How can artificial intelligence help mathematicians?

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CERMICS - Ecole des Ponts - Institut Polytechnique de Paris

AI for Mathematics and Theoretical Computer Science

–

April, 9th

–

Simons Institute and SLMath

Introduction

Social networks

Emails and apps

Virtual assistants

Platforms

Introduction

ChatGPT

MISTRAL AI_

Claude

Gemini

Introduction

From computer scientists awareness...

- Convolutional Neural Networks (LeNet-5, 1998,..., ResNet, 2015)
- Generative adversarial networks (2014)
- Transformer (2017)
- AlphaZero (2018)

... to global awareness

- ChatGPT (followed by Llama, Mistral, etc.)
- Diffusion model (Dall·E, Midjourney, Stable Diffusion)
- 2024 Nobel Prizes in Physics **and** in Chemistry

Introduction

An idea that started taking roots:

How can AI help mathematicians ?

Basic usage:

- asking a math question to an LLM / asking to prove a small lemma

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Basic usage:

- asking a math question to an LLM / asking to prove a small lemma
  - terrible answers (2022-2023)
  - but models are evolving quickly

Introduction

An idea that started taking roots:

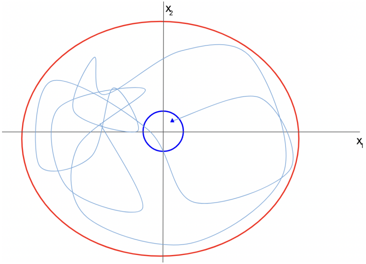
How can AI help mathematicians ?

Basic usages:

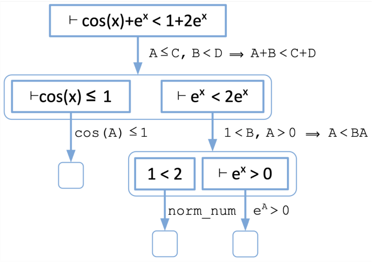
- Finding citations (information retrieval)
- Learning about a new field; explaining a paper (in-context or retrieval)
- Reviewing process (!)

AI for mathematics $\neq$ generic LLM

Outline of the talk



1. AI tools for math discovery



2. When will AI prove theorems?

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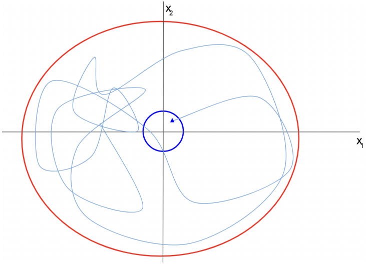

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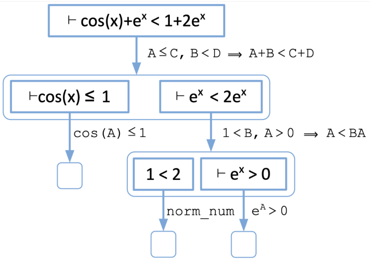

2. When will AI prove theorems?

Solving maths problems with a computer

Solving maths problems with a computer

Conjecture (Euler, 1769)

If there exist integers a_1, a_2,...,a_k, b, and n such that

$$a_1^n + a_2^n + \ldots + a_k^n = b^n,$$

then $k \geq n$.

A problem open for almost 200 years

Solving maths problems with a computer

Lander and Parkin (1966)

Solving maths problems with a computer

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$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

Solving maths problems with a computer

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least n nth powers are required to sum to an nth power, $n > 2$.

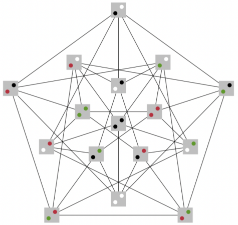
REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

Solving maths problems with a computer

Proof of Keller's Conjecture (Brakensiek, Heule, Mackey, Narváez, 2019)

- A proof with many "simple cases" to check
- Many = far too many for a human



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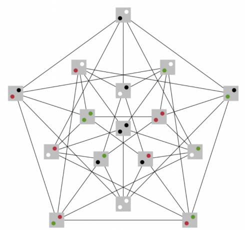

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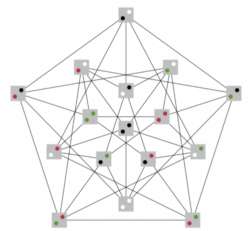

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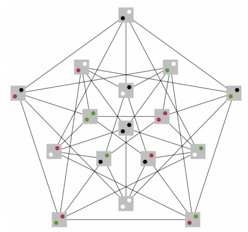
Several impressive results using SAT solvers (see the talk of Giles Gardam and Bernardo Subercaseaux)



Conclusion: Computers have been used to prove theorems for a long time.

