Approximating the Held-Karp Bound for Metric TSP in Nearly Linear Work and Polylogarithmic Depth

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Joint work with
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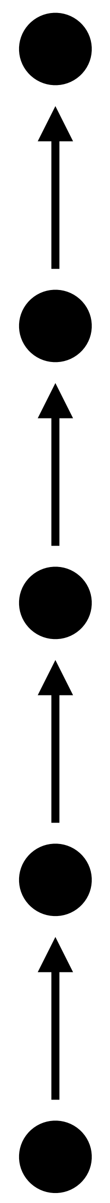
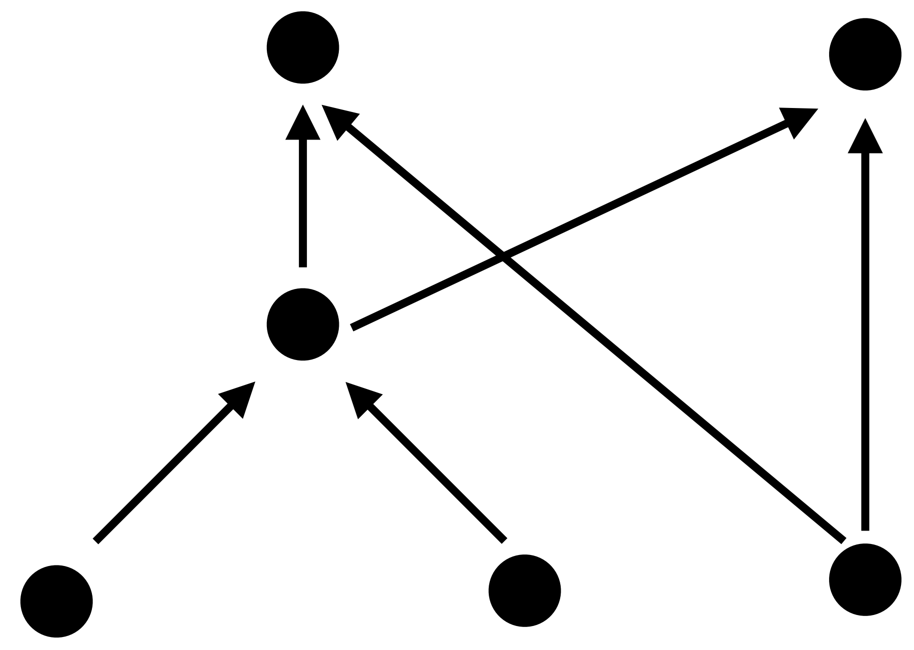
Work-Depth Model

- **Work** = total number of operations
- **Depth** = the length of a longest chain of dependent operations
- Fast parallel algorithm = nearly linear work and polylog depth





- **Metric TSP**
  - **Input**: a complete graph graph G = (V,E,c) where $\forall u, v, w \in V, c_{uv} \leq c_{uw} + c_{wv}$
  - **Output**: a min-cost Hamiltonian cycle
  - **Implicit input**: the instance is implicitly defined as the metric completion of the underlying graph G = (V,E,c)
  - **Output:** a min-cost Eulerian multigraph of G
- **APX-hard** [Lampis'12]
- 1.5-approximation algorithm by [Christofide'76]
- 1.5 - 10^{-36} approximation by [Karlin, Klein, Gharan'22]

Subtour Elimination LP

$$
\begin{aligned}
\mathrm{SE}(\hat{G}, \hat{c}) = \min \sum_{u,v} \hat{c}_{\{u,v\}} y_{\{u,v\}} & \\
\text{s.t.} \sum_{u} y_{\{u,v\}} = 2 & \qquad \forall v \in V \\
\sum_{u \in S, v \notin S} y_{\{u,v\}} \geq 2 & \qquad \forall \emptyset \subsetneq S \subsetneq V \\
0 \leq y_{\{u,v\}} \leq 1 & \qquad \forall u, v \in V.
\end{aligned}
$$

The optimal value of the SE coincides with the Held-Karp bound

The Held-Karp bound is defined based on the notion of 1-trees [Held and Karp'70]

The integrality gap of SE is conjectured to be 4/3 [Goemans'95]

k-ECSM: Given an undirected graph $G = (V, E)$ with n nodes, m edges and edge costs $c \in \mathbb{R}^m_{>0}$, find a minimum cost k-edge-connected spanning multi-subgraph.

- LP relaxation:

$$\begin{aligned} \min \; & c^\top x \\ \text{s.t.} \; & \sum_{e \in \delta(S)} x_e \geq k & \forall \emptyset \subsetneq S \subsetneq V \\ & x_e \geq 0 & \forall e \in E \end{aligned}$$

Fact: Held-Karp Bound = LP value of SE = LP value of 2ECSM

Cunningham [via Monma, Munson, and Pulleyblank, 1990] and Goemans and Bertsimas [GB93]:

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- Covering LP with $\Omega(2^n)$ constraints.

Goal: Compute a $(1 + \varepsilon)$-approximate LP solution in nearly linear work and polylog depth.

*Slides from Zhuan Khye Koh

Multiplicative Weights Update (MWU)

- Many MWU variants [Luby Nisan '93] [Plotkin Shmoys Tardos '95] [Garg Könemann '07] [Fleischer '00] [Young '01] [Young '14] [Allen-Zhu Orrechia '15] [Mahoney Rao Wang Zhang '16] ...

MWU "compiles" kECSM into a sequence of mincut problems

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- We consider epoch-based MWU.

- In iteration t, let $w^{(t)} \in \mathbb{R}^m_{\geq 0}$ be the edge weights. Given a lower bound λ on the mincut and $\varepsilon > 0$, define

$$\mathcal{C}^{(t)} := \{C \text{ cut } : w^{(t)}(C) < (1+\varepsilon)\lambda\}.$$

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Sequential MWU: Select one cut from $\mathcal{C}^{(t)}$

$\implies \tilde{O}(m/\varepsilon^2)$ iterations [Garg Könemann '07] [Fleischer '00].

Parallel MWU: Select all cuts from $\mathcal{C}^{(t)}$

$\implies \tilde{O}(\log(|\mathcal{C}^{(t)}|)/\varepsilon^4)$ iterations [Luby Nisan '93] [Young '01].

- $|\mathcal{C}^{(t)}| \leq \#\ (1+\varepsilon)$-mincuts $= O(n^2)$ [Nagamochi Nishimura Ibaraki '94] [Henzinger Williamson '96].

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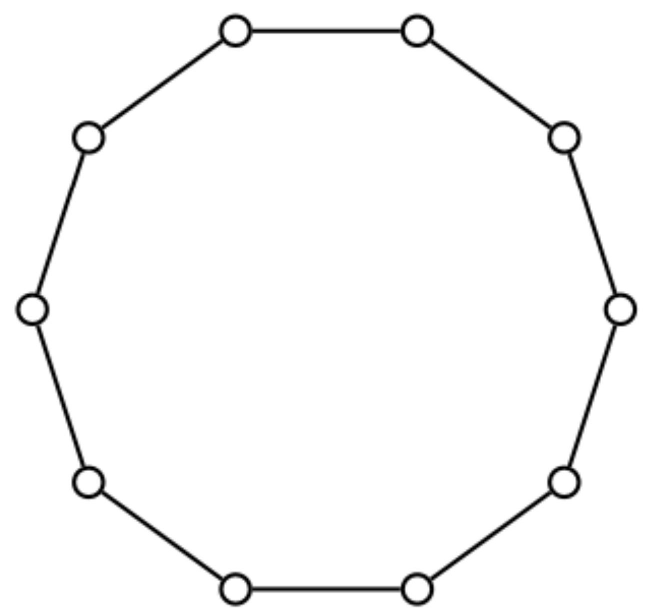
Implementing MWU for k-ECSM LP

- Sequential MWU has a $\tilde{O}(m/\varepsilon^2)$-time implementation [Chekuri Quanrud '17].

- Parallel MWU incurs $\Omega(n^2)$ work, because $|\mathcal{C}^{(t)}| = \Omega(n^2)$ for some graphs.



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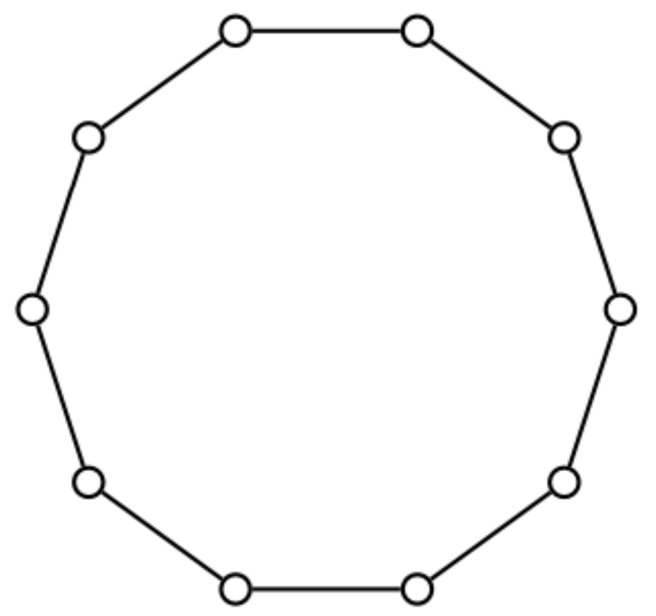

New Selection Rule

Let $\mathcal{S} = (S_1, \ldots, S_\ell)$ be a sequence of sets of cuts. In iteration t, select

$$S_{i(t)} \cap \mathcal{C}^{(t)}$$

where $i(t)$ is the smallest index such that $S_{i(t)} \cap \mathcal{C}^{(t)} \neq \emptyset$.

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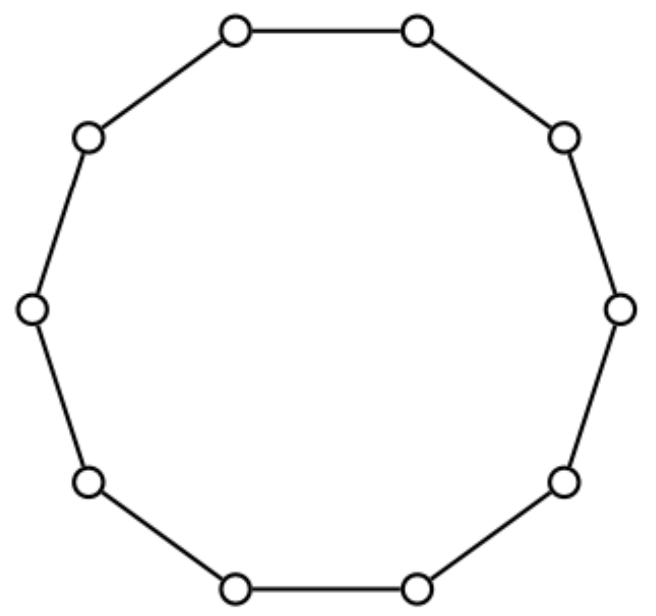

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- If $\cup_{i=1}^{\ell} S_i \cap \mathcal{C}^{(t)} = \emptyset \implies \mathcal{C}^{(t)} = \emptyset$, then $\mathcal{S}$ is a core-sequence of the epoch.

