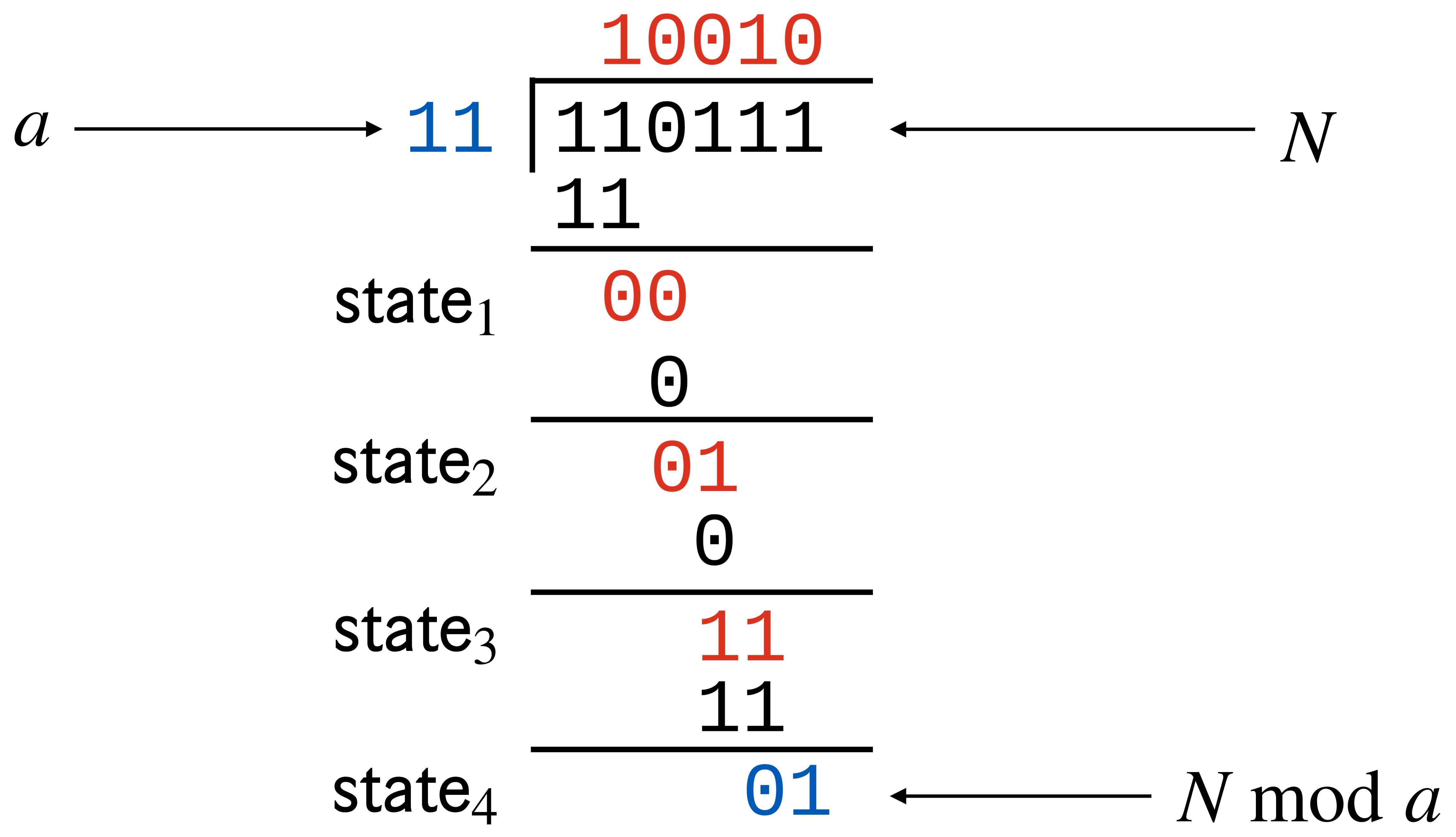

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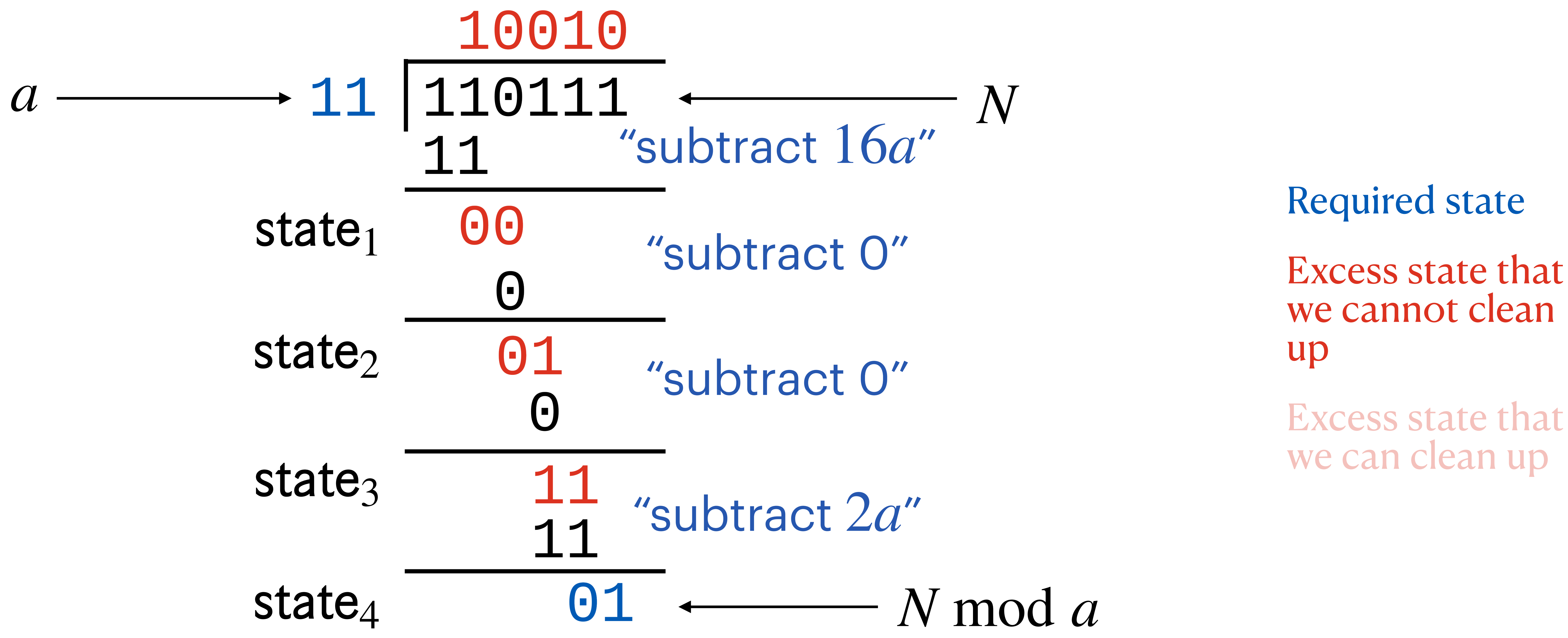

- Goal of long division: find k such that $N \approx ka$, and output $N - ka$
- Does this by subtracting $2^r a$ from state, starting with large r (most significant bit of k) and going down to $r = 0$ (least significant bit of k)

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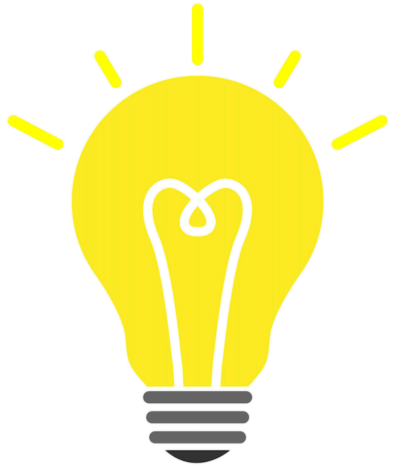

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    - Equivalently: $N \equiv ka \pmod{2^{n-m}}$

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Tracing Through Our Algorithm

Recall:

state $= ka$

```
2^0 a =      11
    N =  110111
    k =       1
state =      11
```

Tracing Through Our Algorithm

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2^0 a =	11	2^1 a =	11
N =	110111	N =	110111
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This now gets us down to $O(m)$ space!

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This is already nice, but we can do better…

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   a. Recursion step just does some multiplications and bit shifts on m-bit integers

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  - Gates: $O(n/m) \times \tilde{O}(m) = \tilde{O}(n)$
  - Depth: $O(n/m) \times \tilde{O}(1) = \tilde{O}(n/m)$

Summary and Open Questions

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   - Contributes an additional depth of $\tilde{O}(m)$ (dominant term in depth if $m \gg \sqrt{n}$)

Summary: Our Factoring Circuit for P^2Q