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Can AI be useful to solve more complicated problems?

Problems where the difficulty is not just a high number of case-checking?

AI in Mathematics Today

Three examples

- Stability of dynamical systems
- Control theory
- Combinatorics

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Discovering Lyapunov function
(Alfarano, Charton, H.)

$$\begin{cases} \dot{x}_0 = -7x_0^5 - 4x_0^3x_1^2 - 5x_0^3 \\ \dot{x}_1 = 7x_0^4 - 3x_1 - 2x_2 \\ \dot{x}_2 = -8x_0^2 - 9x_2 \end{cases}$$

$$V(x) = 2x_0^4 + 2x_0^2x_1^2 + 3x_0^2 + 2x_1^2 + x_2^2$$

(see tutorial by Sean Welleck)

Stability of Dynamical Systems

A system of differential equations

$$\dot{x}(t) = f(x(t)),$$

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Question (System Stability)

Is it true that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial condition satisfies $\|x(0)\| \leq \delta$ then the solution $x(t)$ exists for all $t \in [0, +\infty)$ and

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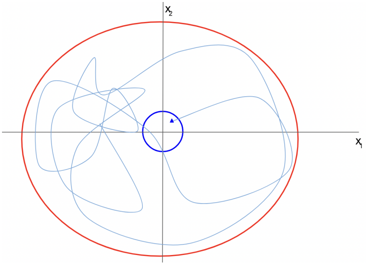
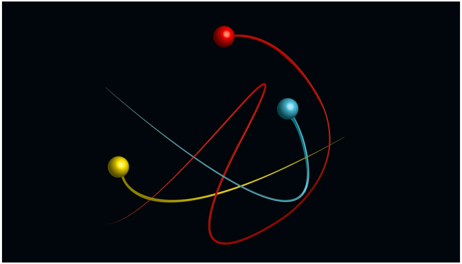


Stability of Dynamical Systems

A problem that has interested mathematicians for over a hundred years.



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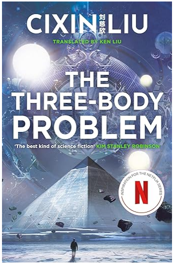


Stability of Dynamical Systems

A significant advancement: Lyapunov functions

Theorem

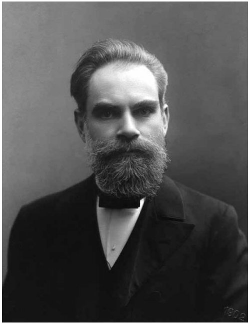
If there exists a function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ *such that for all* $x \in \mathbb{R}^n$

$$V(x) > V(0), \quad \text{and} \quad \nabla V(x) \cdot f(x) \leq 0,$$

and

$$\lim_{\|x\| \to +\infty} V(x) = +\infty,$$

then the system is stable.



A. Lyapunov
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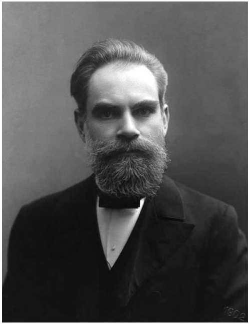

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The system is stable, a Lyapunov function is

$$V(x) = x_1^6 + 2(x_2^6 + x_3^4)$$

Stability of Dynamical Systems

A globally asymptotically stable polynomial vector field with no polynomial Lyapunov function

Publisher: IEEE   Cite This   PDF

Amir Ali Ahmadi ; Miroslav Krstic ; Pablo A. Parrilo   All Authors

39 Cites in Papers

399 Full Text Views

Stability of Dynamical Systems

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there is still no systematic way to construct a Lyapunov function.

