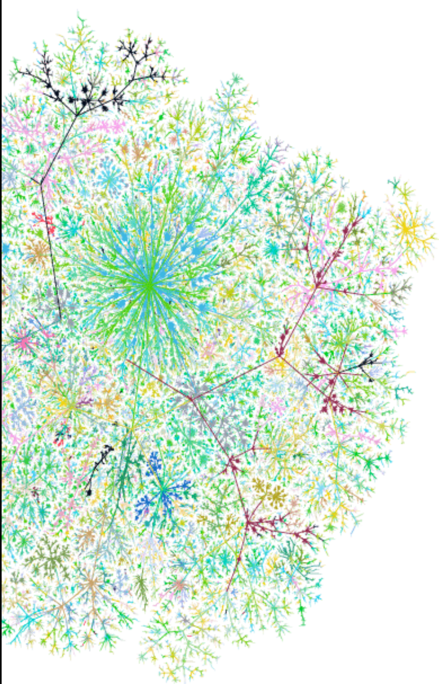
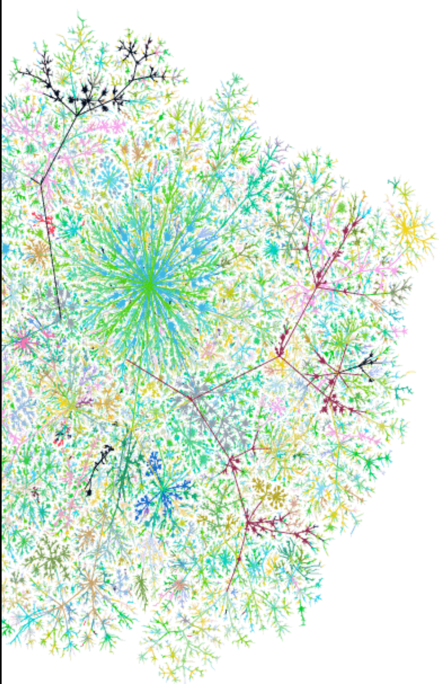


Max Coverage *via* Simplification, Streaming, Sampling, *and* Submodular Stuff

Andrew McGregor
University of Massachusetts

includes joint work with **Hoa Vu**, **David Tench**, **Amit Chakrabarti**, & **Anthony Wirth**

Hi to anyone reading Clement's live tweets!



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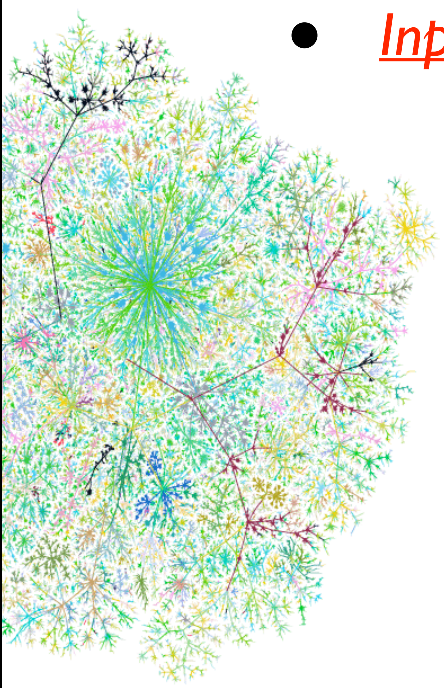

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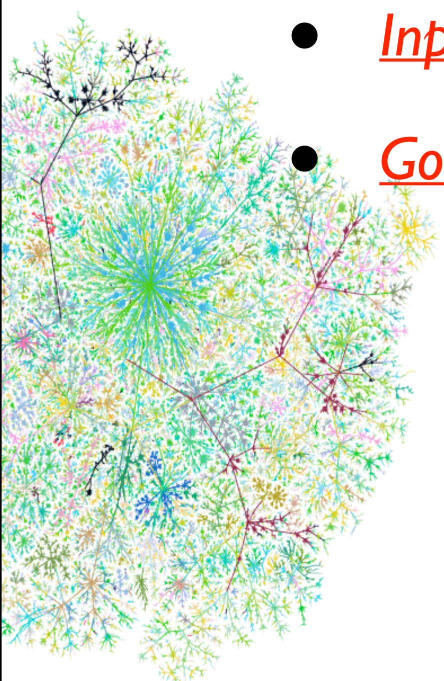
MaxCover Problem



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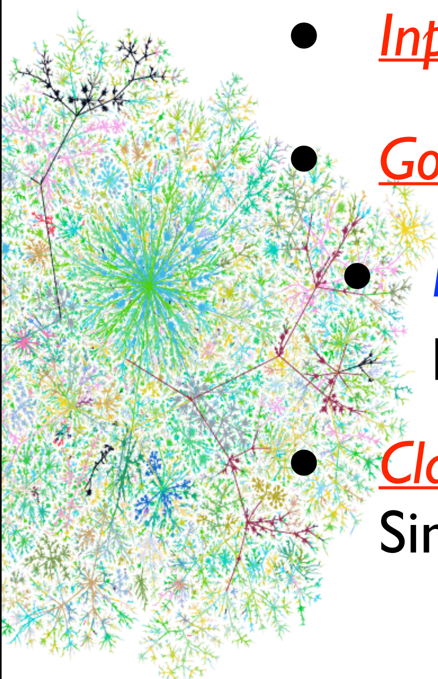

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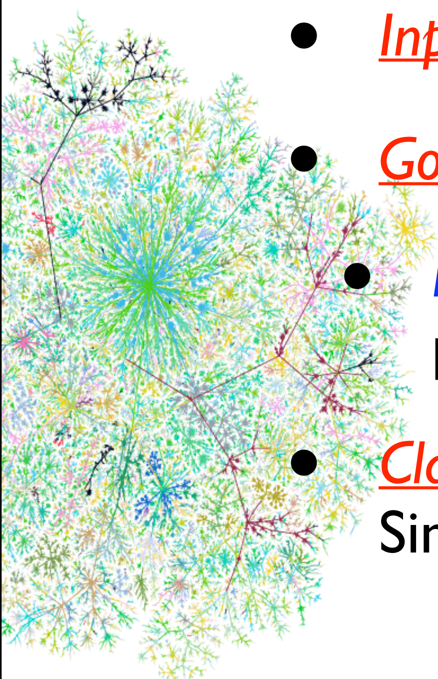

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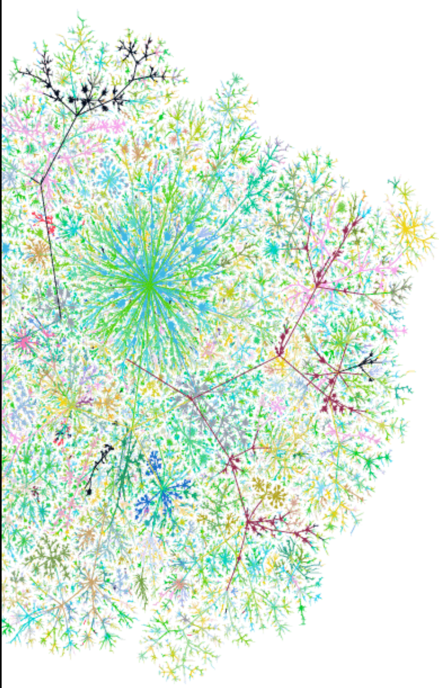
- *Data Streams* Growing body of work on MaxCover and SetCover considers input given as a stream of sets

Assadi et al. [STOC 16], *Warneke et al.* [ESA 23], *Indyk-Vakilian* [PODS 2019], *Yu-Yuan* [SDM 13]
Saha-Getoor [SDM 09], *McGregor-Vu* [ICDT 17], *Bateni et al.* [SPAA 17], *Assadi* [PODS 17]
Demaine et al. [DISC 14], *Indyk et al.* [APPROX 17], *Chakrabarti-Wirth* [SODA 16]
Assadi et al. [SODA 18], *Har-Peled et al.* [PODS 16] *etc.*

Talk Summary



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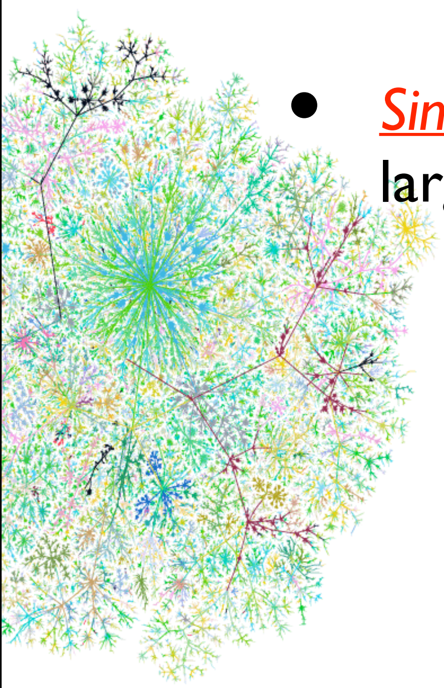

- *Simplification* If sets are small, can throw out many sets. If sets are large, can subsample universe to ensure sets are small-ish.