*Slides from Zhuan Khye Koh

Core-Sequence MWU

- Special cases:
  - $\mathcal{S} = (S_1, \ldots, S_\ell)$ where $|S_i| = 1$ for all $i \in [\ell] \implies$ sequential MWU.
  - $\mathcal{S} = (\mathcal{C}^{(t)}) \implies$ parallel MWU.

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Theorem [KW**Y**'25]

If every epoch has a core-sequence of length $\leq \ell$, in which every set has size $\leq k$, then core-sequence MWU returns a $(1+\varepsilon)$-appproximate solution in

$$\tilde{O}\left(\frac{\ell \log(k)}{\varepsilon^4}\right)$$

iterations.

- Ideally, we want a short core-sequence consisting of small sets.

Core-Sequence for the k-ECSM LP

Theorem [KW**Y**'25]

For the k-ECSM LP, every epoch has a core-sequence of length $O(\log n)$, in which every set has size $O(n)$.

Theorem [KW**Y**'25]

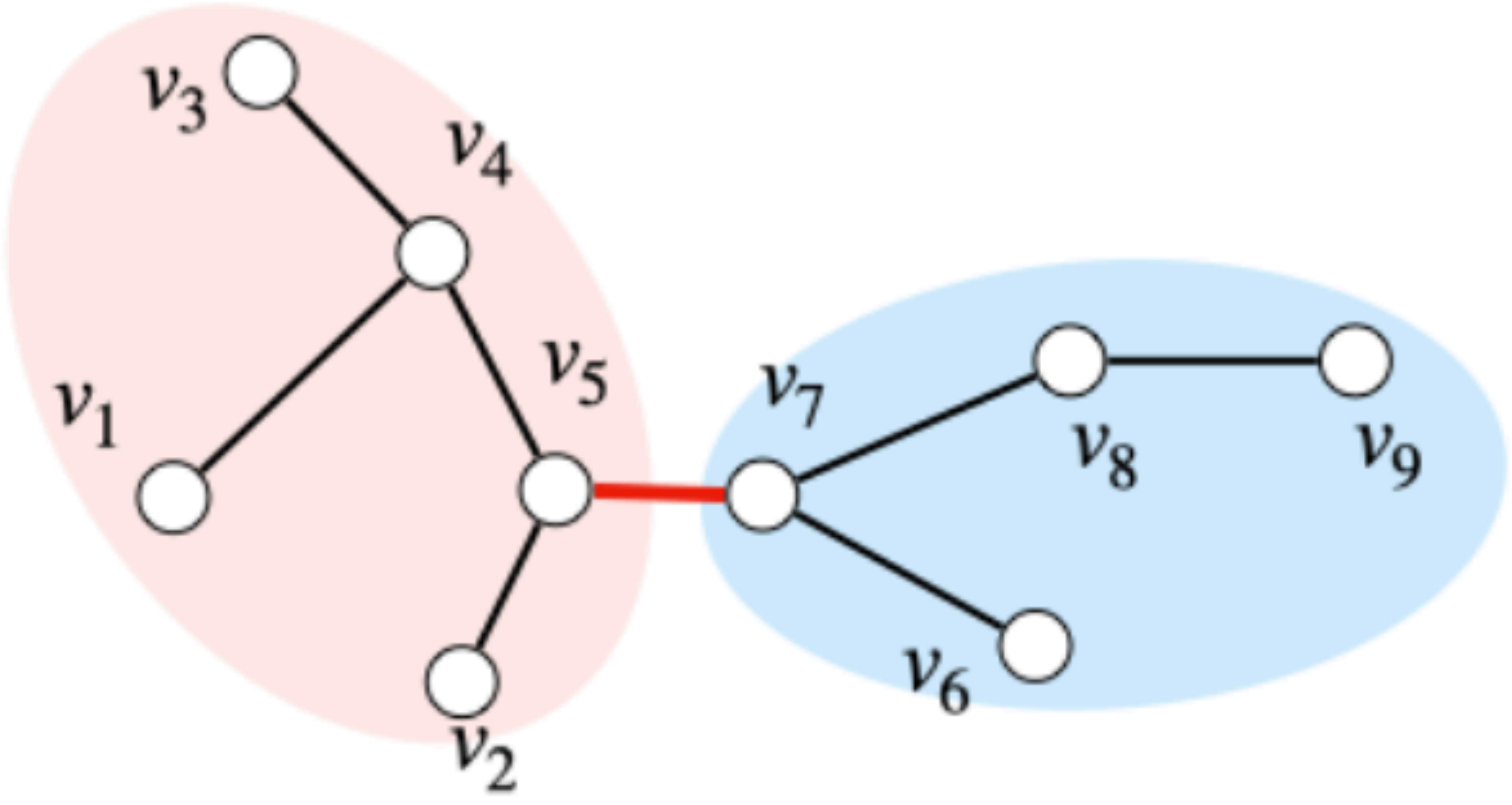
There is a parallel PTAS for the k-ECSM LP using $\tilde{O}(m/\varepsilon^4)$ work and $\tilde{O}(1/\varepsilon^4)$ depth.

- 2-ECSM LP optimum $=$ Held–Karp bound for metric TSP.
- Extends to the k-edge-connected spanning subgraph (k-ECSS) LP.

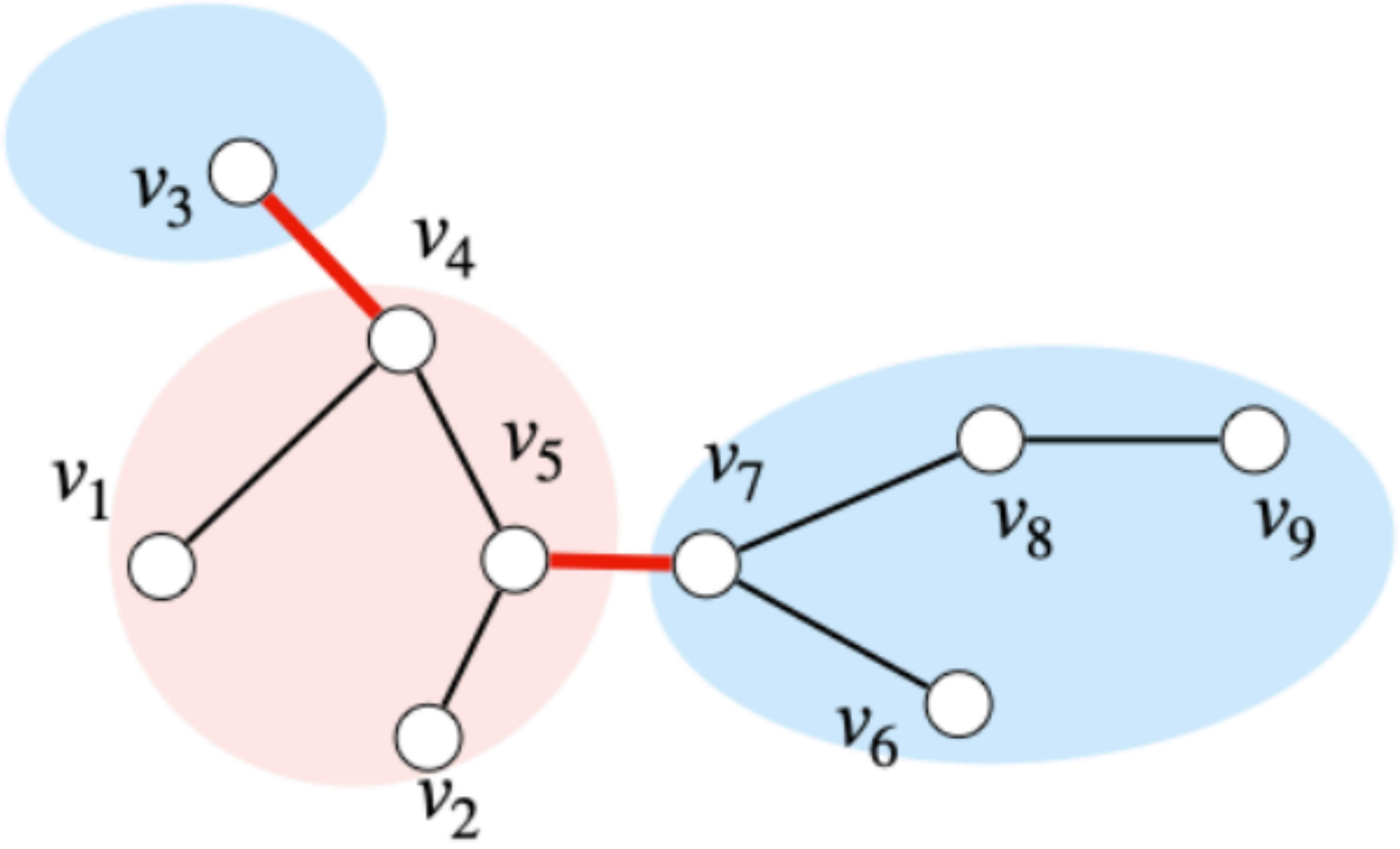
Using weight thresholding technique from [CHNSS**Y**'22]

Combinatorial Tools

Def: A cut C k-*respects* a tree T if $|E(T) \cap C| = k$



A 1-respecting cut in a tree



A 2-respecting cut in a tree

Combinatorial Tools

The Tree Packing Theorem: There is a set of O(log n) spanning trees such that every (1+eps)-mincut 1-or-2 respects some tree

[Karger'00]

(cf. [CQ'17])