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$\longrightarrow$ We resort to intuition

Stability of Dynamical Systems

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Task: guessing Lyapunov functions

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Stability of Dynamical Systems

Train an AI to have an intuition of Lyapunov functions (Alfarano, Charton, A.H., 2024).

Neural network architecture: Transformer ($\sim$1000 smaller than GPT-3)

Procedure:

1. Generate a set of systems and associated Lyapunov functions.
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- Find a mathematical way to get:

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  Then sample at random.

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Limitations: even with a perfect generator, it biases the distribution (and it matters).

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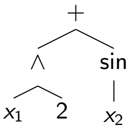
$(x_1^2+\sin(x_2)) \quad \rightarrow$



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Stability of Dynamical Systems

Results

It works! The AI learns a mathematical intuition of Lyapunov functions.

[1] Existing method for some polynomial systems.

Stability of Dynamical Systems

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Type	n equations	SOS algorithms[1]	AI
polynomial (train distrib.)	2-5	15%	**99%**
polynomial (fwd distrib.)	2-3	47%	**84-93%**
Non-polynomial	2-3	$\sim$ 0%	**87%**

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Master students accuracy: $\sim$ 10%

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My accuracy (!): $\sim$ 25%

[2] Existing method for some polynomial systems

Meta says its AI could help mathematicians

Tada Images/Shutterstock

An AI system developed by Meta can find solutions to maths problems that have eluded mathematicians for over a century, researchers at the firm claim.

The problems involve mathematical tools called Lyapunov functions, named after mathematician Aleksandr Lyapunov, which analyse whether a system will remain stable over time, meaning its behaviour can be predicted. One famous example of such a system is the motion of three celestial bodies as a result of their mutual gravitational interactions – describing the behaviour of this "three-body problem" is extremely challenging.

Read more Incredible maths proof...

AIがリアプノフ関数の発見に革新！安定性理論に新たなブレイクスルー（2024-10）【論文解説シリーズ】

Summary of the approach

Paradigm: Training a Transformer to have a mathematical intuition on a problem

Key points:

- Generating data in a backward fashion (solution $\rightarrow$ problem)
- Test out-of-distribution on "real" instances of the problem.

Used in many frameworks: explicit solutions to ODE; local controllability; eigenvalues of random matrices; GCD; equilibrium of bio-networks, etc.

Try it yourself:

`https://github.com/ahayat16/Lyapunov/`

and customize on your favorite math problem

AI in Mathematics today

Three examples

- Stability of dynamical systems
- Control theory
- Combinatorics

Control of a Differential System

Evolution of the mosquito population

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \dfrac{E}{K}\right) - (\nu_E + \delta_E) E, \\ \dot{M} = (1-\nu)\nu_E E - \delta_M M, \\ \dot{F} = \nu \nu_E E \dfrac{M}{M + M_s} - \delta_F F, \\ \dot{M}_s = u - \delta_s M_s, \end{cases}$$

$E(t)$ represents mosquito eggs, $F(t)$ fertilized females, $M(t)$ males, $M_s(t)$ sterile males. u is the flow of sterile mosquitoes that we release. This is what is called *control*.

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$$u = f(M + M_s, F + F_s)$$

with $F_s = FM/M_s$.

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Question

Is it possible to find f such that the system is stable and

$$\lim_{t \to +\infty} \|E(t), M(t), F(t)\| = 0 \text{ and } \lim_{t \to +\infty} \|M_s(t)\| = \varepsilon,$$

with ε as small as desired?