Older results: [*Demaine et al.* DISC 14], [*McGregor, Vu.* Theory Comput. Syst. 19], [*McGregor et al.* ICDT 21]

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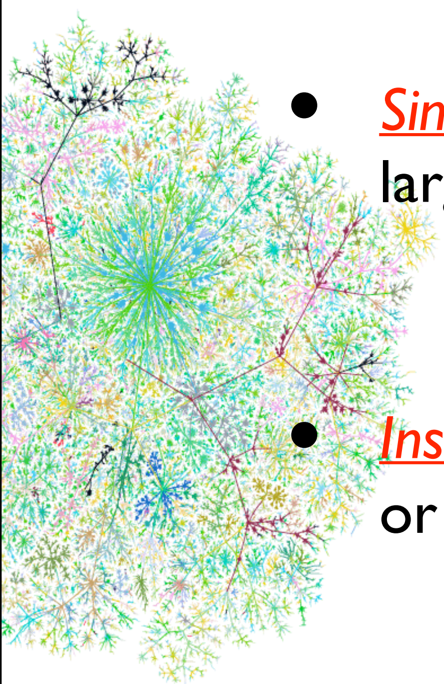

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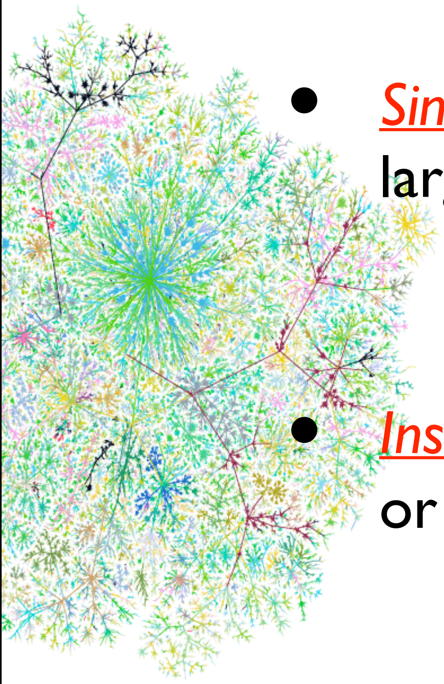

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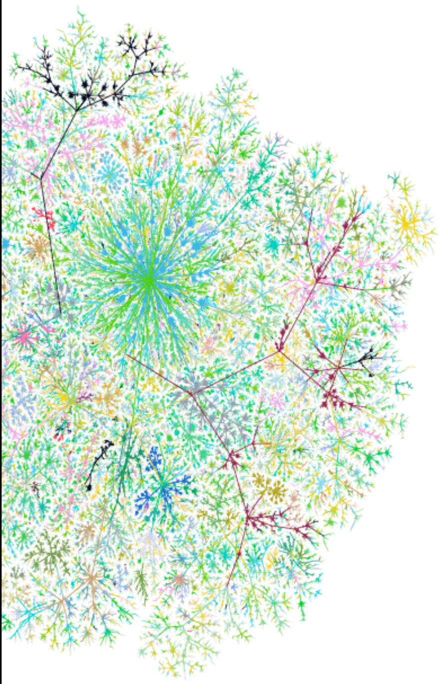
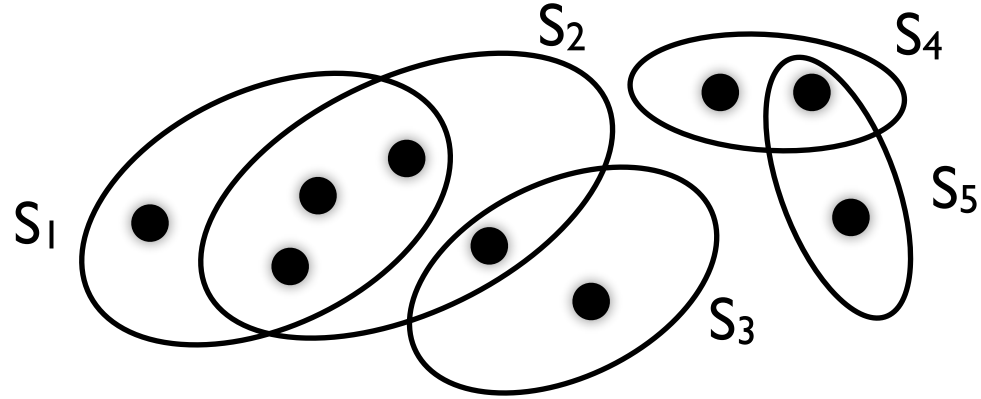
- *Dynamic Streams* $\tilde{O}(\varepsilon^{-2} k)$ space, $O(\log m+\varepsilon^{-1} \log m/\log \log m)$ passes suffice for 1-1/e-ε approx.

  [*Chakrabarti, McGregor, Wirth.* ESA 24]
  Prev. best space bound $\tilde{O}(n+\varepsilon^{-4} k)$ [*Assadi, Khanna.* SODA 18]



1: <u>**Simplification**</u>
2: **Insert Streams**
3: **Dynamic Streams**

Universe Sampling



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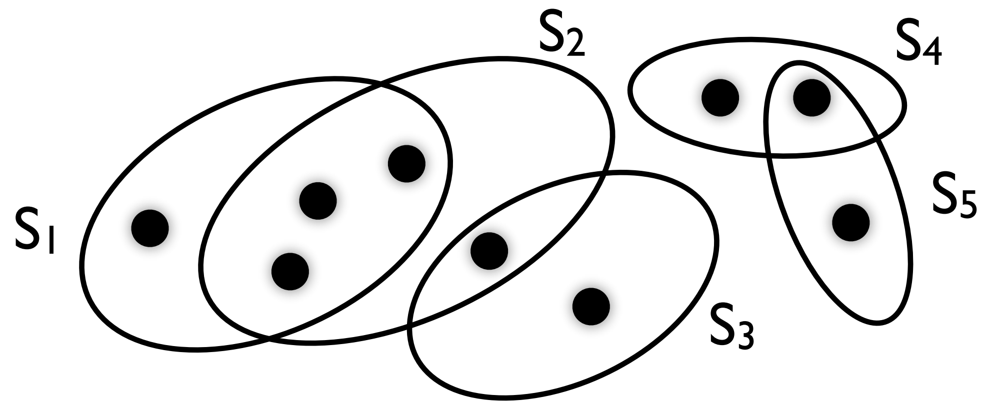

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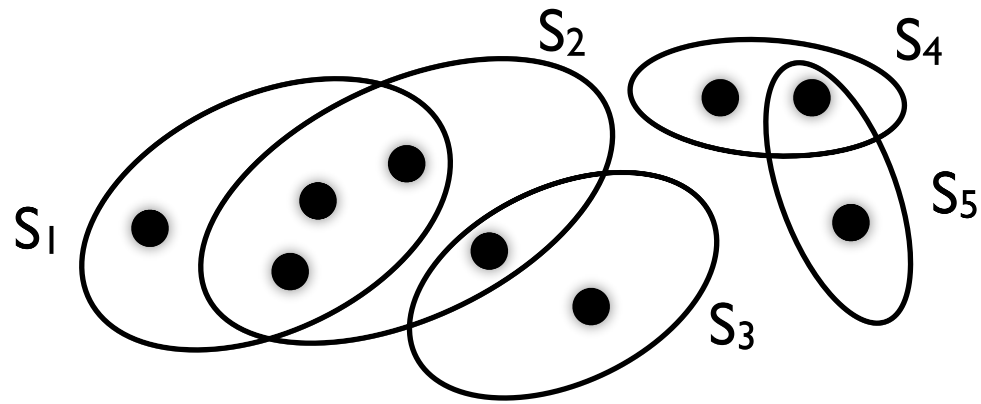

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    - Sample elements with probability $p \approx (k\varepsilon^{-2} \log m)/\mathrm{OPT}$.

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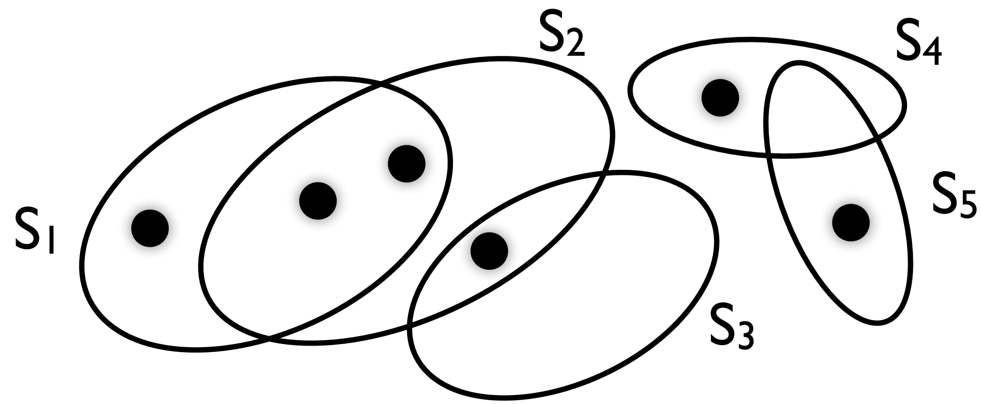

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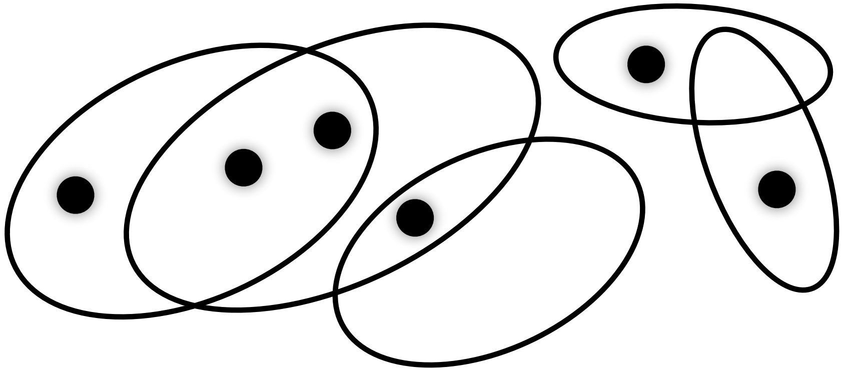

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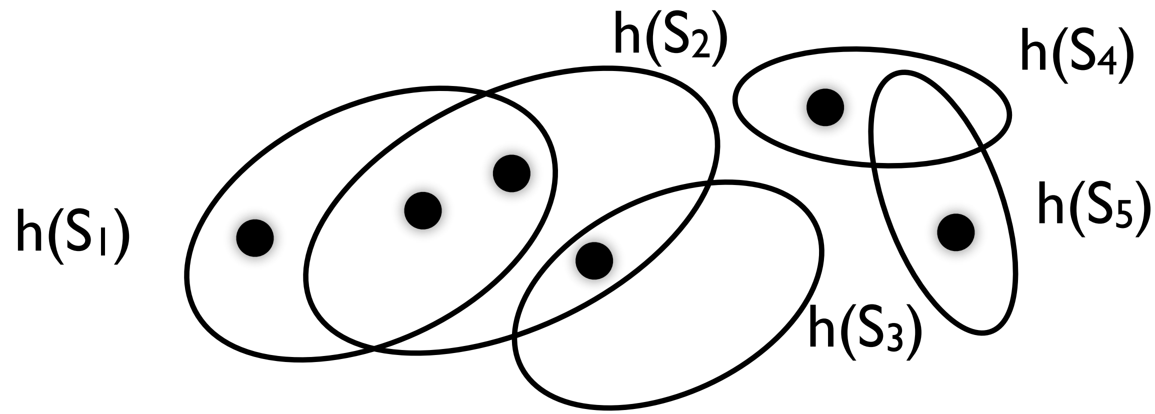

