Polynomial-time preparation of low-temperature Gibbs states for 2D toric code

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Workshop: Mathematical Challenges of Quantum Algorithms for Open Quantum Systems

Joint work with Bowen Li, Lin Lin, Ruizhe Zhang
arXiv/2410.01206

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Outlines:

- Motivation

Why we care about this problem?

- Introduction

2D toric code, Davies, Spectral gap

- Main result and implications

New Davies, spectral gap under low temperature

Mixing time of Lindbladian

for

low-temperature Gibbs states preparation

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Motivation 1:

Fast mixing implies efficient state preparation
from any initial state

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Efficiently implementable Lindbladian: for general (non-commuting) Hamiltonian

- Thermal state:

  Cost $\sim O(\beta t_{\text{mix}} \text{polylog}(1/\epsilon))$

  [Chen, Kastoryano, Brandão, Gilyén, arXiv:2303.18224]
  [Chen, Kastoryano, Gilyén, arXiv:2311.09207]
  [**Ding**, Li, Lin, arXiv/2404.05998] …….

- Ground state:

  Cost $\sim O(\text{poly}(t_{\text{mix}}/\epsilon))$

  [**Ding**, Chen, Lin, PRR, arXiv/2308.15676]

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Motivation 1:

Fast mixing implies efficient state preparation
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Efficiently implementable Lindbladian:

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- Ground state:

  Cost $\sim O(\mathrm{poly}(t_{\mathrm{mix}}/\epsilon))$

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Question: What is t_{mix}?

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Existing result:

- General local (commuting) Hamiltonian:

[Kastoryano, Brandao, CMP, 2016], [Bardet, Capel et al, PRL, 2023], [Kochanowasi, Alhambra et al, arXiv/2404.16780, 2024], [Rouze, Franca et al, arXiv:2403.12691, 2024] …

For high temperature ($\beta \ll 1$), or moderate-temperature 1D commuting ($\beta = \Theta(1)$)

Large β-dependence is not clear

Question: What is t_{mix} for large β?

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Motivation 2:

Fast low temperature mixing might be helpful for
passively protected quantum memory

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Motivation 2:

Fast low temperature mixing might be helpful for
passively protected quantum memory

$$\partial_t \rho_t = \mathscr{L}_e(\rho_t), \quad \rho(0) \in \mathscr{C} \qquad \mathscr{D} : \rho \to \mathscr{C}$$

Noise

Code space
(Logic information)

Decode
(obtain logic information)

Self correcting quantum memory:

$$\inf_t \left\{ \left\| \mathscr{D}(\rho(t)) - \rho(0) \right\| \leq \epsilon \right\} = \Omega(\exp(n))$$

Similar to slow mixing.

Motivation 2:

Fast low temperature mixing might be helpful for
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$$\partial_t \rho_t = \mathscr{L}_e(\rho_t) + \mathscr{L}_p(\rho_t), \quad \rho(0) \in \mathscr{C}$$

Noise

Protector

Motivation 2:

Fast low temperature mixing might be helpful for
passively protected quantum memory

$$\partial_t \rho_t = \mathscr{L}_e(\rho_t) + \mathscr{L}_p(\rho_t), \quad \rho(0) \in \mathscr{C}$$

Noise

Protector

Passively protected quantum memory:

$$\left\| \mathscr{D}(\rho_t) - \rho_0 \right\| \leq \exp(-\Theta(n)) \text{ for any } t > 0$$

Preserve logic information

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Example:

Logic qubits are encoded in ground space $\mathscr{G}(H)$ of H

$$\begin{cases} \partial_t \rho_t = \mathscr{L}_e(\rho_t) + \mathscr{L}_{\beta=\infty}(\rho_t), \\ \rho_0 \in \mathscr{G}(H) \end{cases}$$

$$\mathscr{D}(\rho) = \frac{P_{\mathscr{G}} \rho P_{\mathscr{G}}}{\mathrm{Tr}(P_{\mathscr{G}} \rho P_{\mathscr{G}})}$$