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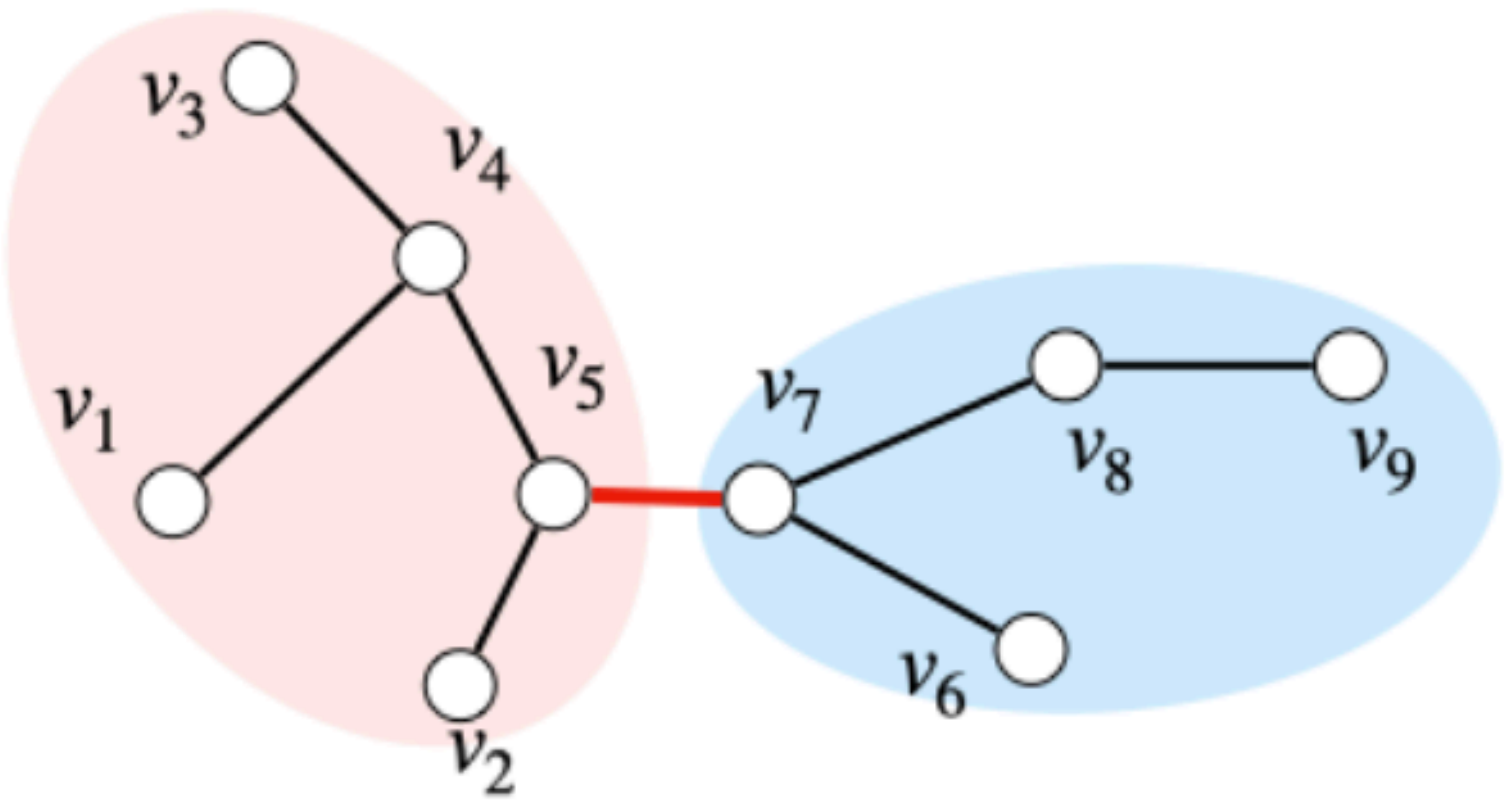

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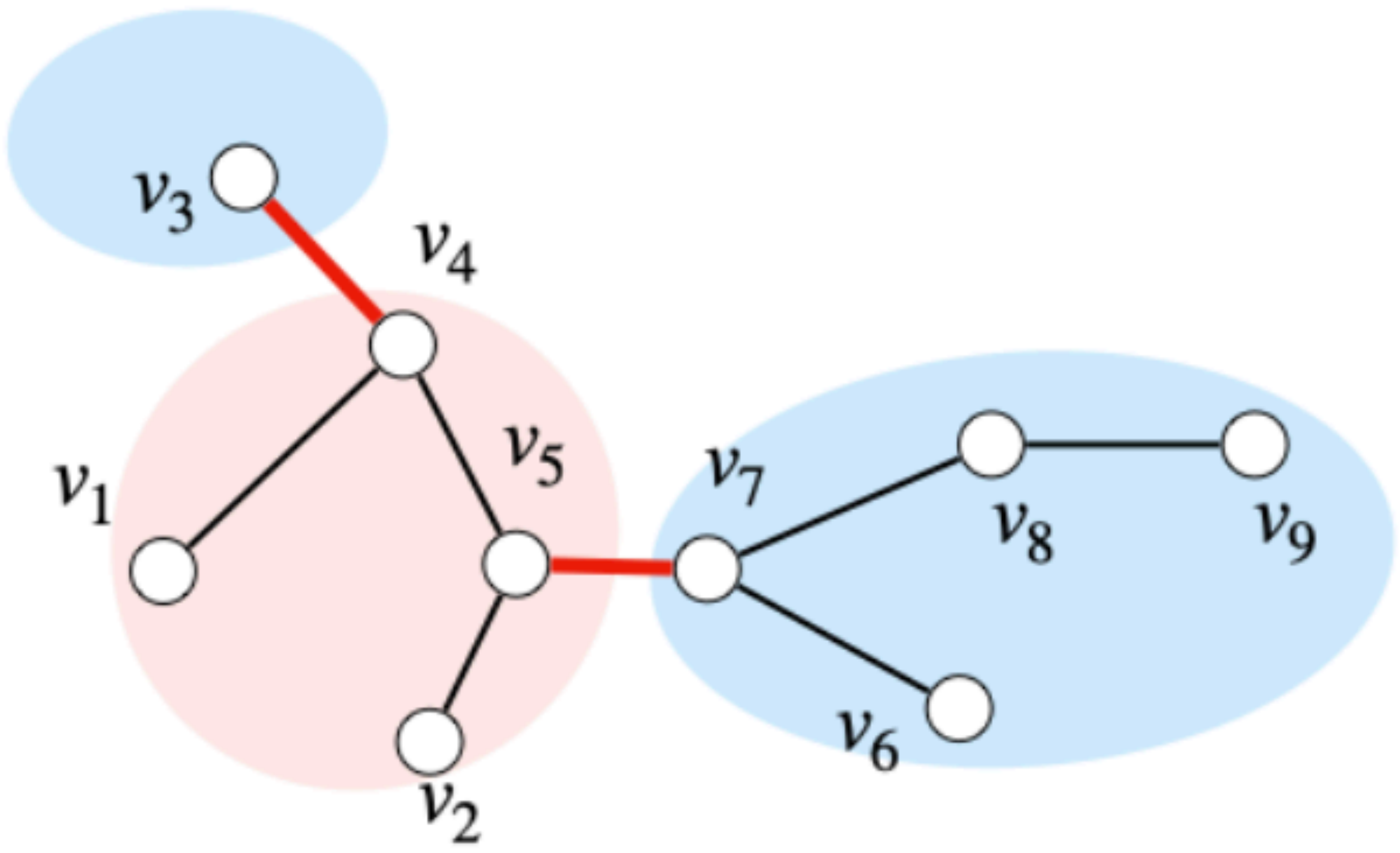

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Can be computed in nearly linear work and polylog depth [Geissmann and Gianinazzi'18]

Sauce: A Corollary of [Tutte'61][Nash-Williams'61]

$$\lambda_G/2 \leq \tau_G \leq \lambda_G$$

$\tau_G :=$ The maximum number of disjoint spanning trees

$\lambda_G :=$ The edge connectivity

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→

For $T \in \mathcal{T}$,

$$C_T^{(t)} := \left\{ \{e_1, e_2\} \subseteq E(T) : \mathrm{w}(\mathrm{cut}_T(e_1, e_2)) < (1+\varepsilon)\cdot\lambda \right\}$$

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$\tilde{O}(n)$ work per iteration, $\tilde{O}(1)$ iterations

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n² work, 1 iteration if select all pairs

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Intuition: Select a representative/maximal set of cuts so that

Updating weights of these cuts = increase weights of every cut

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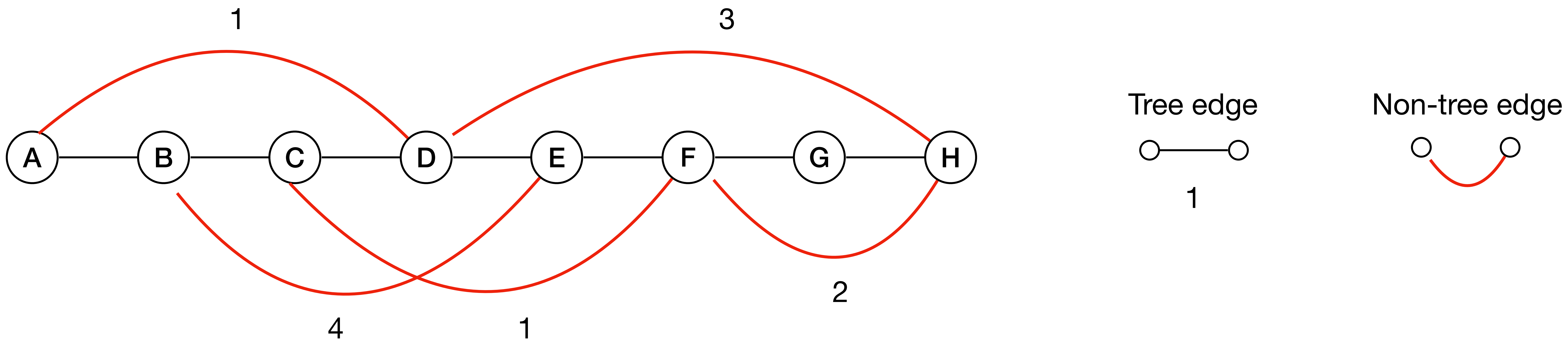
This talk: assume T is a path and show 'good' core sequence exists

(In general, reduce to path via heavy/light decomposition [MN'20].)