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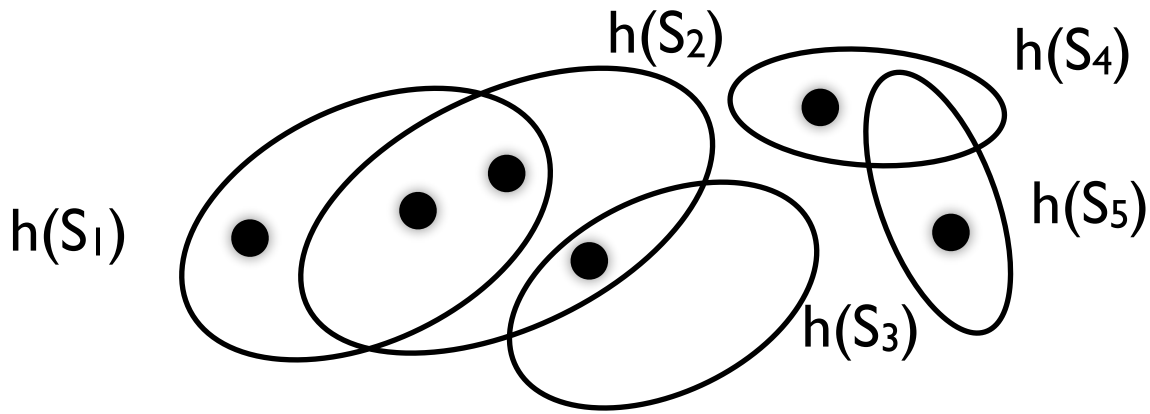

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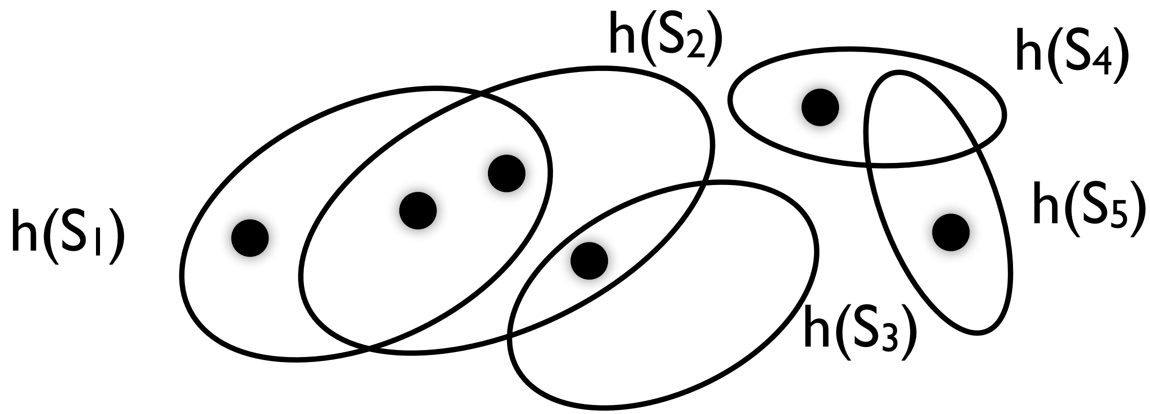

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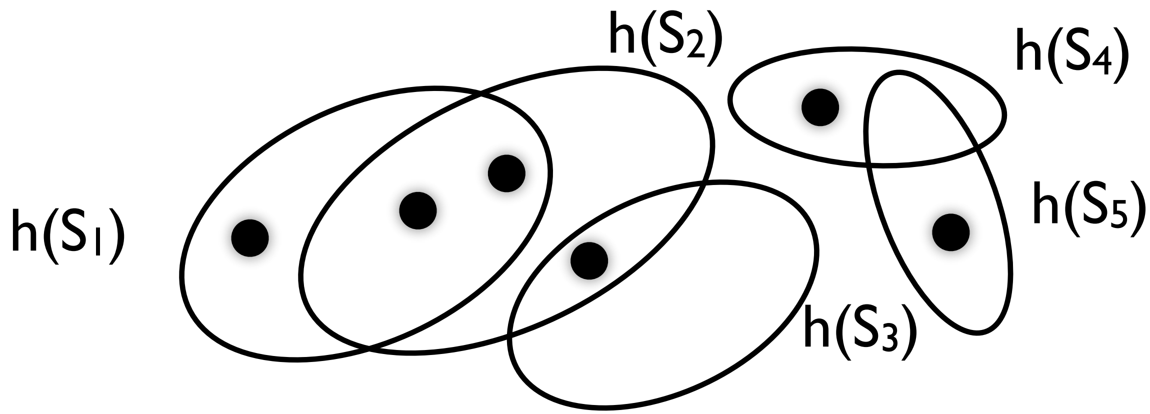

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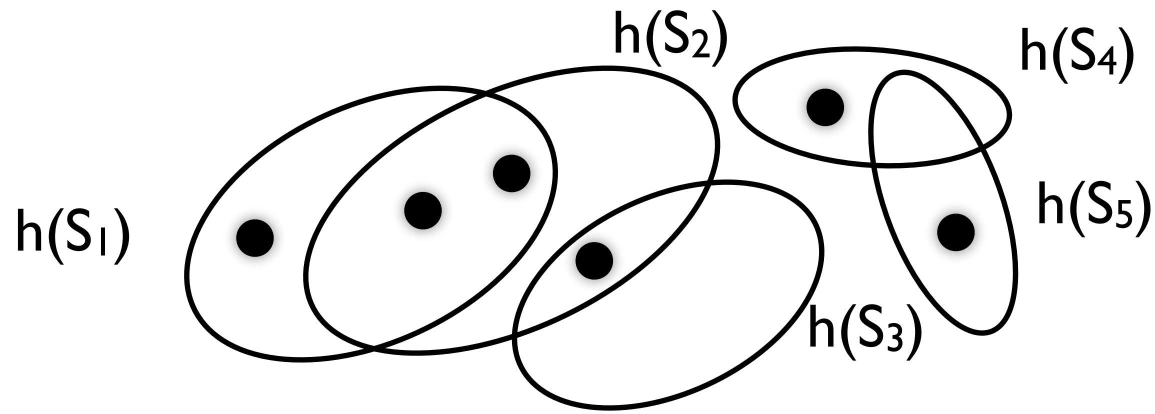

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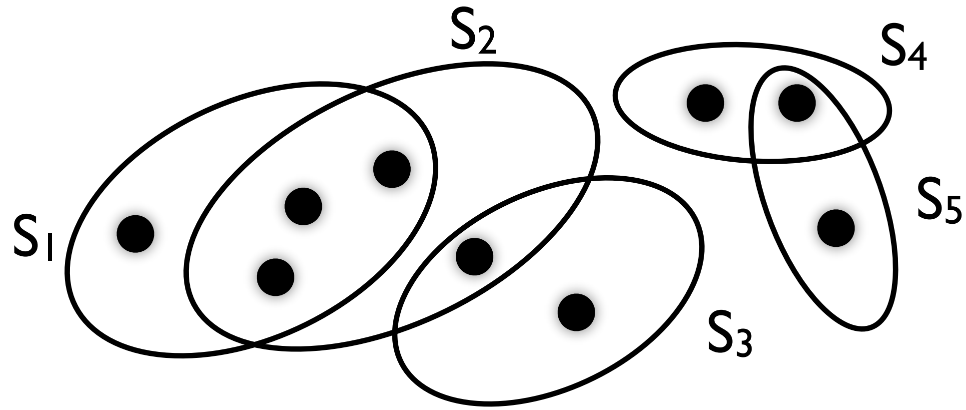
- Optimum solution for $h(S_1),\ldots, h(S_m)$ covers $O(k\varepsilon^{-2} \log m)$ elts.

Simple Set Sampling (Kernelization)



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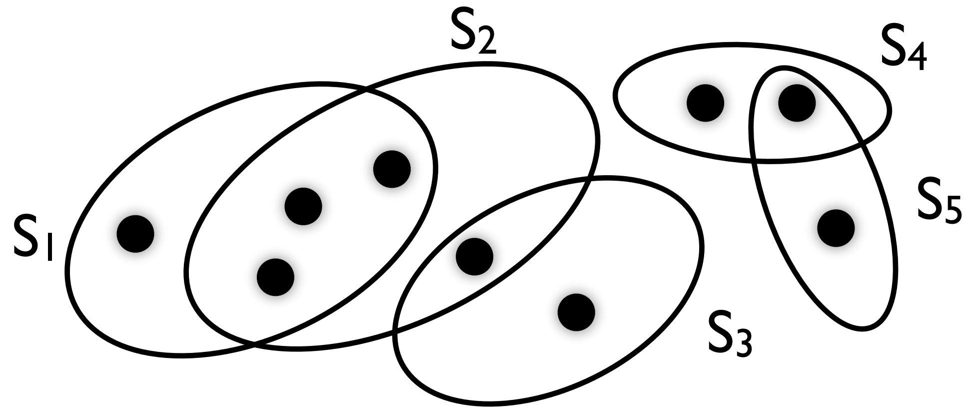