Goal: $\left\| \mathscr{D}(\rho_t) - \rho_0 \right\| \sim \exp(-\Theta(n)), \quad \forall t > 0$

Example: Logic qubits are encoded in ground space $\mathscr{G}(H)$ of H

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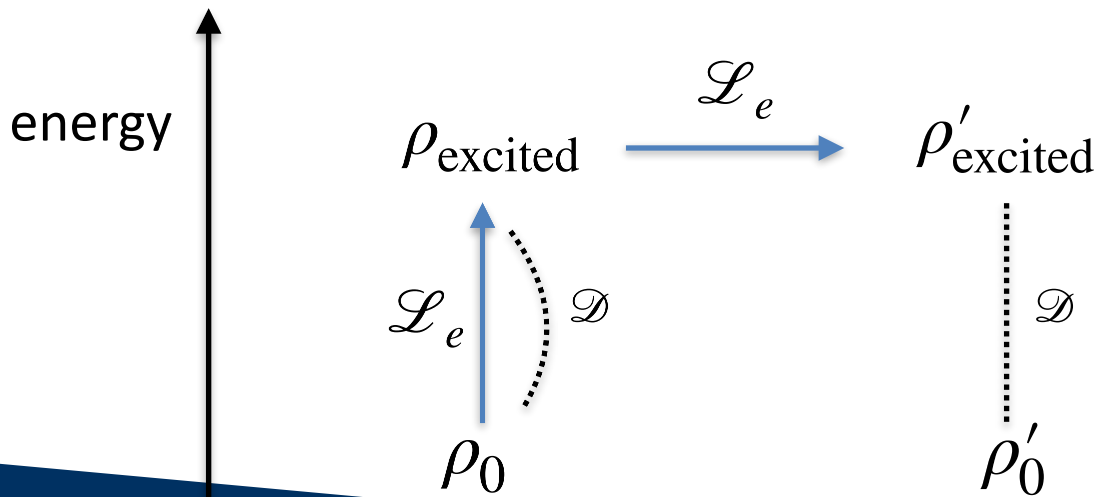
Goal: $\left\| \mathscr{D}(\rho_t) - \rho_0 \right\|_{L_t^\infty} \sim \exp(-\Theta(n))$

How does the logic information change?



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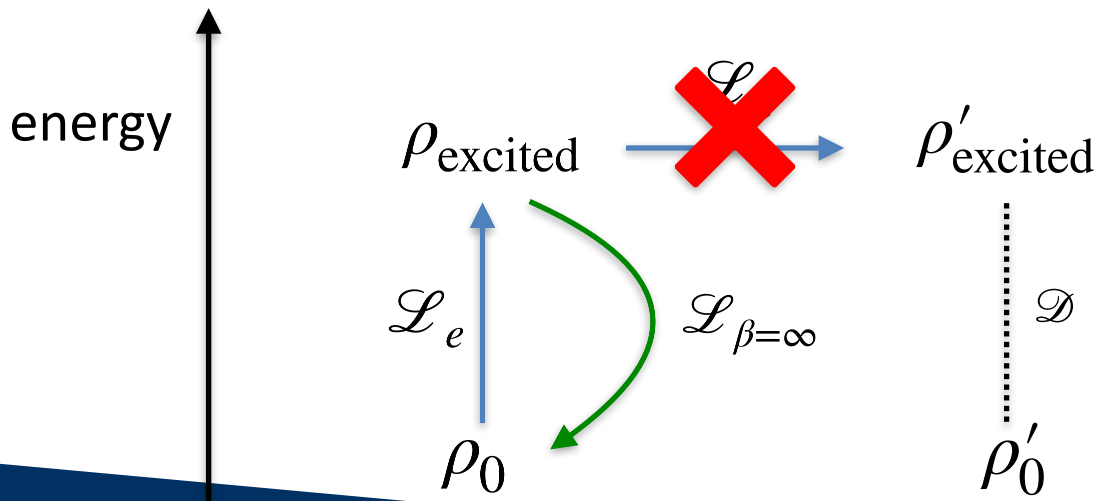
Goal: $\left\| \mathscr{D}(\rho_t) - \rho_0 \right\|_{L_t^\infty} \sim \exp(-\Theta(n))$