T = path



$w(\mathsf{cut}_T(e_1, e_2))$

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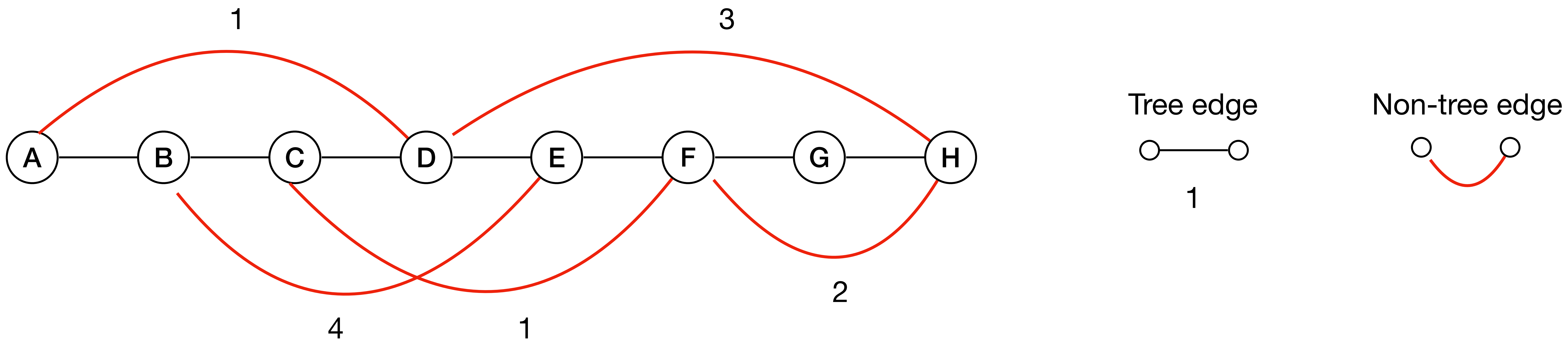

$w(\mathrm{cut}_T(BC, DE)) = ?$

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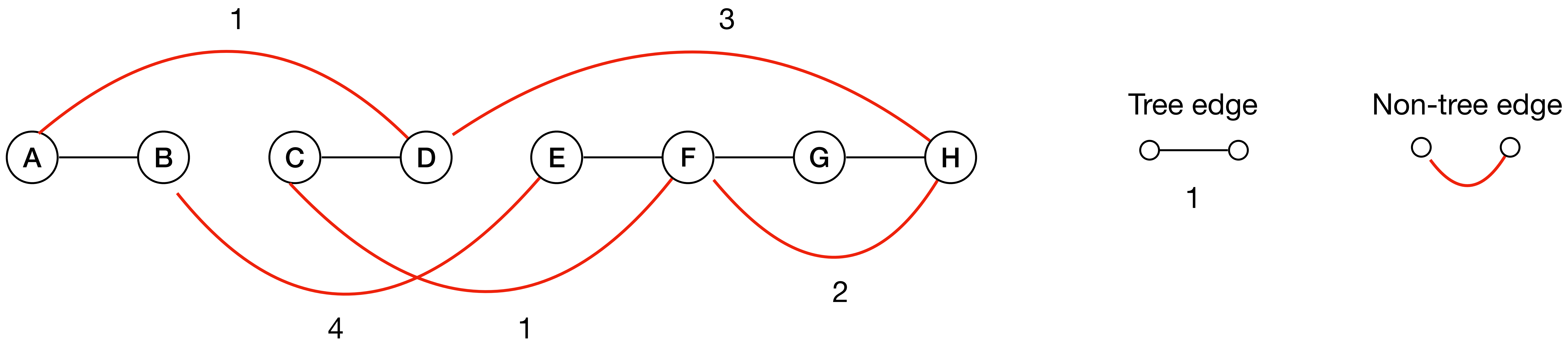

$$w(\text{cut}_T(BC, DE)) = 1 + 3 + 1 + 2 = 7$$

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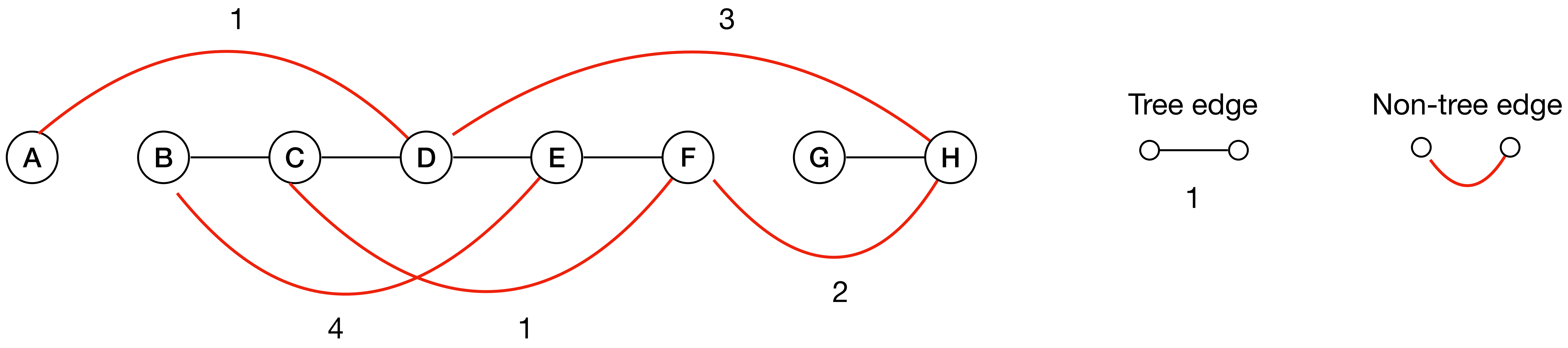
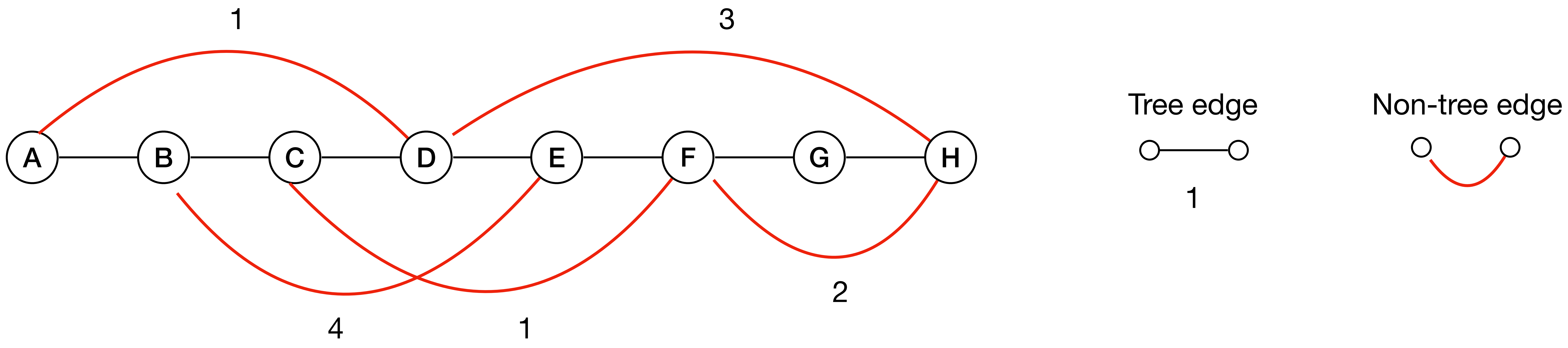


$$w(\text{cut}_T(BC, DE)) = 7$$

$$w(\text{cut}_T(AB, FG)) = 1 + 3 + 2 + 2 = 8$$

T = path



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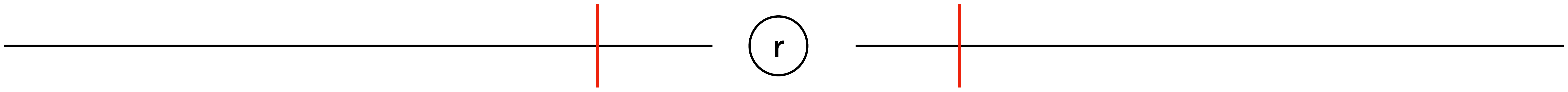
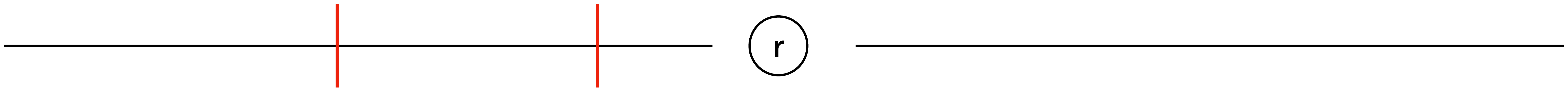
$$|C_T^{(t)}| \leq n^2$$