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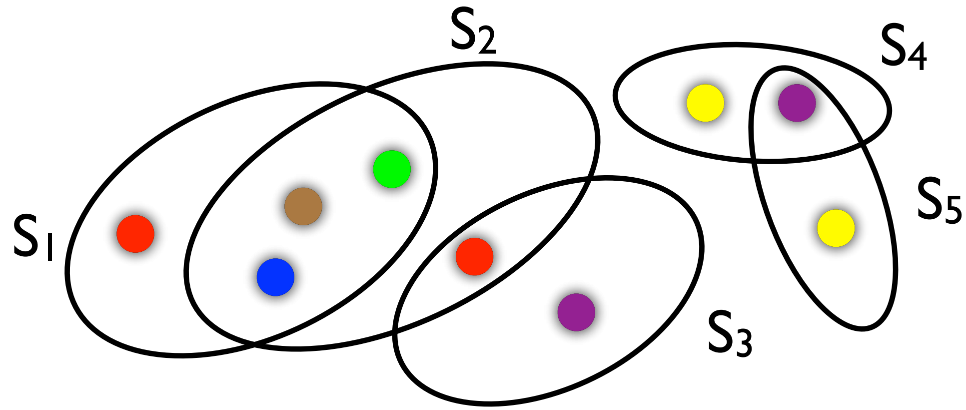

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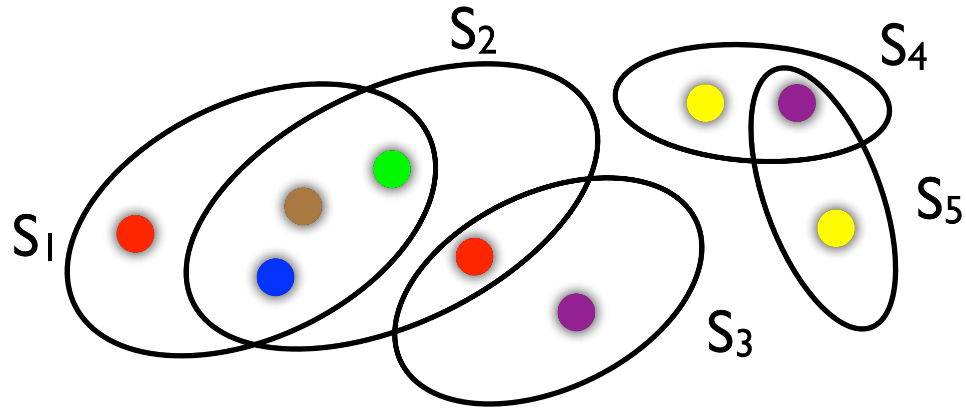

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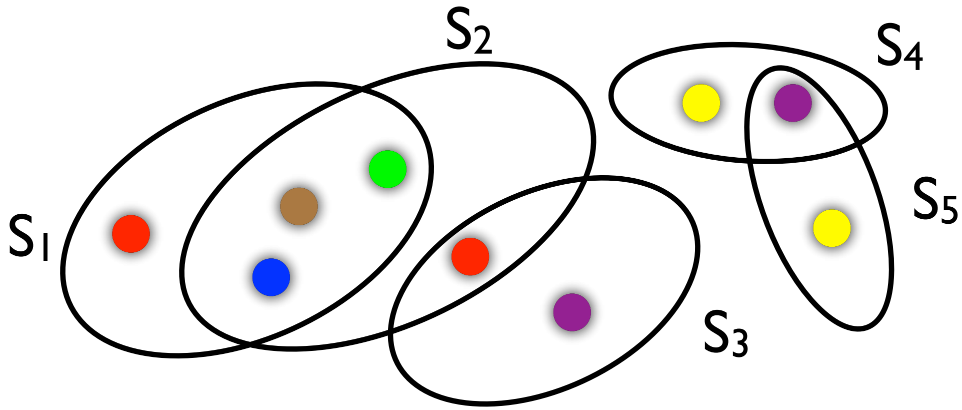
Simple Set Sampling (Kernelization)

Remove one of $\{S_1, S_2\}$ and one of $\{S_4,S_5\}$.



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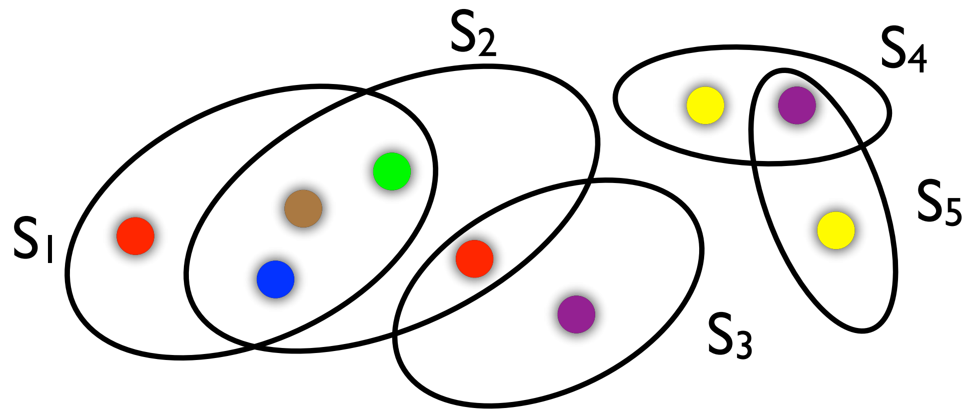

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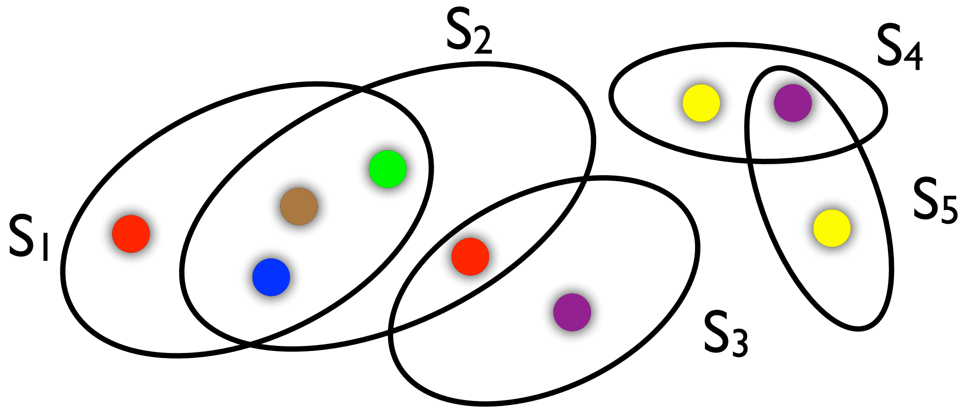

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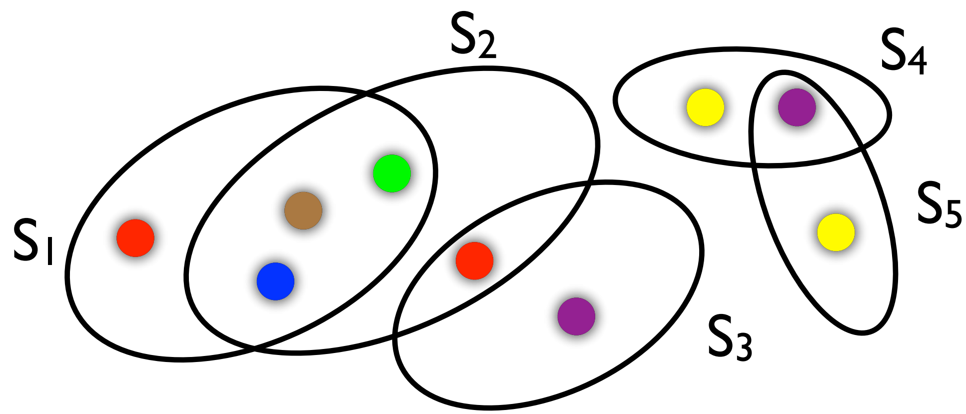

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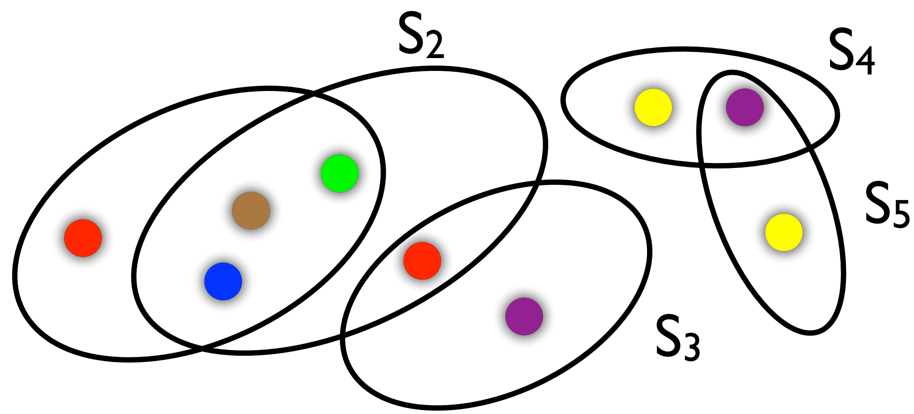

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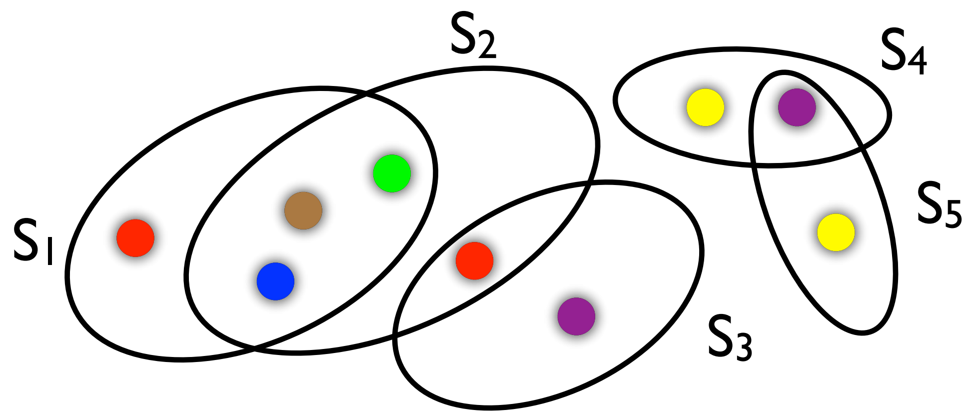