$\mathscr{L}_{\beta=\infty}$ prohibits the transition in the second step:



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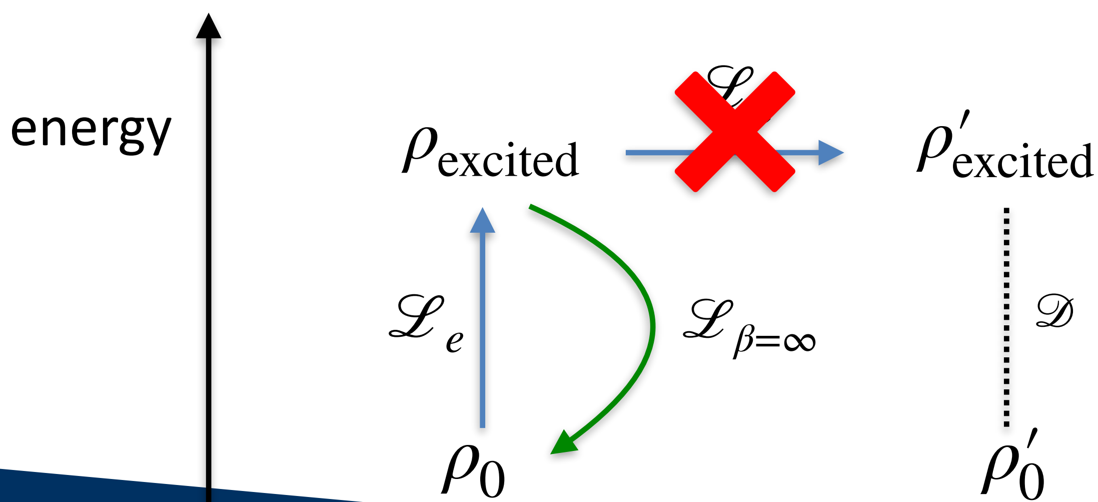

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Candidate for a Passively Protected Quantum Memory in Two Dimensions

Simon Lieu, Yu-Jie Liu, and Alexey V. Gorshkov
Phys. Rev. Lett. **133**, 030601 – Published 16 July 2024

2D Ising

Mixing time of Lindbladian

for

low-temperature Gibbs states preparation

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Spectral gap of Davies generator

for

low-temperature 2D toric code

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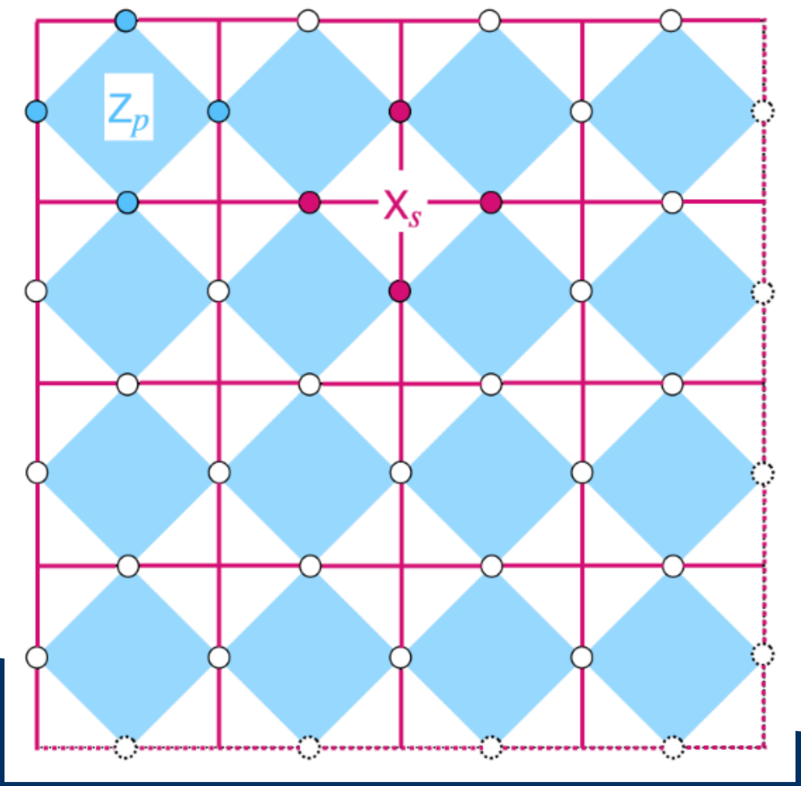
2D toric code