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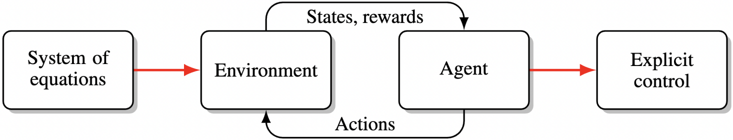
An open question

Control of a Differential System

Principle of the approach (Agbo Bidi, Coron, A.H., Lichtlé, 2023)



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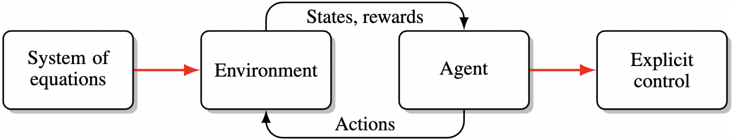
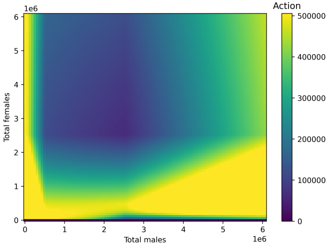


1. **Transform the equations** using a well-chosen numerical scheme.
2. **Train a Reinforcement Learning (RL) model**. The AI trains by trial and error and tries to maximize a well-chosen objective.
3. **Deduce the mathematical control**, from the numerical control.
4. **Verify** that it is a solution to the problem.

Control of a Differential System



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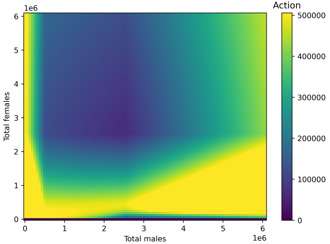
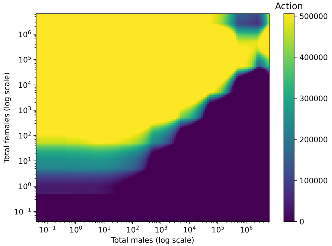


$\xrightarrow{\text{N.Lichtlé}}$



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Control of a Differential System

$$u_{\text{reg}}(M+M_s, F+F_s) = \begin{cases} u_{\text{reg}}^{\text{left}}(M+M_s, F+F_s) & \text{if } M+M_s < M^*, \\ u_{\text{reg}}^{\text{right}}(M+M_s, F+F_s) & \text{otherwise,} \end{cases}$$

Control of a Differential System

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where $I_1(x) = \frac{\log M^*}{\log(F+F_s)}$ and $I_2(x,y) = \frac{\log(M+M_s)}{\log(F+F_s)}$,

Control of a Differential System

Final control

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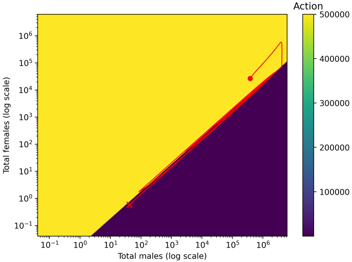

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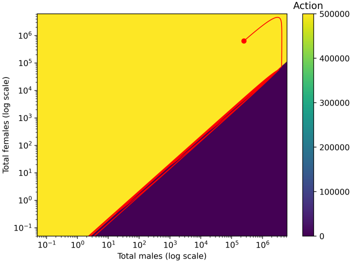

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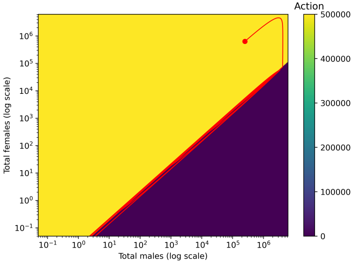

$\varepsilon = 0$

We can see a mathematical bifurcation with our "AI-augmented intuition".

Control of a Differential System

Robustness of the solution

Can the solution still work if there are uncertainties on the parameters ?

$$\begin{cases} \dot{E} = \beta_E F \left(1 - \dfrac{E}{K}\right) - (\nu_E + \delta_E)E, \\ \dot{M} = (1-\nu)\nu_E E - \delta_M M, \\ \dot{F} = \nu \nu_E E \dfrac{M}{M + M_s} - \delta_F F, \\ \dot{M}_s = u - \delta_s M_s, \end{cases}$$

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$$\begin{aligned} \hat{\beta}_E &\sim \mathcal{U}(0.65, 9.03) \\ \hat{\nu}_E &\sim \mathcal{U}(0.01, 0.35) \\ \hat{\delta}_E &\sim \mathcal{U}(0.03, 1.0) \\ \hat{\delta}_F &\sim \mathcal{U}(0.03, 0.1) \\ \hat{\delta}_M &\sim \mathcal{U}(0.09, 0.2) \\ \hat{\nu} &\sim \mathcal{U}(0.39, 1.0) \end{aligned}$$