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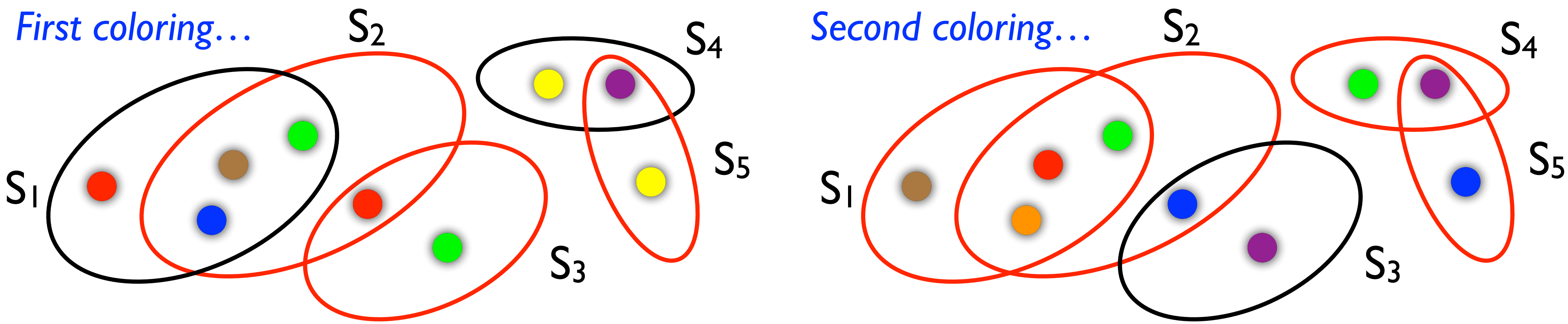
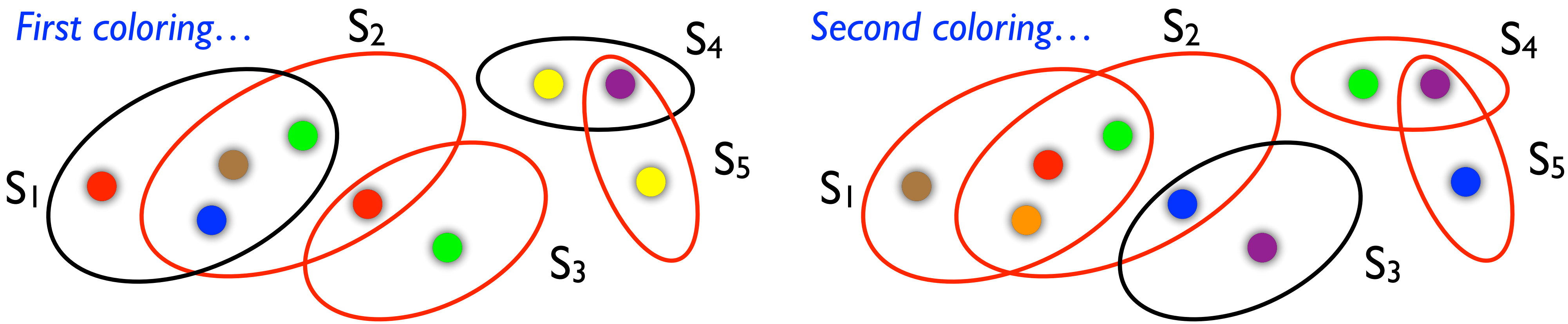
$$|\text{color}(O_1 \cup \ldots \cup O_k)| = |\text{color}(R_1 \cup \ldots \cup R_k)| \leq |R_1 \cup \ldots \cup R_k|$$

Superior Set Sampling



- <u>*Superior Set Sampling*</u>
    - Let $\mathrm{color}_1, \ldots, \mathrm{color}_t : [n] \rightarrow [c]$ be t random hash functions

Superior Set Sampling



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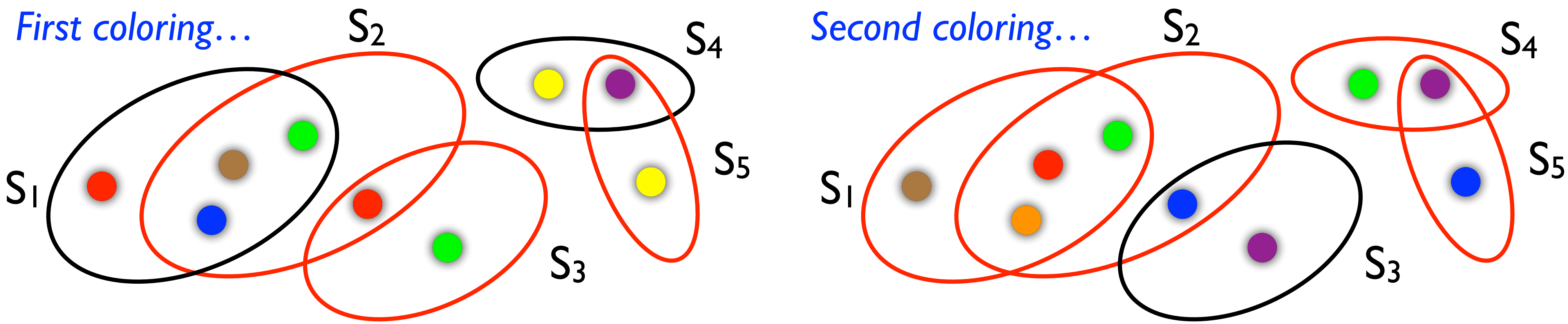
  - For $j \in [t]$, retain maximal collection of sets with distinct $color_j(\cdot)$

Superior Set Sampling



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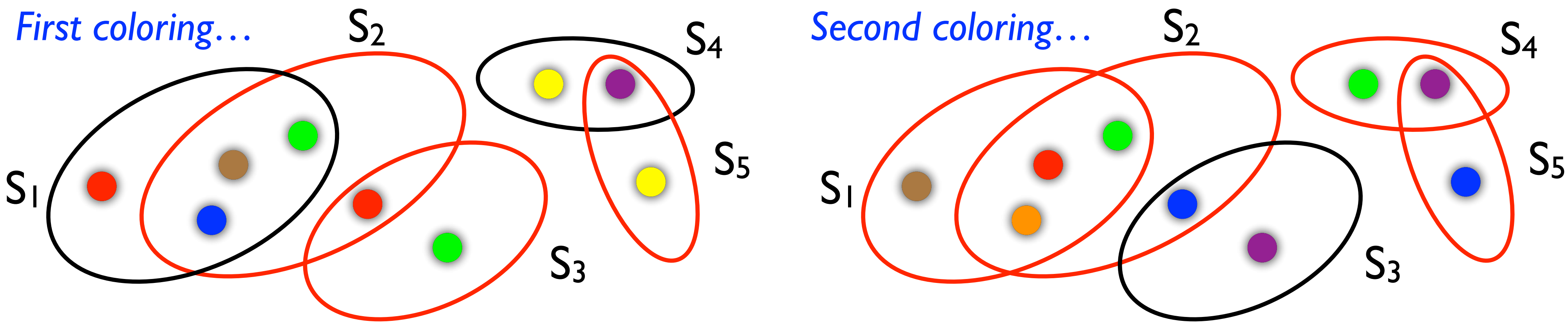

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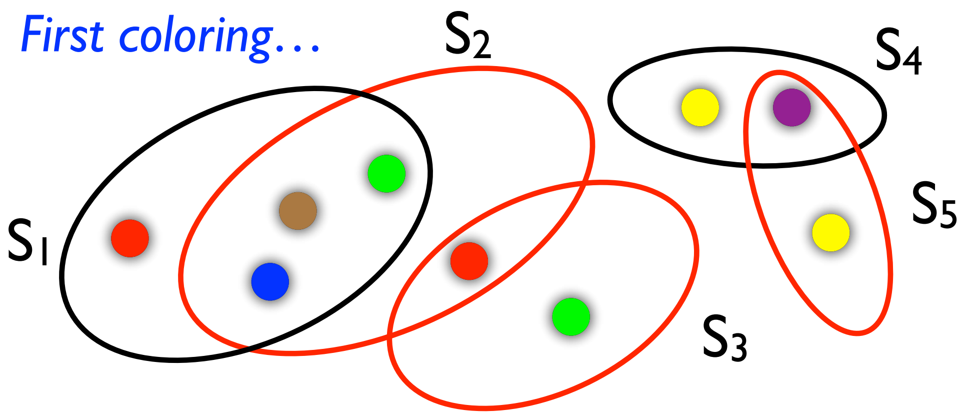
- *Proof Approach* Fix opt solution. In each coloring, for each opt set with constant prob we sample that set or a set just as good…

Superior Set Sampling: Proof Idea

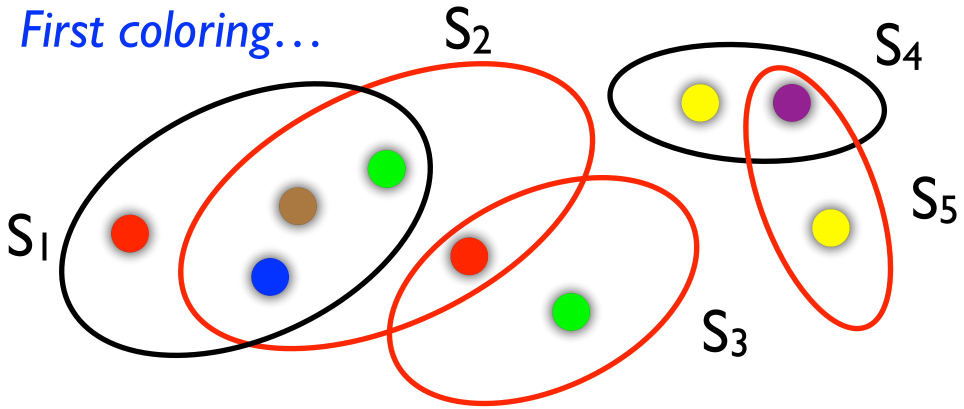


$O_1=S_1$, $O_2=S_3$, $O_3=S_4$
$R_1=S_2$, $R_2=S_3$, $R_3=S_5$

- Let O_i be opt set and R_i set retained with color(R_i)=color(O_i).