$$H^{\text{toric}} = -\sum_s \mathsf{X}_s - \sum_p \mathsf{Z}_p\,,$$

$$\mathsf{X}_s = \prod_{i \in s} \sigma_i^x\,,$$

$$\mathsf{Z}_p = \prod_{i \in p} \sigma_i^z$$

$$n = 2L^2$$



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2D toric code

$$H^{\text{toric}} = -\sum_s \mathsf{X}_s - \sum_p \mathsf{Z}_p \,,$$

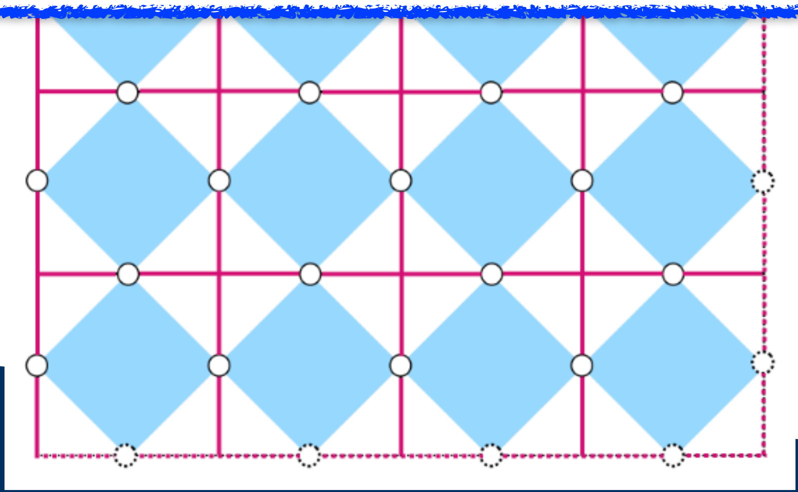
Prepare:

$$\sigma_\beta \propto \exp\left(-\beta H^{\text{toric}}\right), \qquad \beta \gg 1$$

$$\mathsf{X}_s = \prod_{i \in s}$$

$$\mathsf{Z}_p = \prod_{i \in p} \sigma_i^z$$

$$n = 2L^2$$



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Davies generator:

$$\partial_t \rho = \sum_{a\in\mathscr{A}} \sum_{\omega \in B_H} \gamma(\omega)\left(S_a(\omega)\,\rho\, S_a^\dagger(\omega) - \frac{1}{2}\left\{ S_a(\omega)^\dagger S_a(\omega), \rho \right\} \right)$$

- $S_a(\omega) = \sum_{\lambda_i - \lambda_j = \omega} P_{\lambda_i} A^a P_{\lambda_j}$ Transite $|\psi_j\rangle$ to $|\psi_i\rangle$
- $\gamma(\omega) = \exp(-\beta\omega)\gamma(-\omega)$ Transition correction

+ $\{A^a\} = \left\{ \sigma_i^x, \sigma_i^y, \sigma_i^z \right\}_{i=1}^n$ $\Rightarrow$ $\sigma_\beta \propto \exp(-\beta H)$ is the unique fixed point

local Paulis

$\rho(t) \to \sigma_\beta$

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GNS detail balance condition:

$$\langle Y, X \rangle_{\sigma_\beta} = \mathrm{tr}(Y^\dagger X \sigma_\beta)$$

GNS inner product

$$\mathscr{L}_\beta(X) = \sum_{a \in \mathscr{A}} \sum_{\omega \in B_H} \gamma(\omega) \left(S_a^\dagger(\omega)\, X\, S_a(\omega) - \frac{1}{2} \left\{ S_a(\omega)^\dagger S_a(\omega), X \right\} \right)$$