Yes

AI tools for math discovery

Three examples

- Stability of dynamical systems
- Control theory
- Combinatorics

PatternBoost: finding interesting constructions

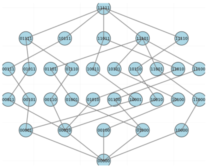
Paradigm: couple a classical algorithm and a transformer. [Charton, Ellenberg, Wagner, Williamson, 2024]

Step 1: a classical algorithm generate "good" constructions from random seeds

Step 2: train a transformer on the best such constructions

Step 3: use the transformer as seeds.

Repeat.



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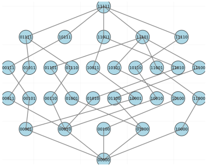
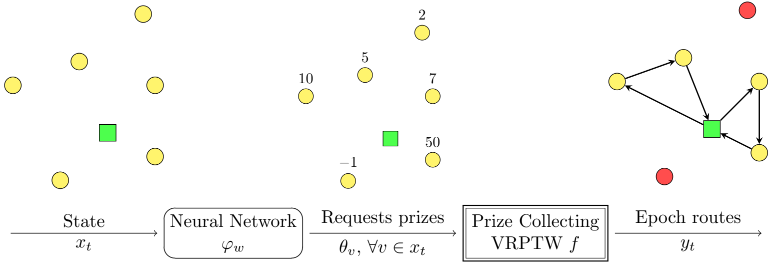


In particular, closed a 30-years conjecture.

Combinatorial Optimization Augmented Machine Learning

Paradigm: couple a combinatorial optimization algorithm and a neural network and train the ensemble [Baty, Jungel, Klein, Parmentier, Schiffer, 2024].



Perform better than existing mathematical algorithm and machine learning algorithm

(winner of EURO NeurIPS challenge 2022)

Many other examples

- in topology feedforward and MPNNs to guess links between different mathematical quantities, in particular hyperbolic and algebraic invariant of knots (a conjecture that was later proved) [Davies et al., 2021]
- in partial differential equations PINNs to find an exact self-similar solution to 3D Euler [Wang, Lai, Gómez-Serrano, Buckmaster, 2023]
- in group theory path-finding for large graphs with ResMLP and RL [Chervov et al. 2025]
- ... and in many others fields

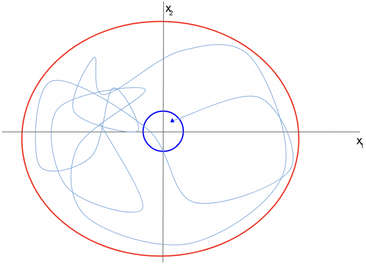
AI tools for math discovery

- AI are already useful in the practice of mathematics and can help solve difficult problems.
- AI are trained to have better intuition than humans on a specific problem.
- This augmented intuition allows us to bypass the difficulty of the problem.

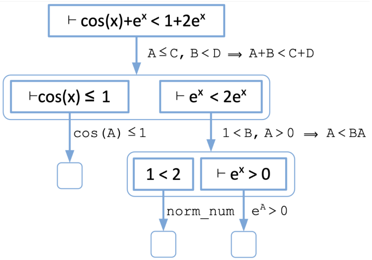
Future of Mathematical AI

Can an AI prove a mathematical result on its own?

Outline of the talk



1. AI tools for math discovery



2. When will AI prove theorems?

AI for Mathematical Proof

Can a trained AI find a proof for a mathematical statement?