Superior Set Sampling: Proof Idea

First coloring…



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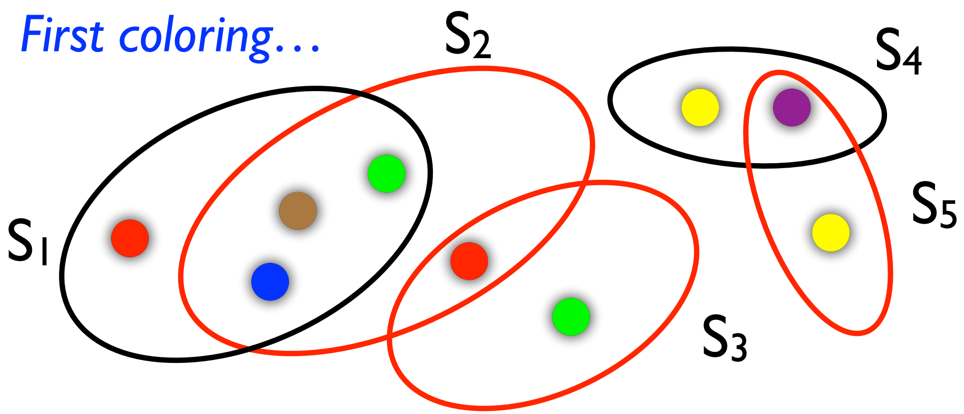

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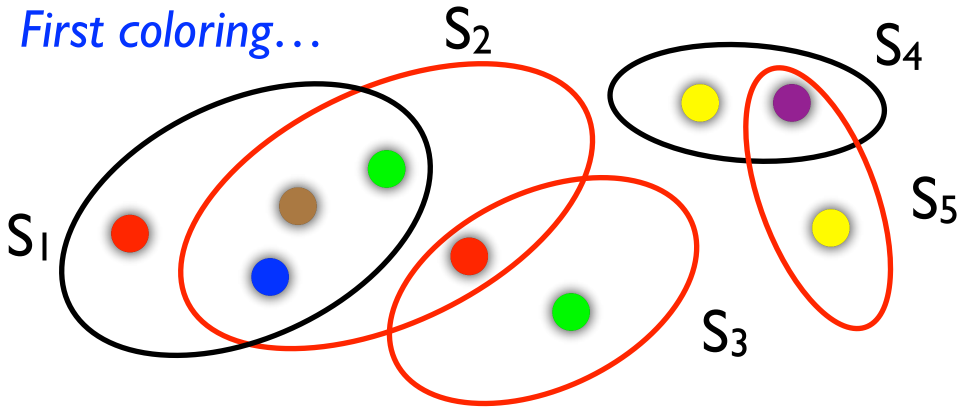

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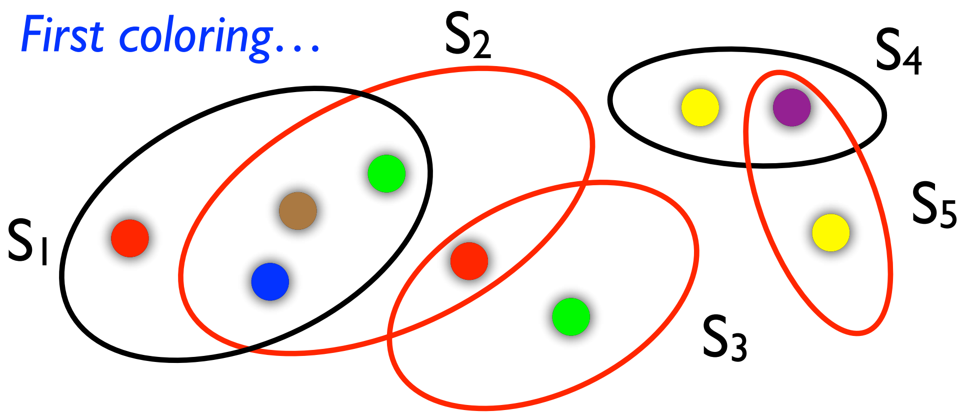

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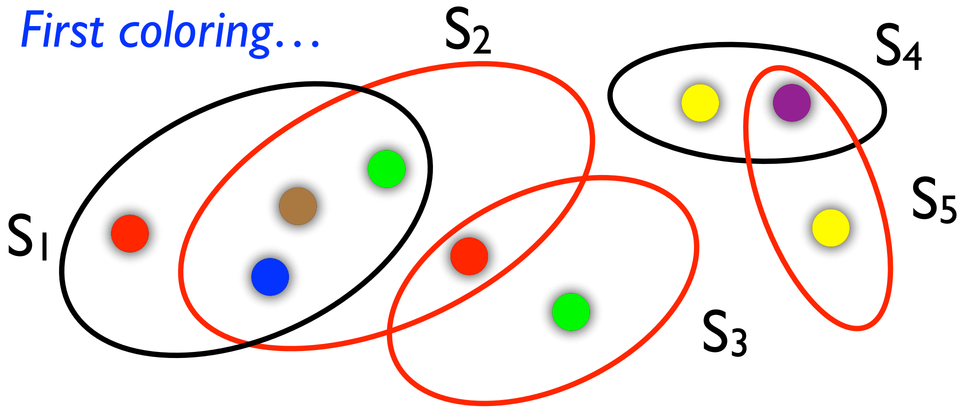

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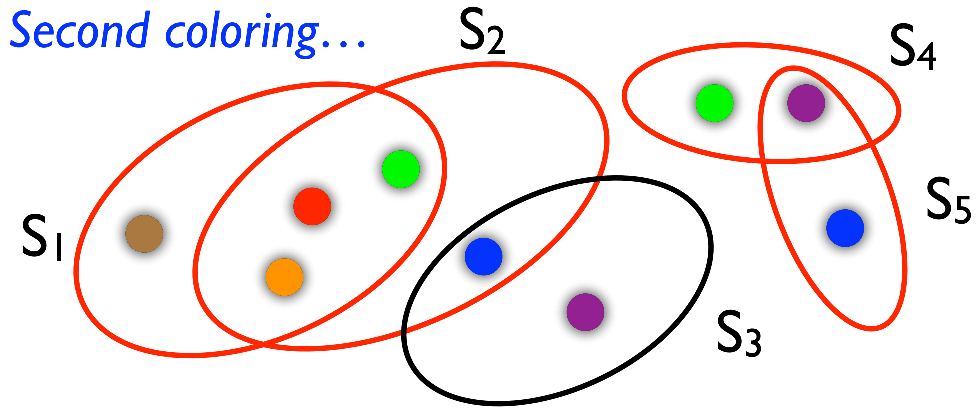

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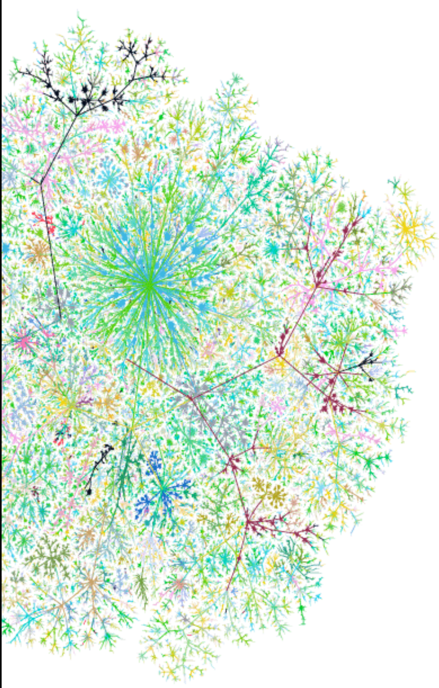

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2: Insert Streams

3: Dynamic Streams

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- *Theorem* 1 pass, $\tilde{O}(\varepsilon^{-3} k)$ space, $1/2-\varepsilon$ approx algorithm.
- *Proof* Combine sub-sampling approach with threshold algorithm. Extra $\tilde{O}(\varepsilon^{-1})$ arises from having to guess OPT up to $1+\varepsilon$ factor.

Multi-Pass Algorithm

- **In pass $p = 1, \ldots, O(1/\varepsilon)$:**
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- *Theorem* $O(\varepsilon^{-1})$ pass $1-1/e-\varepsilon$ approx using $\tilde{O}(\varepsilon^{-2}\ k)$ space.
- *Proof* Approx factor follows from analysis of greedy algorithm. Universe subsampling and fact we only need 2-approx of OPT.

Random Order Algorithm

- Recall, function $f: 2^{[m]} \to \mathbb{R}$ is sub-modular if for $X \subset Y \subseteq [m]$, $i \notin Y$

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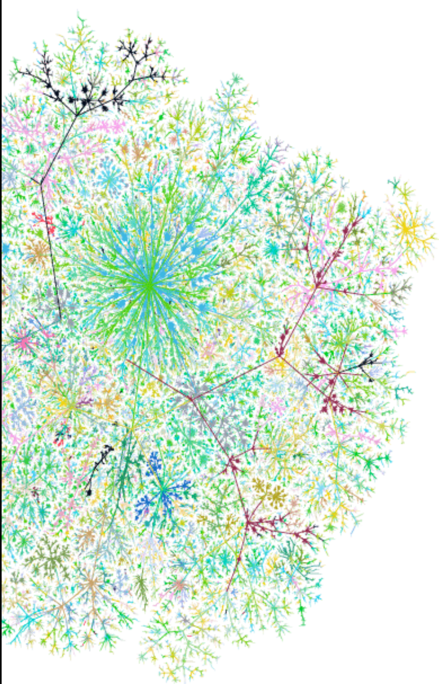
- *With more work…* Can reduce space dependence to linear in k.
  Chakrabarti, McGregor, Wirth [ESA 24]



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- A more careful analysis, gives O(log m/log log m) passes.