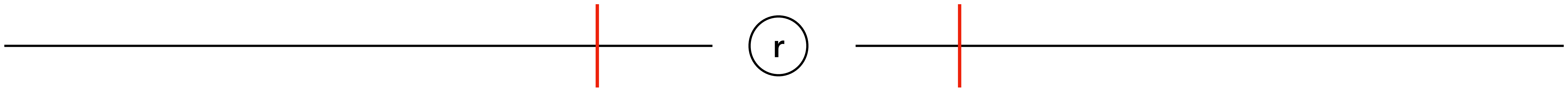


r-crossing



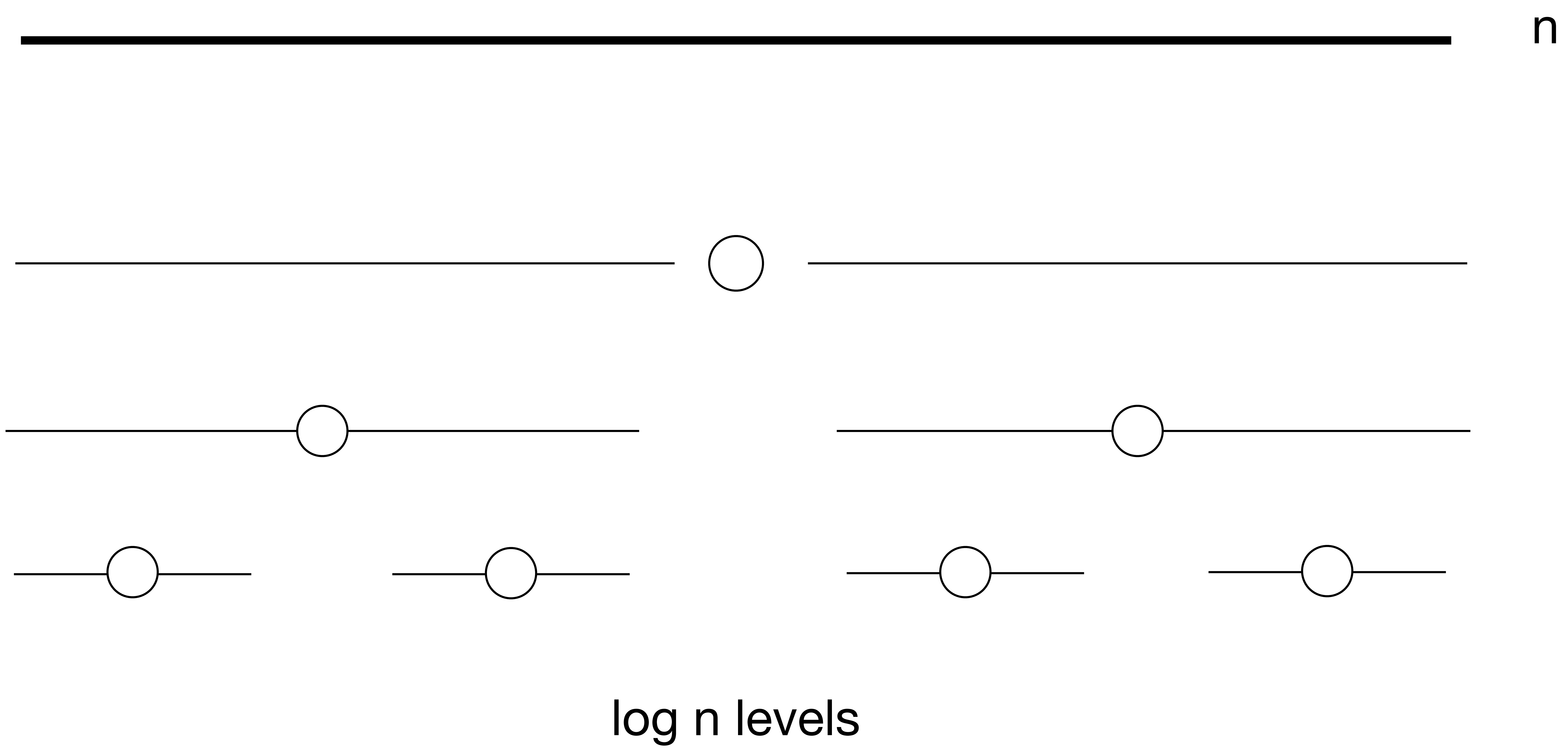
NOT r-crossing

Lemma: Fix a node r in a path, if every **small** pair is r-crossing, then there are O(n) **small** pairs



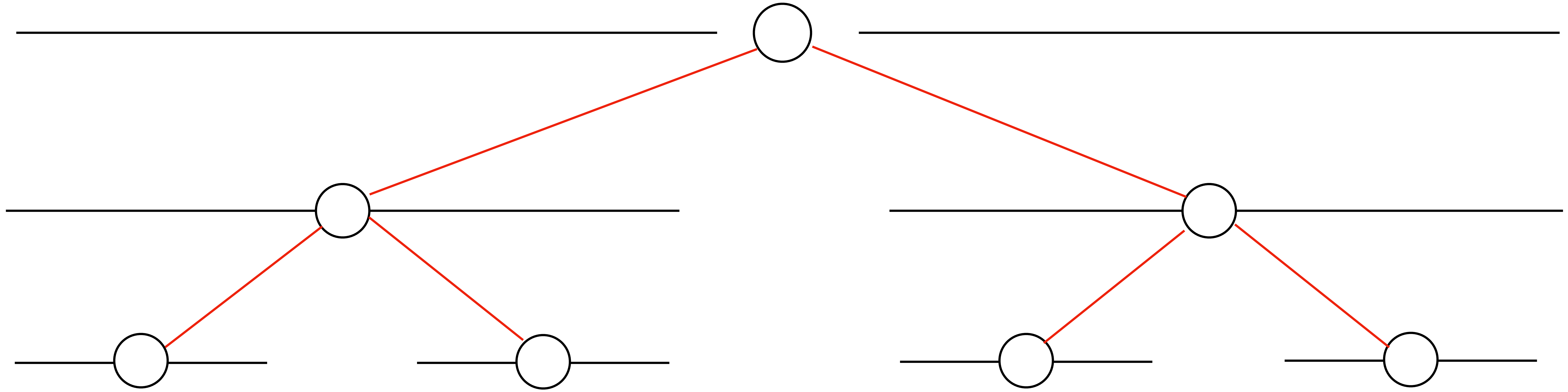
r-crossing

Representation of all pairs



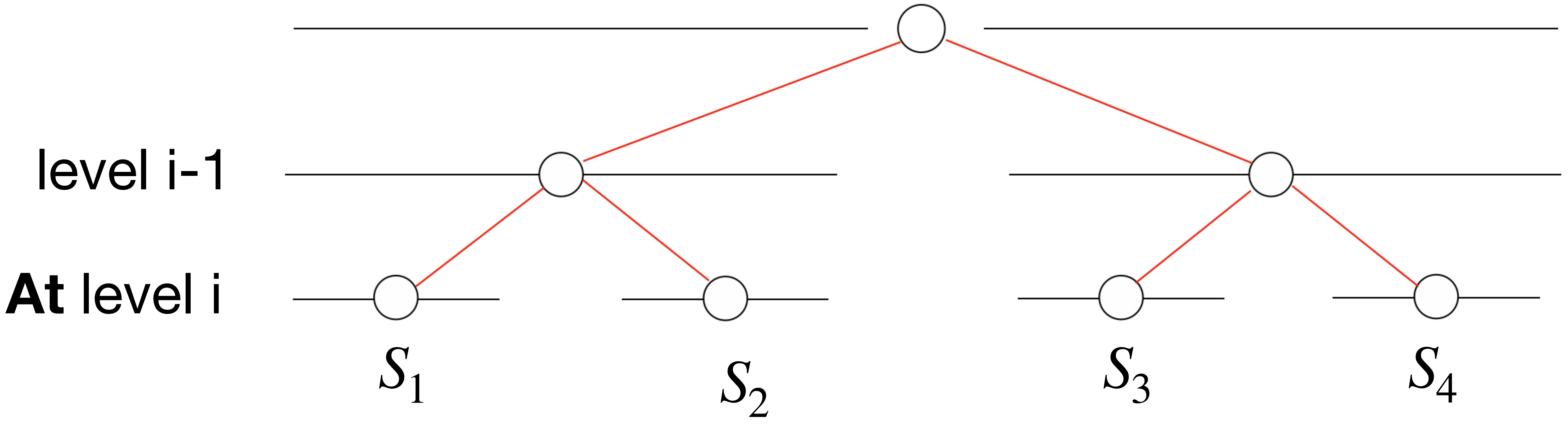
Representation of all pairs

The recursion tree



Property: every pair of edges in T is mapped to the first node that splits it

Lemma 1 : Fix a node r in a path, if every small pair is r-crossing, then there are O(n) small pairs



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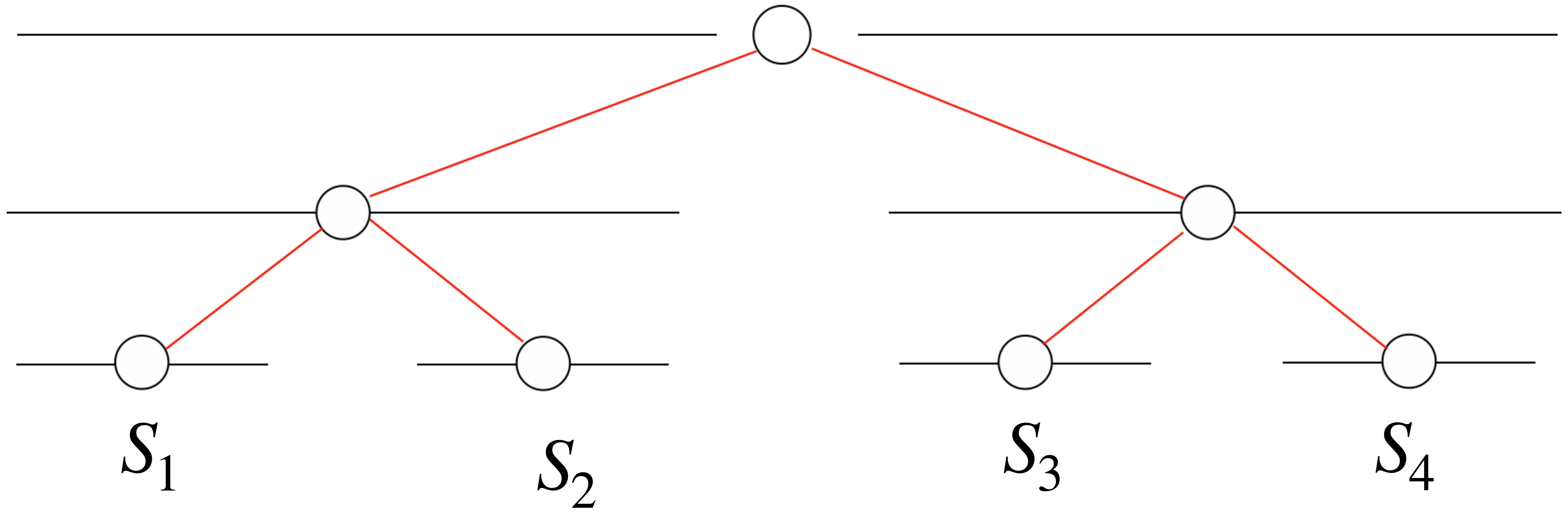
level i-1