- A much harder problem
- Shocking question: calls into question our very vision of mathematics
- A science-fiction future that is probably closer than we imagine

AI for Mathematical Proof

First approach: training a Transformer (GPT-f, Polu, Sutskever, 2020)

Question

Let $a > 0$ and $b > 0$, such that $ab = b - a$, show that

$$\frac{a}{b} + \frac{b}{a} - ab = 2$$

→ GPT → Proof

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GPT

Proof

Ilya Sutskever: The OpenAI Genius Who Told Sam Altman He Was Fired

Company's chief scientist led a board coup against one of the most prominent figures in Silicon Valley

Ilya Sutskever

SUIVRE

Co-Founder and Chief Scientist of OpenAI

Adresse e-mail validée de openai.com - Page d'accueil

Machine Learning Neural Networks Artificial Intelligence Deep Learning

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Citations	462122	356473
indice h	84	79
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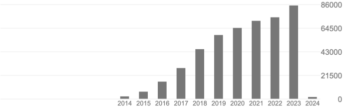

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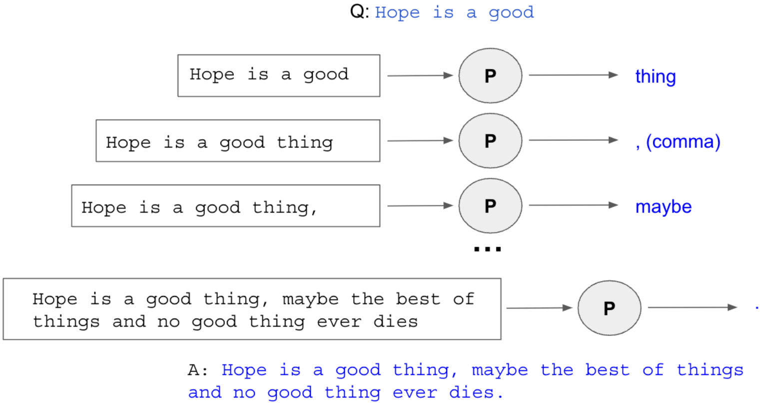
AI for Mathematical Proof

What does an autoregressive transformer do (in a nutshell) ?



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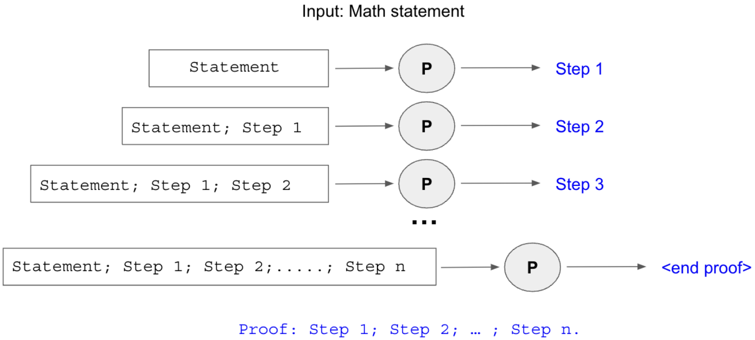

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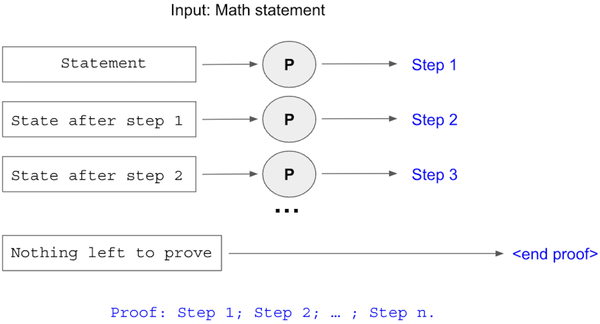

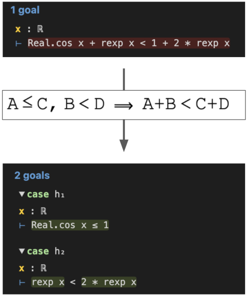

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sorry,
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Procedure: train it with examples: (exercises, proofs)

- The hope is that by showing it enough examples, the AI will be capable of learning to reason, just by learning to predict the next step each time.

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Enough = sufficiently diverse and sufficiently numerous

→ Limitation: lack of data

AI for Mathematical Proof

We have very few data available (especially formal).