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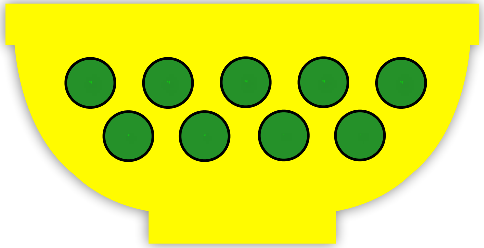

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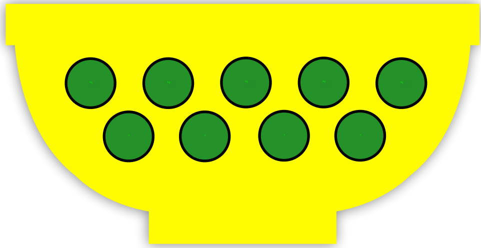

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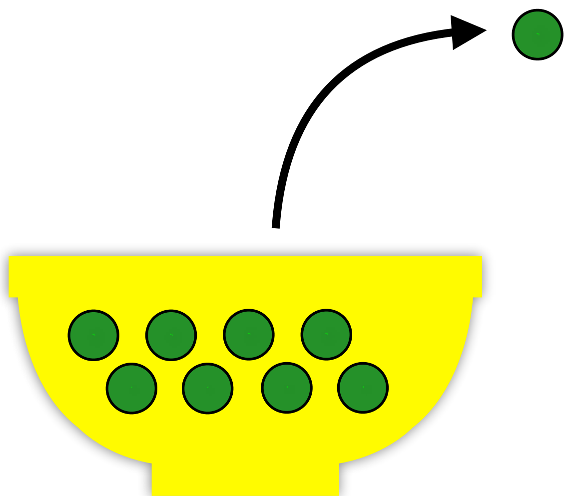

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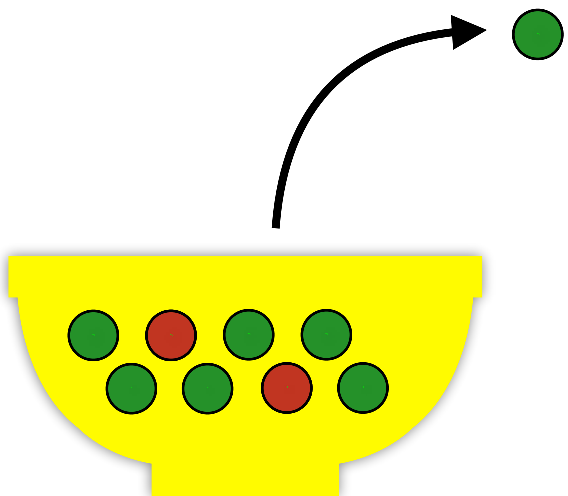

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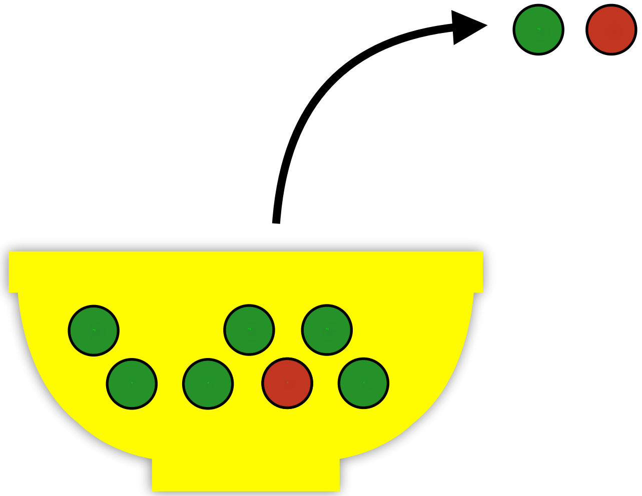

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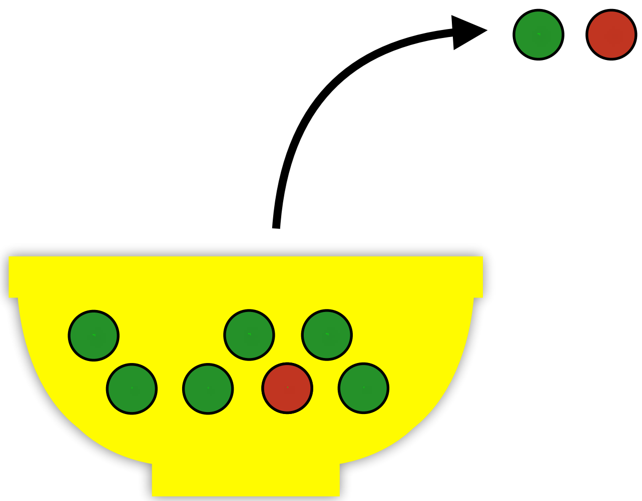

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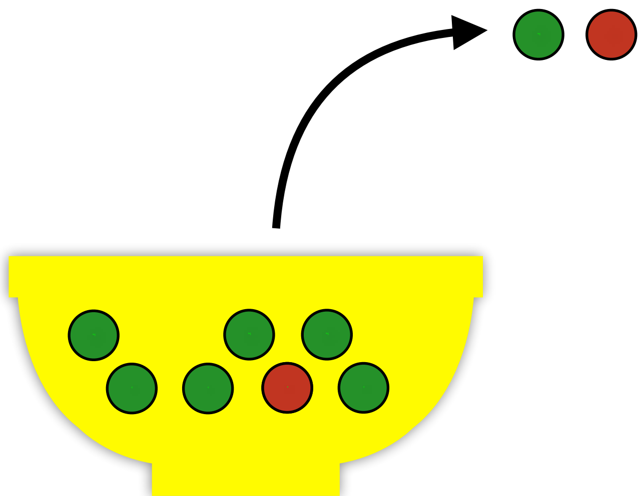

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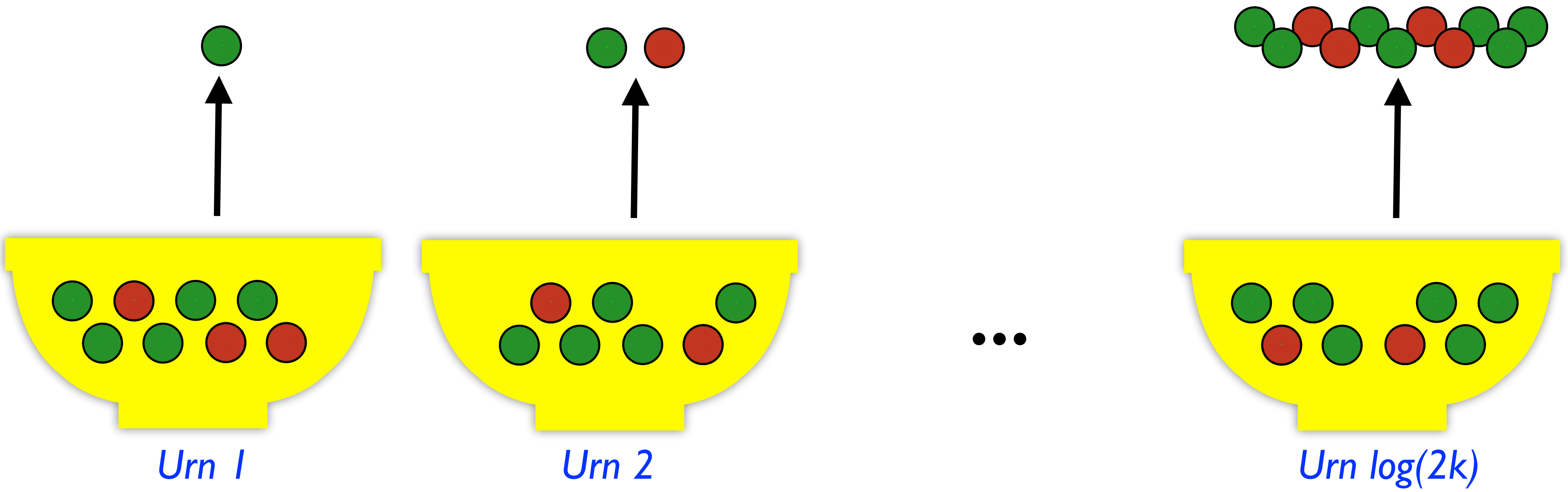
Proof Idea of Lemma: Cascading Urns



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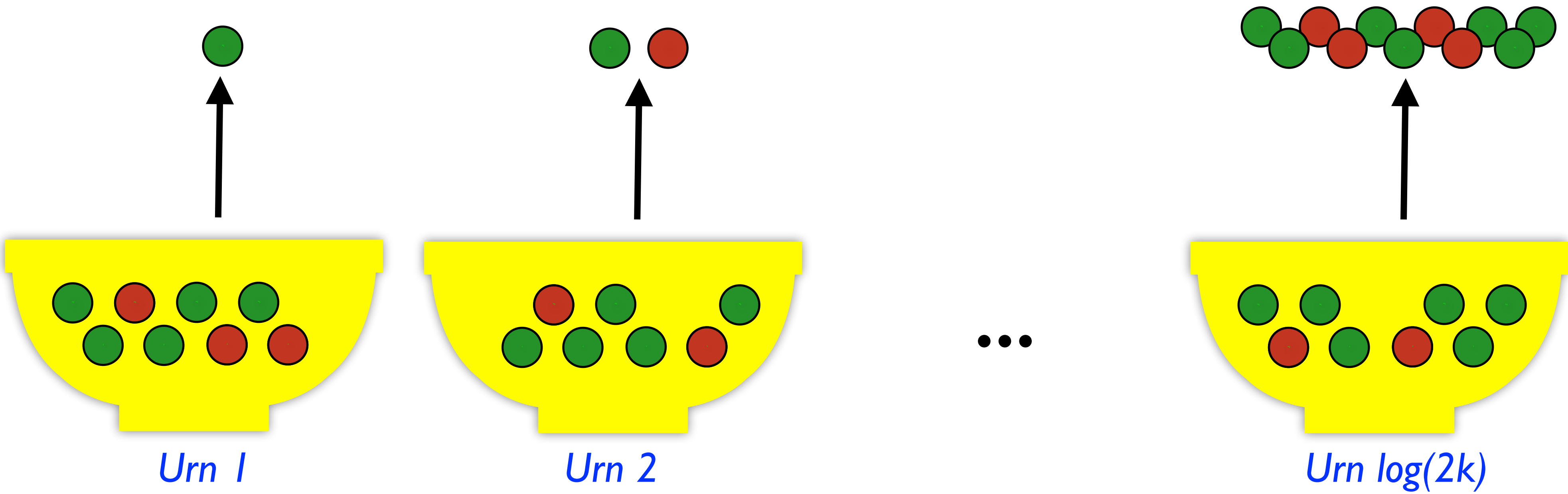