is self-adjoint under $\langle \cdot\,,\cdot \rangle_{\sigma_\beta}$

GNS DBC

Spectral gap

$$\mathscr{L}_\beta(X) = \sum_{a\in\mathscr{A}} \sum_{\omega\in B_H} \gamma(\omega)\left(S_a^\dagger(\omega)\,\rho\, S_a(\omega) - \frac{1}{2}\left\{S_a(\omega)^\dagger S_a(\omega), \rho\right\}\right)$$

is self-adjoint under $\langle \cdot\,, \cdot \rangle_{\sigma_\beta}$

Eigenvalue of $\mathscr{L}_\beta$: $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots\ldots$

$$\mathrm{Ker}\left(\mathscr{L}_\beta\right) = \{\mathrm{cI}\}$$

$$\mathrm{Gap}\left(\mathscr{L}_\beta\right) = \lambda_0 - \lambda_1 = \inf_{X\neq 0\,,\mathrm{tr}(\sigma_\beta X)=0} \frac{\left\langle X, -\mathscr{L}_\beta(X)\right\rangle_{\sigma_\beta}}{\langle X, X\rangle_{\sigma_\beta}}$$

characterize the convergence speed of $\exp(\mathscr{L}^\dagger t)$

Main Result

Existing result:

- General local (commuting) Hamiltonian:

[Kastoryano, Brandao, CMP, 2016], [Bardet, Capel et al, PRL, 2023], [Kochanowasi, Alhambra et al, 2024], [Rouze, Franca et al, 2024] …

For high temperature ($\beta \ll 1$), or moderate-temperature 1D commuting ($\beta = \Theta(1)$)

β-dependence is not clear

- For 2D toric code:

[Alicki et al, 2009]

$$\mathrm{Gap}\left(\mathscr{L}_\beta\right) \geq \exp(-\Theta(\beta))$$

Independent of system size

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Existing result:

- For 2D toric code:
  [Alicki et al, 2009]

$$\mathrm{Gap}\left(\mathscr{L}_\beta\right) \geq \exp(-\Theta(\beta))$$

Independent of system size

- Empirical study and energy barrier:

[Freeman, Herdman, et al, PRB, 2014], [Brown, Loss et al, RMP, 2016], [Temme, CMP, 2017], ….

$$\mathrm{Gap}\left(\mathscr{L}_\beta\right) = \exp(-\Theta(\beta))$$

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Existing result:

- For 2D toric code:

Question: Can we modify the local Davies generator such that the gap is independent of β?

Yes. New gap:

$$\mathrm{Gap}\left(\mathscr{L}_\beta\right) \geq \max\left\{\exp(-\Theta(\beta)), \mathrm{poly}(1/n)\right\}$$

New Davies:

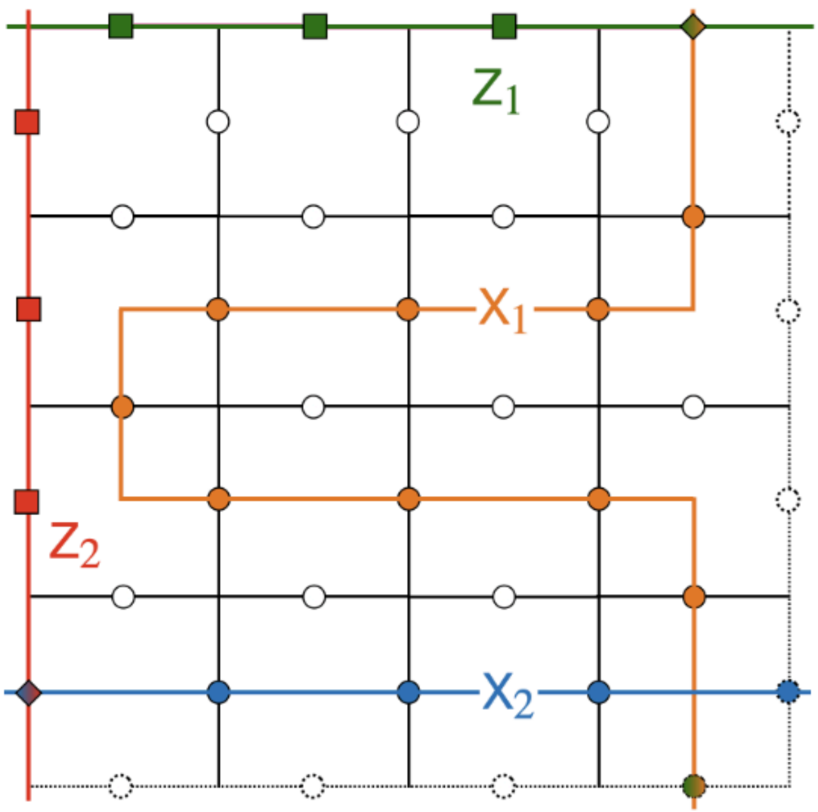
$$\partial_t \rho = \mathscr{L}^\dagger(\rho) = \mathscr{L}^\dagger_{\text{local}}(\rho) + \mathscr{L}^\dagger_{\text{global}}(\rho)$$

Old: $\{A^a\} = \{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$

New Davies:

$$\partial_t \rho = \mathscr{L}^\dagger(\rho) = \mathscr{L}^\dagger_{\text{local}}(\rho) + \mathscr{L}^\dagger_{\text{global}}(\rho)$$

$\{A^a\} =$ logic operators$\left\{\overline{\mathrm{X}}_i, \overline{\mathrm{Z}}_i\right\}_{i=1}^{2}$



New Davies:

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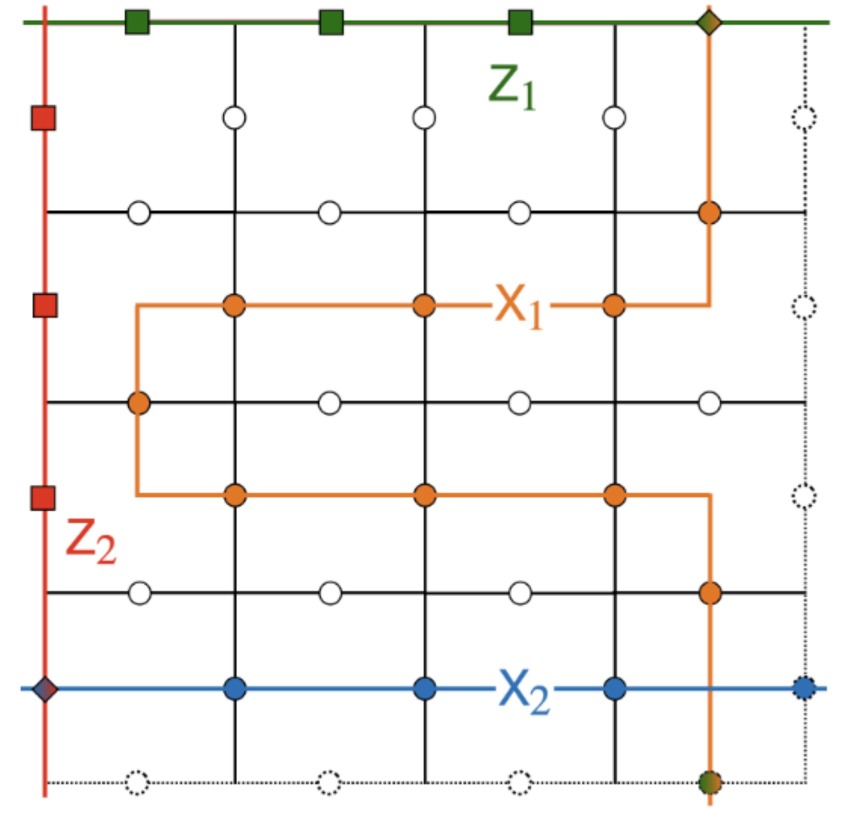