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LeanLlama F. Glöckle et al. 2023 (Temperature-scaled large language models for Lean proofstep prediction)

AI for Mathematical Proof

Second approach: treat mathematics as a game (Lample, Lachaux, Lavril, Martinet, Hayat, Ebner, Rodriguez, Lacroix, 2022); (Gloeckle, Limperg, Synnaeve, Hayat, 2024)

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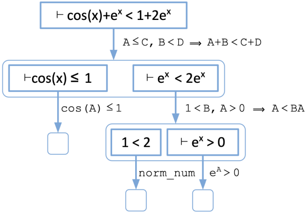
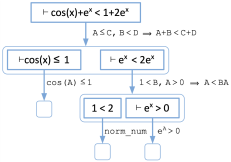


You won !

AI for Mathematical Proof

Main difficulties:

- two-player game vs. solo against a goal.
- In chess, when you play a move you always have a single game. In mathematics: one statement $\to$ multiple statements
- Difficult in mathematics to know automatically in the middle of a proof what the probability of succeeding is.
- The number of possibilities is much, much larger in mathematics



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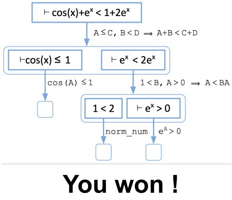
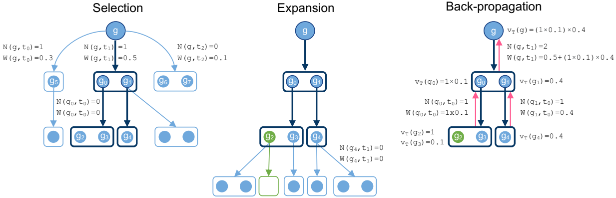


Much more difficult than chess

AI for Mathematical Proof

In practice

- Two transformers: P_θ which predicts a tactic, c_θ which predicts the difficulty of proving a statement (goal, hypothesis, etc.).
- An intelligent proof search that sees the proof as a tree and combines P_θ, c_θ and a tree expansion.



- Continuously training of P_θ and c_θ on successful proofs

AI for Mathematical Proof

Results

Exercises at the undergraduate level...

...30 to 60% of middle school / high school exercises up to Olympiad level...

...and a few exercises from the International Mathematical Olympiads

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Unexpected outcomes

French AI start-up Mistral reaches unicorn status, marking its place as Europe's rival to OpenAI

> Mistral's value has increased more than sevenfold in six months. It raised nearly €500 million in November and €105 million in its first funding round.

(Euronews 11/12/2023)

MISTRAL AI_

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Solo game $\implies$ still need for a good base supervised model P_θ (trained on data)

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How to get more formal data ? One idea: train a model to translate from "natural language proof" to formal and verifiable data.

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Autoformalization: both a means and an end:

- a means: more data to train AI models
- an end: checking the correctness of new mathematical theories

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Example of contradictory papers:

Schumacher, G., & Tsuji, H. (2004). Quasi-projectivity of moduli spaces of polarized varieties. Annals of mathematics, 597-639.

Kollár, J. (2006). Non-quasi-projective moduli spaces. Annals of mathematics, 1077-1096.

Today, very good statement autoformalization for olympiad style exercise, challenges for proof autoformalization, very far from research level mathematics (e.g. arxiv).

AI and maths

A fast-moving field

2019	2022	2024
$(r_1 - r_2)r_3 = r_1 r_3 - r_2 r_3$	$\forall\ n \in \mathbb{N},\ \neg 7 \mid 2^n + 1$	Silver medal Inter. Math. Olymp.
HOList, LPLG	GPT-f, Thor/DSP, HTPS	Llemma, LeanDojo, InternML, DeepSeek, ABEL, AlphaProof, etc.

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New approaches post 2024: end-to-end RL and reasoning models (o1, o3, R1, etc.).

Conclusion

- AI methods are already useful in the practice of mathematics
- AI and LLM for proving theorems is only beginning, and there are many ideas... and much to do.
- LLMs will not be the final AI tool which will be used to mathematics.
- The practice of mathematics will probably change... and that's okay.
- AI will not replace mathematicians but will instead enhance them.

Conclusion

Thank you for your attention