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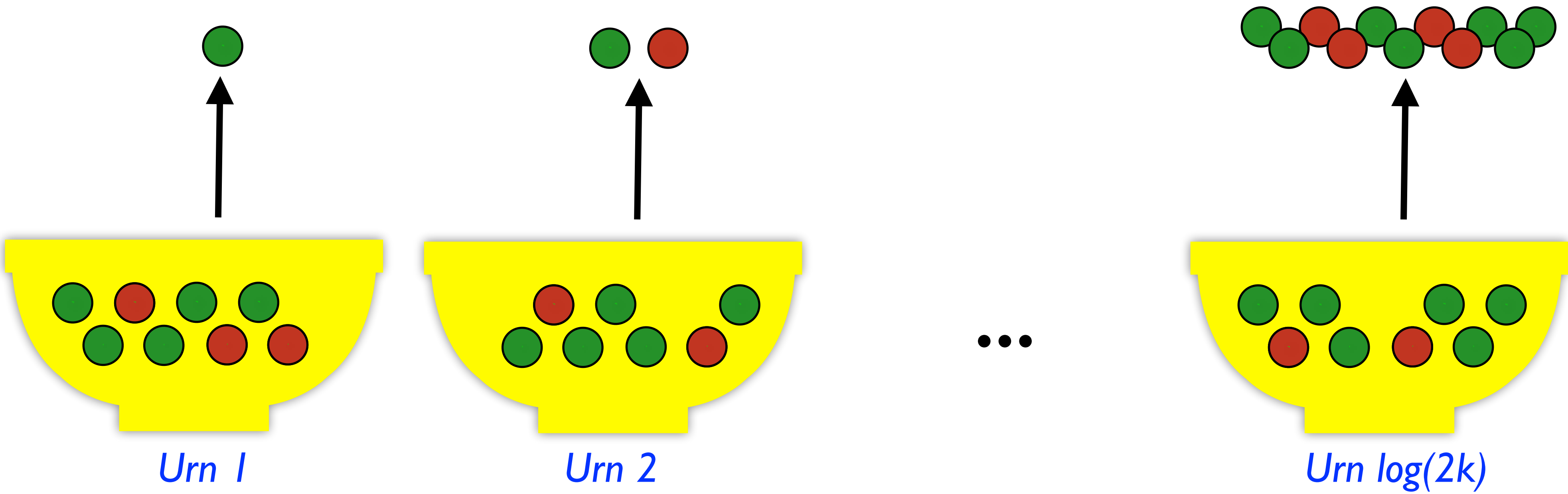

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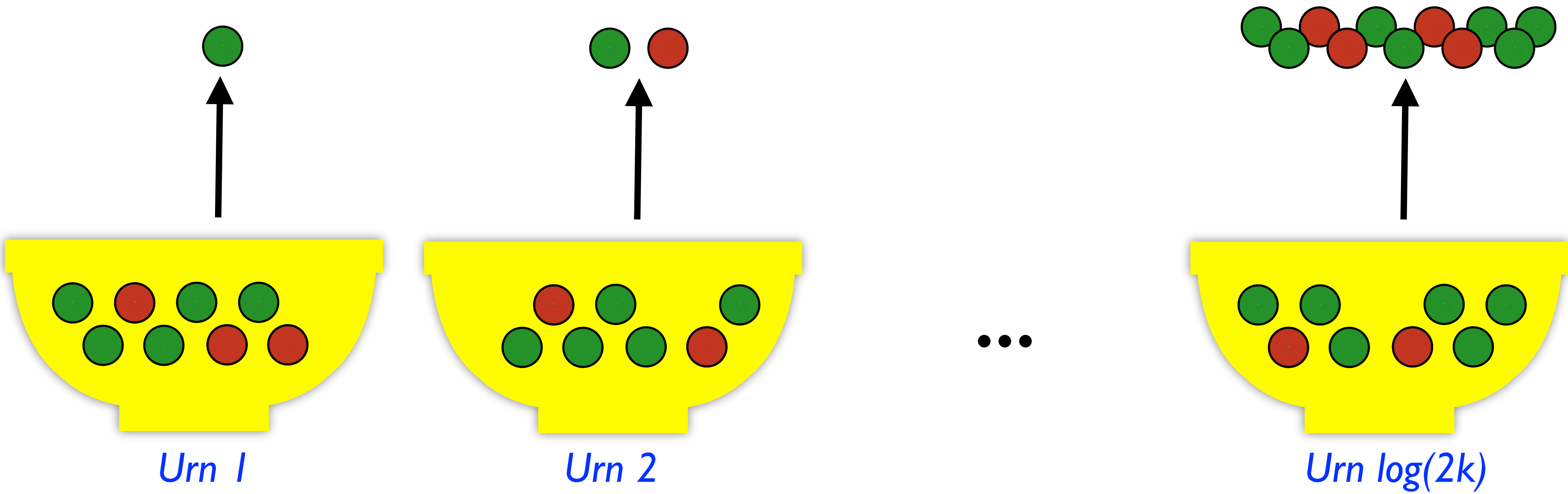

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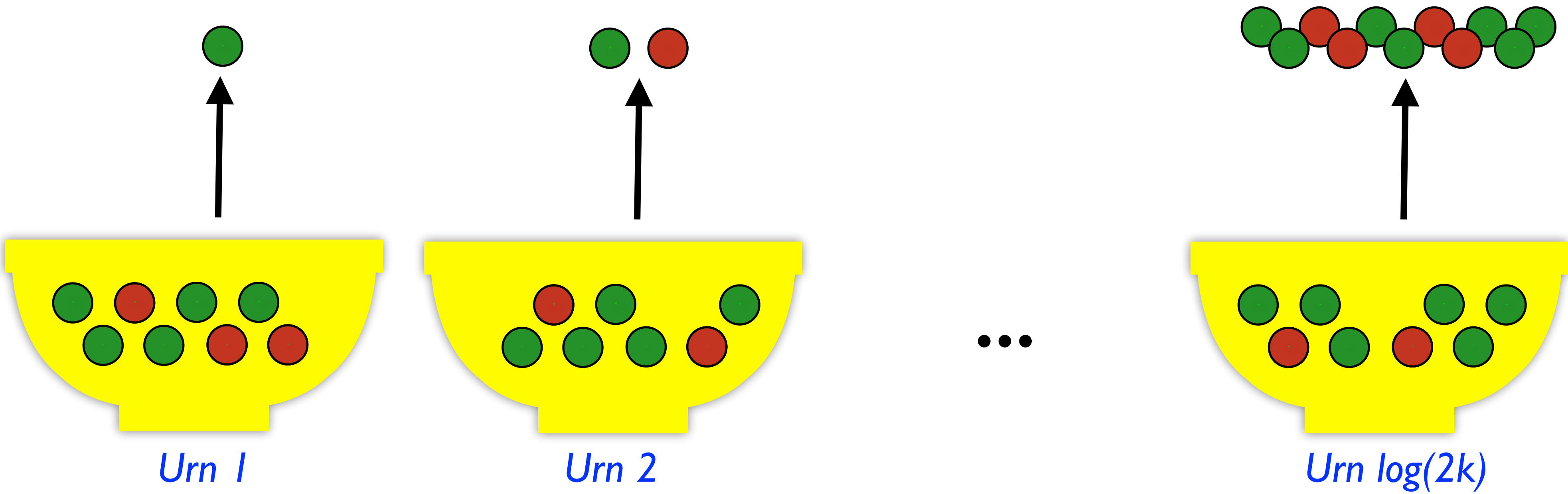

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- At end of each pass useless balls are removed and potentially placed in later urns, i.e., balls "cascade".

Getting 1-1/e-ε Approx

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- **In series for $i\in\{1, \ldots, \log_{1+\varepsilon} e\}$**
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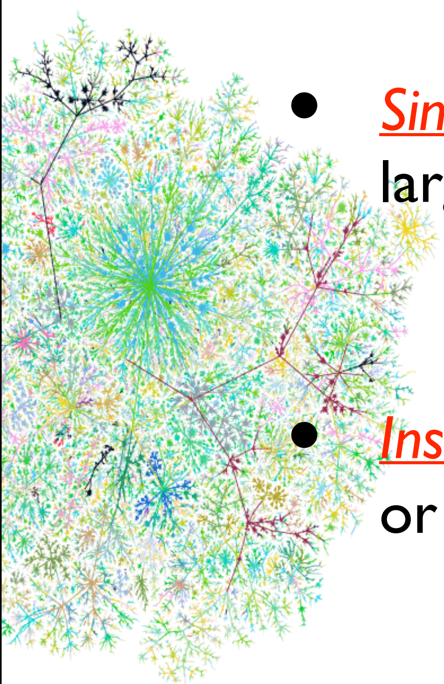
- *Theorem* $O(\log m + \varepsilon^{-1} \log m/\log\log m)$ pass, $\tilde{O}(\varepsilon^{-2}k)$-space, $1-1/e-\varepsilon$ approx. in the dynamic set stream model,

Summary of Talk



- *Simplification* If sets are small, can throw out many sets. If sets are large, can subsample universe to ensure sets are small-ish.

  Older results: [*Demaine et al.* DISC 14], [*McGregor, Vu.* Theory Comput. Syst. 19], [*McGregor et al.* ICDT 21]

- *Insert-Only Streams* $\tilde{O}_\varepsilon(k)$ space suffices for 1/2-ε approx (1-pass) or 1-1/e-ε (1-pass random order $O(1/\varepsilon)$-pass arbitrary order).

  Prev. best space bound for random order $\tilde{O}_\varepsilon(k^2)$ [*Warneke et al.* ESA 23]

- *Dynamic Streams* $\tilde{O}(\varepsilon^{-2} k)$ space, $O(\log m+\varepsilon^{-1} \log m/\log \log m)$ passes suffice for 1-1/e-ε approx.

  [*Chakrabarti, McGregor, Wirth.* ESA 24]
  Prev. best space bound $\tilde{O}(n+\varepsilon^{-4} k)$ [*Assadi, Khanna.* SODA 18]

Thanks!