- Main theorem (K**R**VV24): for $N = P^2Q < 2^n$ such that Q is squarefree and $< 2^m$, we can recover P, Q from N in $\tilde{O}(n)$ gates, $\tilde{O}(m)$ qubits, and $\tilde{O}(n/m + m)$ depth

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Central workhorse: an efficient quantum circuit for computing $N \bmod a$ for classical $N < 2^n$ and quantum $a < 2^m$

Summary: Our Factoring Circuit for P^2Q

Authors	Types of inputs	Gates	Space	Depth
Shor (1994)	Any	$\tilde{O}(n^2)$	$\tilde{O}(n)$	$\tilde{O}(n)$
LPDS (2012)	$N = P^2Q$	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(n)$
KCVY (2021)	N/A	$\tilde{O}(n)$	$\tilde{O}(n)$	$\tilde{O}(1)$
Regev (2023)	Any	$\tilde{O}(n^{1.5})$	$O(n^{1.5})$	$\tilde{O}(n^{0.5})$
RV (2024)	Any	$\tilde{O}(n^{1.5})$	$\tilde{O}(n)$	$\tilde{O}(n^{0.5})$
K**R**VV (2024)	$N = P^2Q\ (Q < 2^m)$	$\tilde{O}(n)$	$\tilde{O}(m)$	$\tilde{O}(n/m + m)$

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- Other quantum factoring algorithms exploiting special structure in N? (Many such algorithms in the classical world)

Bonus Slides

Our Result, Distilled

- Theorem (K**R**VV24): for $a < 2^m$ (in our application, $m = O(\log Q)$) and classically known $N < 2^n$, we can compute

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Open question: other applications of these results?

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simplified presentation based on an observation by Daniel J. Bernstein

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That's it! We computed $N \bmod a$!

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