At level i



At level i

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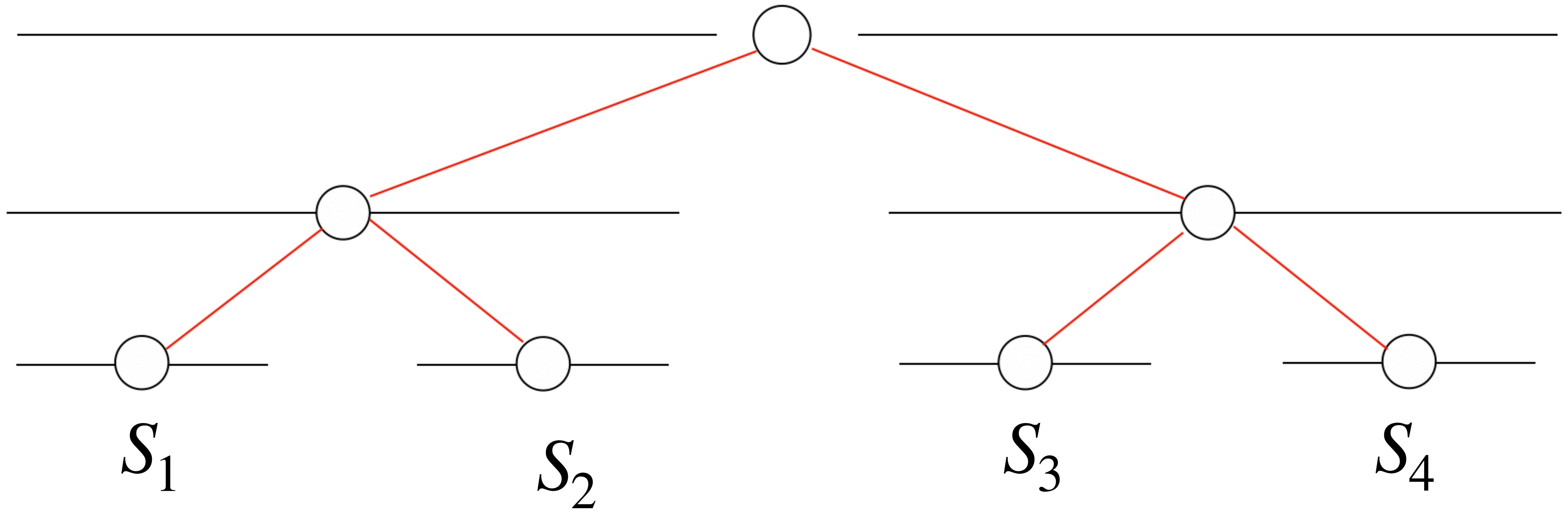

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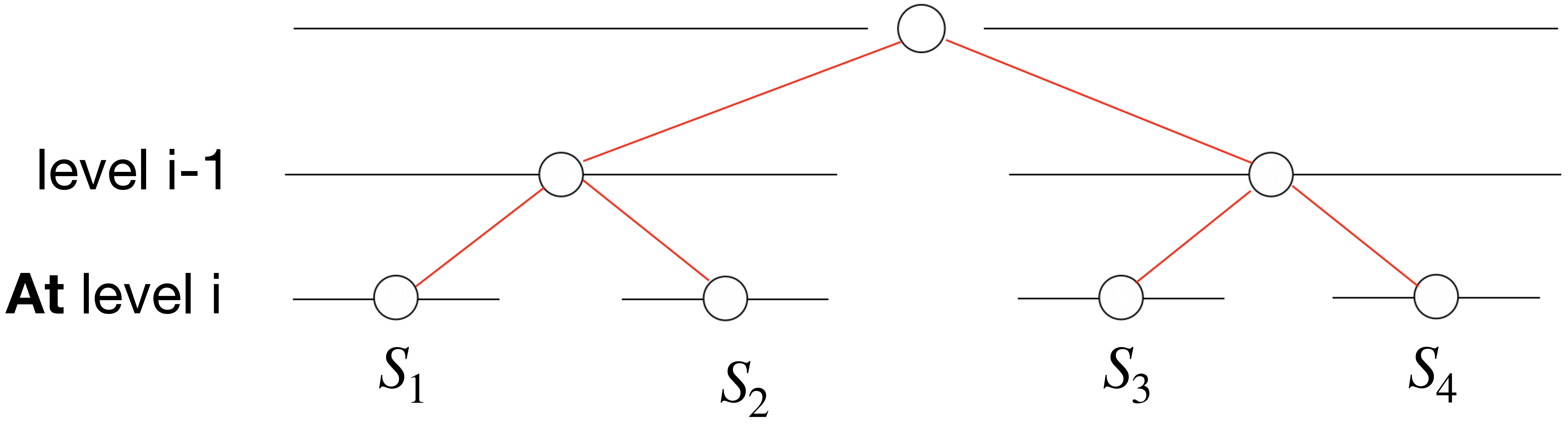

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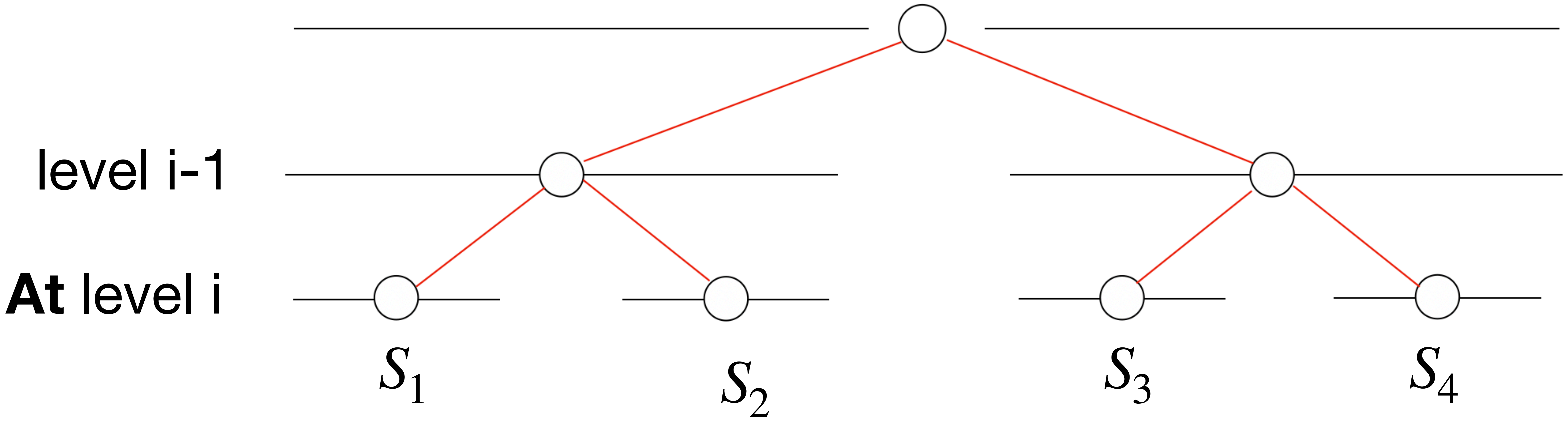

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Repeat at level i-1 and so on $(B_{\log n}, B_{\log n-1}, \ldots, B_1)$ is a good core sequence

Lemma: Fix a node r in a path, if every **small** pair is r-crossing, then there are O(n) **small** pairs

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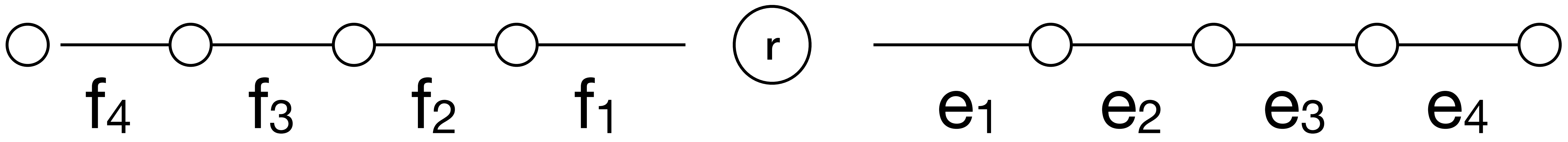

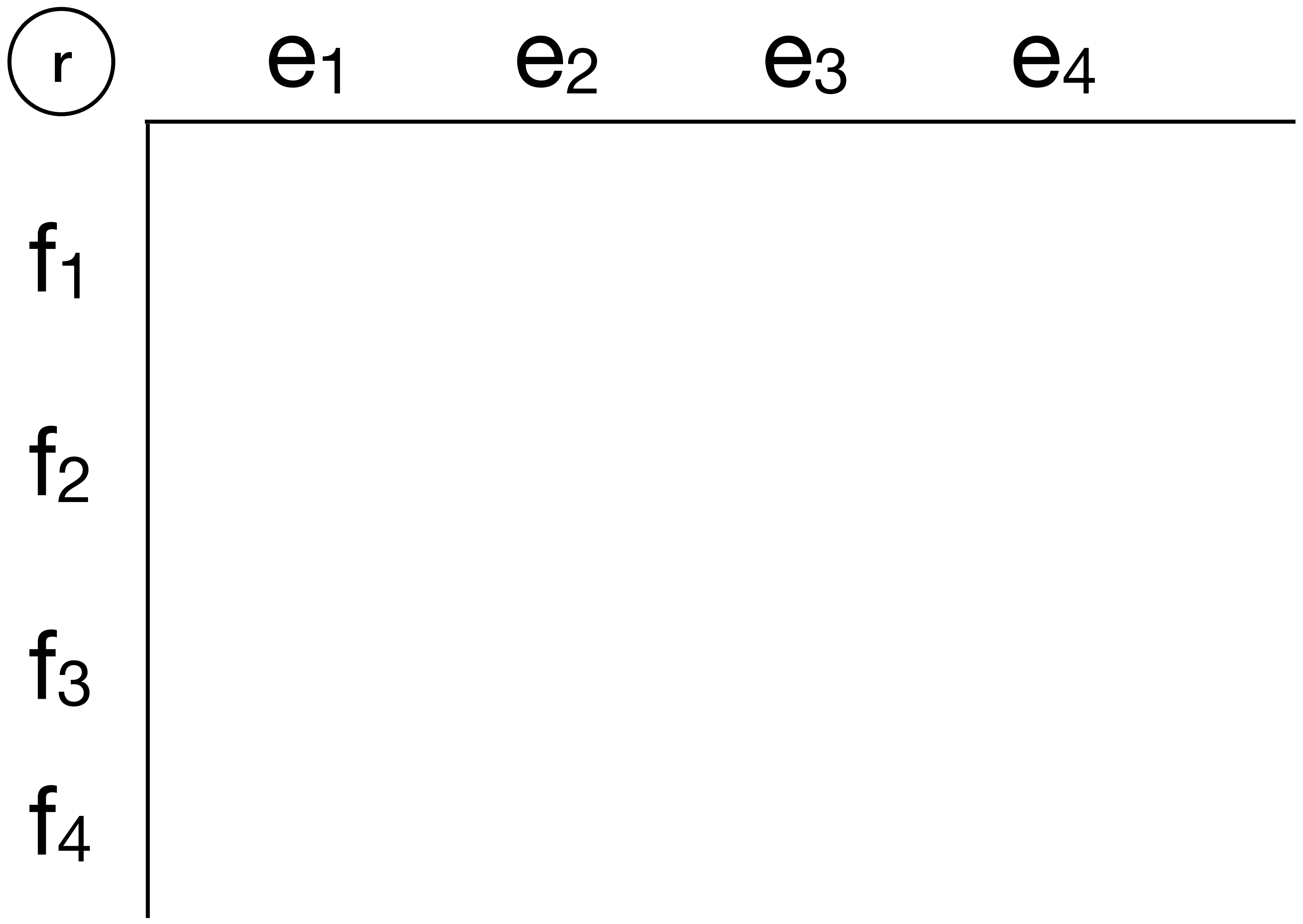

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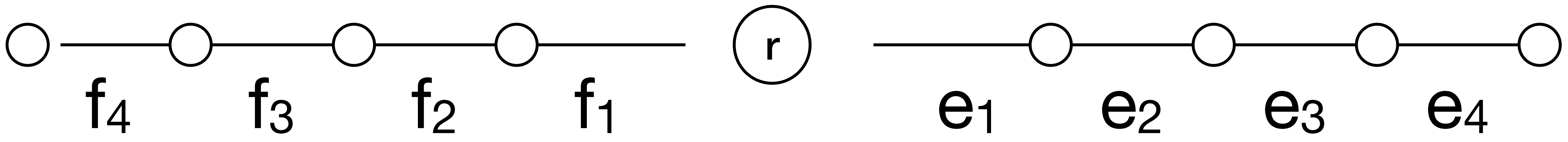

r	e_1	e_2	e_3	e_4
f_1				
f_2				
f_3				
f_4				

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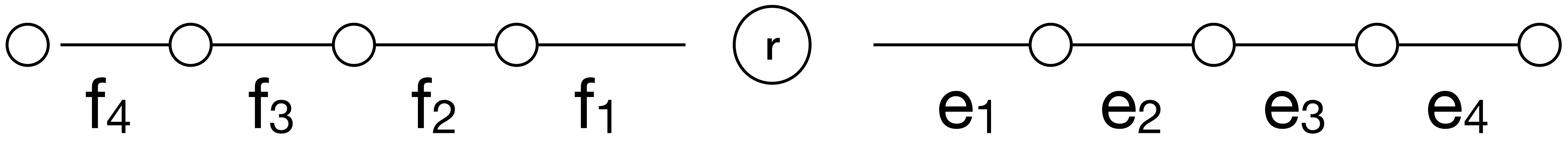


Define M_r

$M_r(f_i, e_j) = w(\text{cut}_T(f_i, e_j))$

r	e_1	e_2	e_3	e_4
f_1				
f_2				
f_3				
f_4				

Lemma: Fix a node r in a path, if every **small** pair is r-crossing, then there are O(n) **small** pairs

Proof:

$$\begin{array}{c|cccc} r & e_1 & e_2 & e_3 & e_4 \\ \hline f_1 & & & & \\ f_2 & & & & \\ f_3 & & & & \\ f_4 & & & & \end{array}$$

M_r

$M_r(f_i, e_j) = w(\text{cut}_T(f_i, e_j))$

define Q_r

$Q_r(f_i, e_j) = 1$ if $M_r(f_i, e_j) < (1 + \varepsilon)\lambda$

$Q_r(f_i, e_j) = 0$ else

Lemma: Fix a node r in a path, if every **small** pair is r-crossing, then there are O(n) **small** pairs

Proof:

r	e_1	e_2	e_3	e_4
f_1				
f_2				
f_3		1		
f_4				

define Q_r

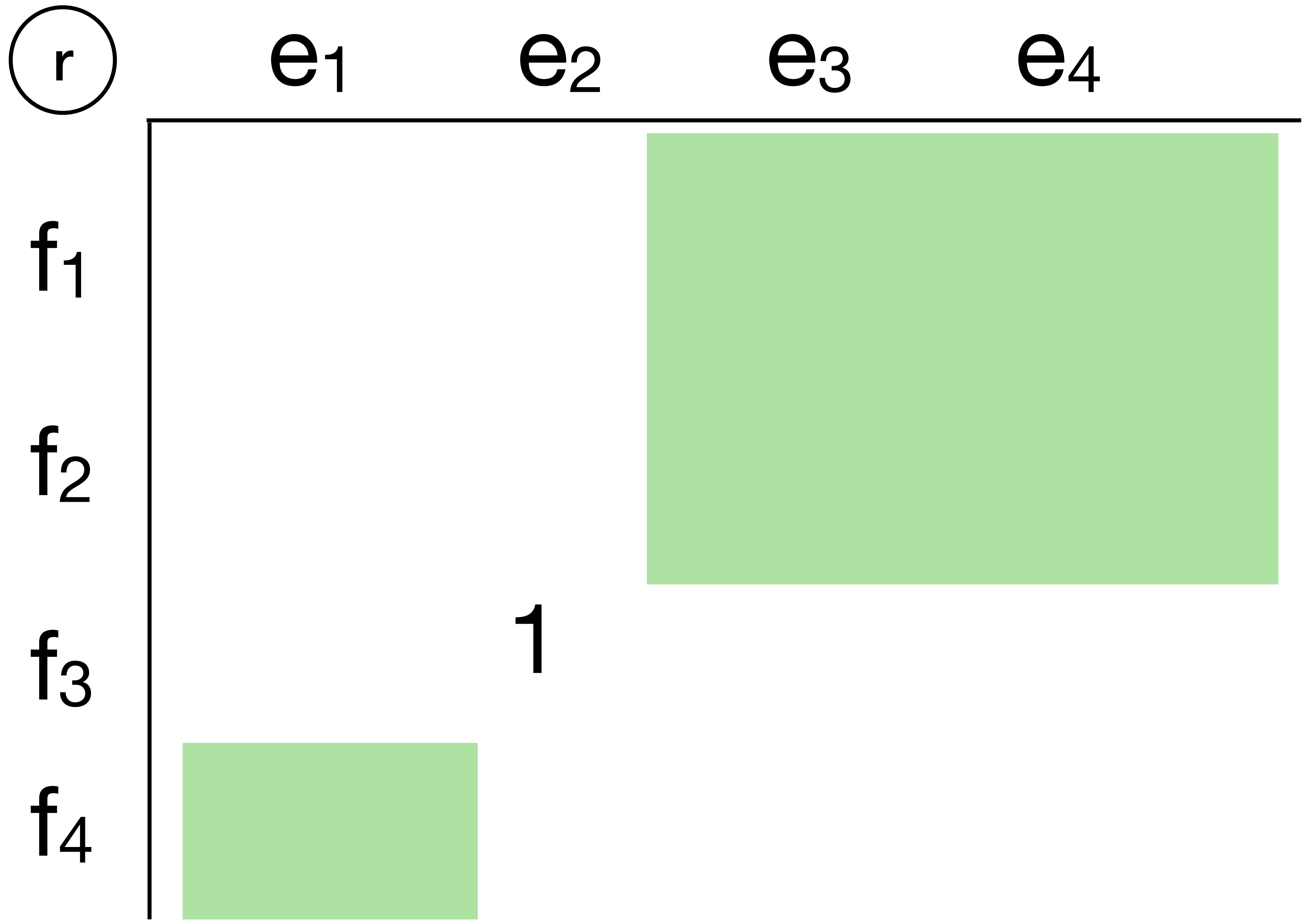
$$Q_r(f_i, e_j) = 1 \text{ if } M_r(f_i, e_j) < (1+\varepsilon)\lambda$$

$$Q_r(f_i, e_j) = 0 \text{ else}$$

claim: Q avoids $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \hline\end{array} \implies |Q| \leq 2n$

Lemma: Fix a node r in a path, if every **small** pair is r-crossing, then there are O(n) **small** pairs

Proof:



define Q_r

$$Q_r(f_i, e_j) = 1 \text{ if } M_r(f_i, e_j) < (1+\varepsilon)\lambda$$
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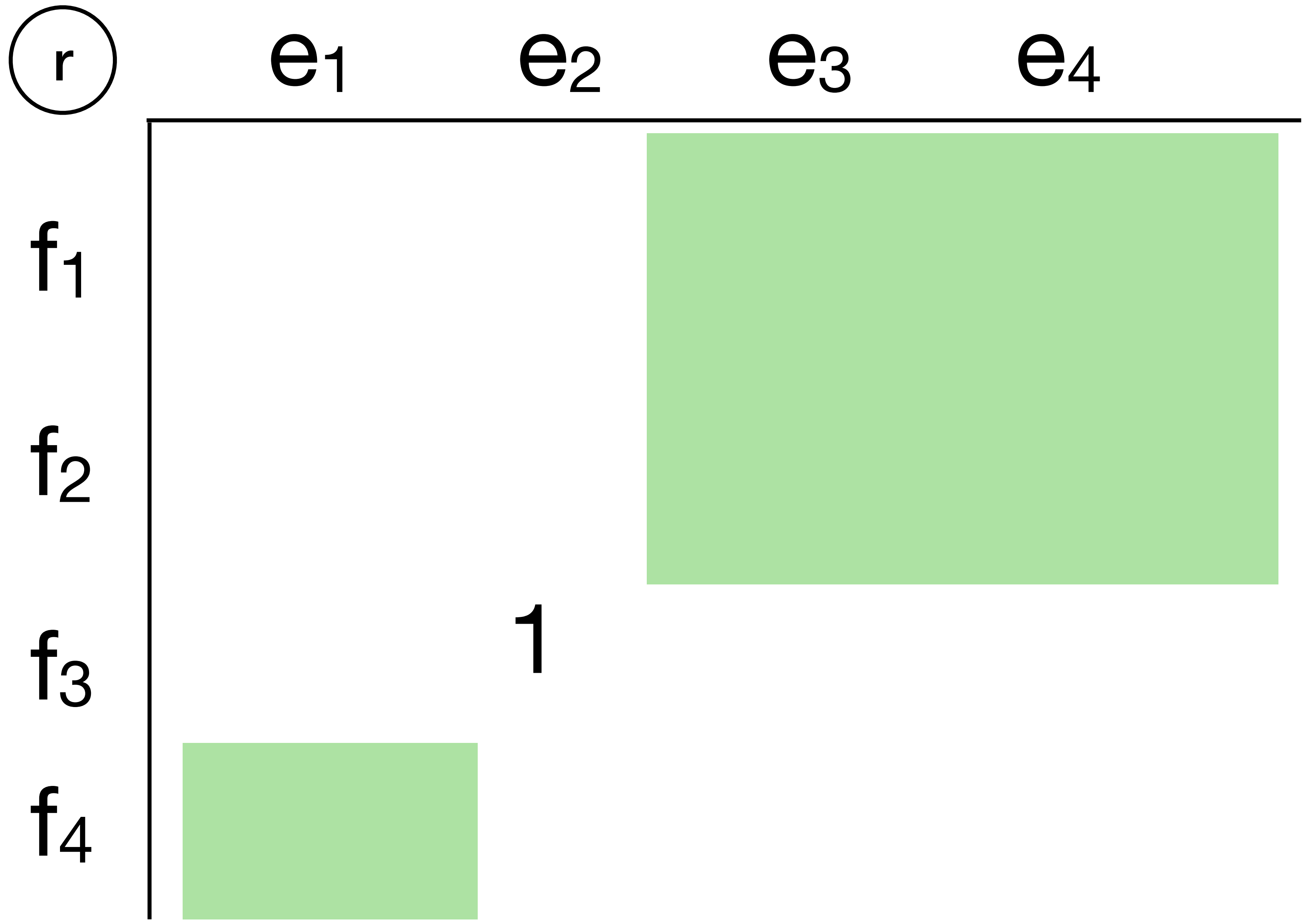

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$$Q_r(f_i, e_j) = 0 \text{ else}$$

claim: Q avoids $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \hline\end{array}$ $\implies |Q| \leq 2n$

claim: Q can be computed

$$ex\left(n, \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & \\ \hline 1 & & \\ \hline\end{array}\right) \leq 6n$$

number of probes in M_r

Lemma: Fix a node r in a path, if every **small** pair is r-crossing, then there are O(n) **small** pairs

Proof:

r	e_1	e_2	e_3	e_4
f_1				
f_2			1	
f_3		1		
f_4				

define Q_r

$Q_r(f_i, e_j) = 1$ if $M_r(f_i, e_j) < (1+\varepsilon)\lambda$

$Q_r(f_i, e_j) = 0$ else

claim: Q avoids $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \hline\end{array}$ $\implies |Q| \leq 2n$

proof: Suppose Q contains $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \hline\end{array}$

claim 2:

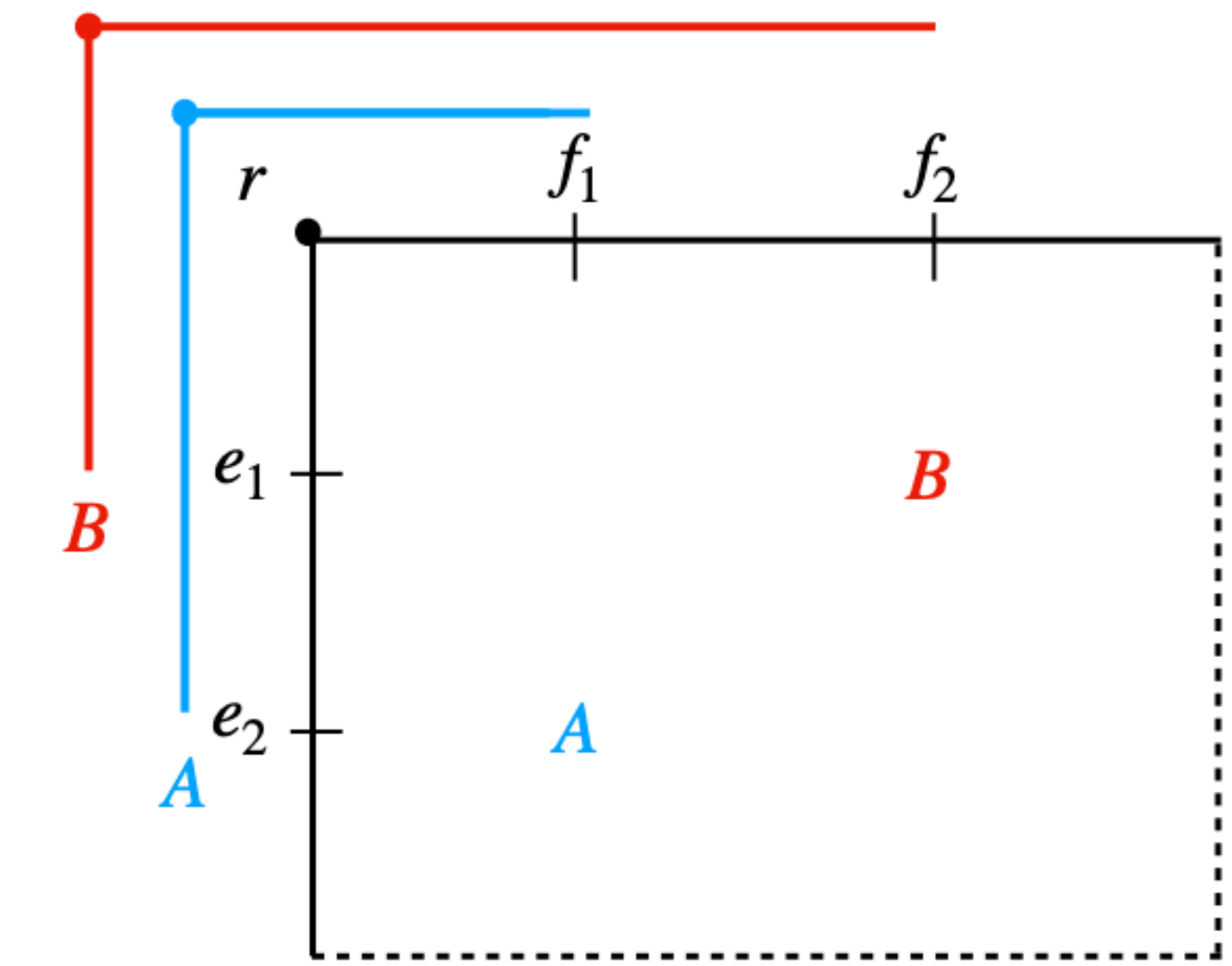
there is a small pair that does **not** cross r

contradiction

Proposition 2.2. *For every pair of subsets $X, Y \subseteq V$, we have*

- *(Submodularity)* $f_w(X) + f_w(Y) \geq f_w(X \cap Y) + f_w(X \cup Y)$, *and*
- *(Posi-modularity)* $f_w(X) + f_w(Y) \geq f_w(X \setminus Y) + f_w(Y \setminus X)$.

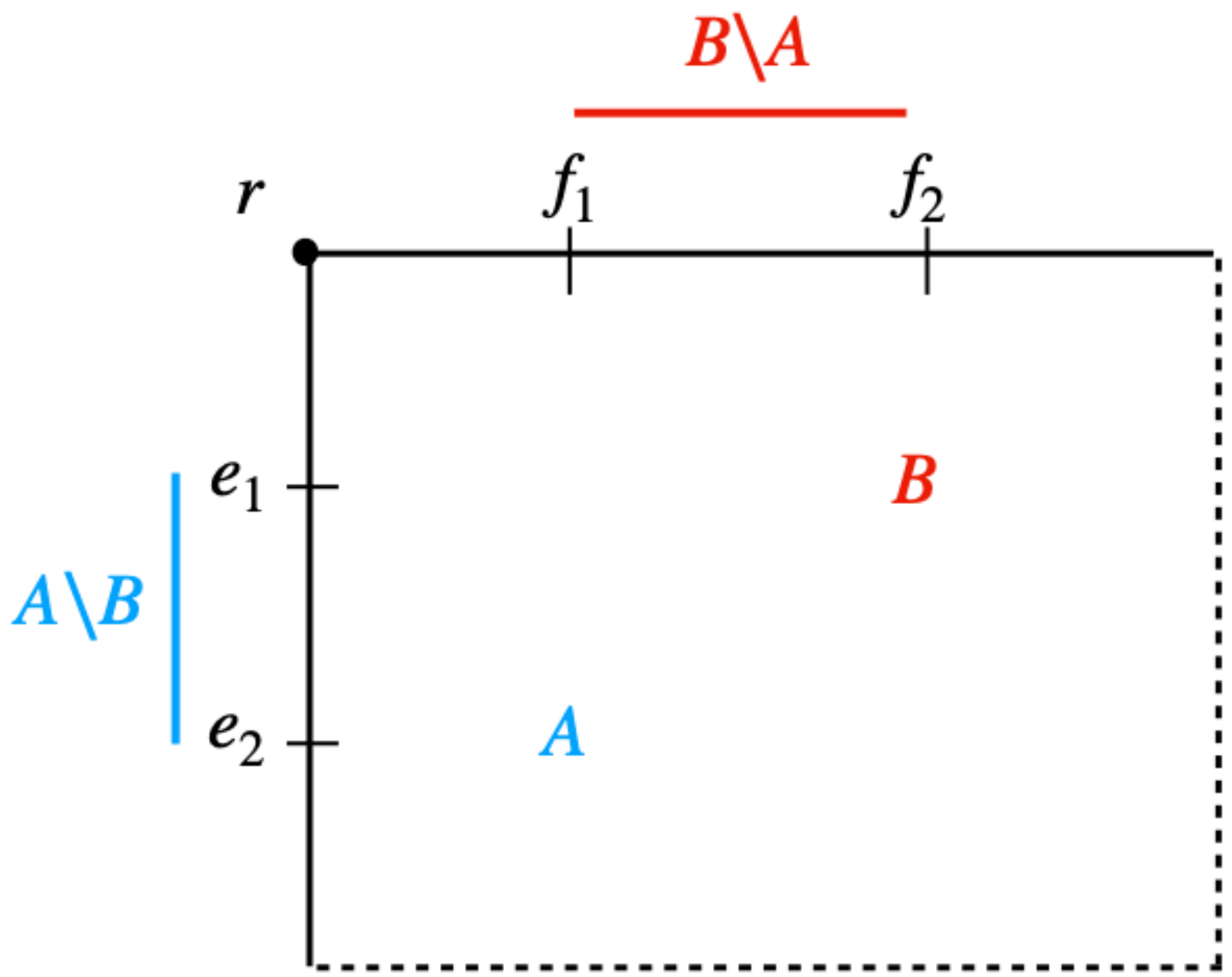
Claim 2: there is a small pair that does **not** cross r



$$2\lambda(1+\varepsilon) > f(A) + f(B) \geq f(A \backslash B) + f(B \backslash A)$$ (Posi-modularity)

Therefore, $\min\{f(A\backslash B), f(B\backslash A)\} < (1+\varepsilon)\lambda$

Contradiction!



Recall **Lemma:** if every **small** pair is r-crossing,
then there are O(n) **small** pairs

Concluding Remark

- Core-sequence is generic
- Improve: $\tilde{O}(\frac{m}{\varepsilon^4}) \rightarrow \tilde{O}(\frac{m}{\varepsilon^2})?$
- Solving kECSM with high accuracy?
- Extension to streaming and distributed algorithms? [MN'20]