$\{A^a\} =$ logic operators $\left\{\overline{\mathrm{X}}_i, \overline{\mathrm{Z}}_i\right\}_{i=1}^{2}$

Thm [Ding, Li, Lin, Zhang, 2024]:
$\mathrm{Gap}\left(\mathscr{L}\right) \geq$
$\max\left\{\exp(-\Theta(\beta)), \mathrm{poly}(1/n)\right\}$

Implications

Implication 1: Polynomial mixing

$$\mathrm{Gap}\left(\mathscr{L}_\beta\right) \geq \quad \max\left\{\exp(-\Theta(\beta)), \mathrm{poly}(1/n)\right\}$$

$$t_{\mathrm{mix}}(\epsilon) := \{t \geq 0\,;\ \|e^{t\mathcal{L}^\dagger}\rho - \sigma_\beta\|_{\mathrm{tr}} \leq \epsilon,\ \forall\, \text{quantum states } \rho\}$$

$$t_{\mathrm{mix}}(\epsilon) = \mathscr{O}\left(\beta\, \mathrm{poly}(n) \log(1/\epsilon)\right)$$

Singularity in χ^2-divergence

Re: Thermal state of 2D toric code can be prepared in $\mathscr{O}(n^2)$ time from maximally mixing state [Hwang, Jiang, arXiv:2410.04909].

Implication 2: Ground state is easy for $\mathscr{L}_{\text{local}}$

Hilbert space decomposition:

$$\mathscr{H} = \mathbb{C}^4 \otimes \mathbb{C}^{2^{n-2}}$$

$$= \mathbb{C}_1^{2^{n-2}} \oplus \mathbb{C}_2^{2^{n-2}} \oplus \mathbb{C}_3^{2^{n-2}} \oplus \mathbb{C}_4^{2^{n-2}}$$

$\mathbb{C}_i^{2^{n-2}}$: syndrome space

Contain one ground state and excited states with the same logic information

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Two parts of thermalization:

Part 1: Thermalize across syndrome space

$$\mathscr{H} = \mathbb{C}^4 \otimes \mathbb{C}^{2^{n-2}}$$

$$= \mathbb{C}_1^{2^{n-2}} \oplus \mathbb{C}_2^{2^{n-2}} \oplus \mathbb{C}_3^{2^{n-2}} \oplus \mathbb{C}_4^{2^{n-2}}$$

Part 2: Thermalize in syndrome space

Implication 2: Ground state is easy for $\mathscr{L}_{\text{local}}$

Part 1: hard for $\mathscr{L}_{\text{local}}$

$$\mathscr{H} = \mathbb{C}^4 \otimes \mathbb{C}^{2^{n-2}}$$

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Part 2: easy for $\mathscr{L}_{\text{local}}$

ndrome space

Thm [Ding, Li, Lin, Zhang, 2024]:

$$\mathrm{Gap}\left(\mathscr{L}_{\text{local}} \Big|_{\text{syndrome}} \right) \geq \max\left\{ \exp(-\Theta(\beta)), \mathrm{poly}(1/n) \right\}$$

First work shows the $\mathrm{poly}(1/n)$ gap in syndrome space.

Implication 2: Ground state is easy for $\mathscr{L}_{\text{local}}$

Part 1: hard for $\mathscr{L}_{\text{local}}$

$$\mathscr{H} = \mathbb{C}^4 \otimes \mathbb{C}^{2^{n-2}}$$

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Part 2: easy for $\mathscr{L}_{local}$ ndrome space

Thm [Ding, Li, Lin, Zhang, 2024]:

$$\text{Gap}\left(\mathscr{L}_{\text{local}}\Big|_{\text{syndrome}}\right) \geq \max\left\{\exp(-\Theta(\beta)), \text{poly}(1/n)\right\}$$

Re: Ground state preparation for 2D toric code is known to be easy.

Implication 3: Quantum memory

Thm [Ding, Li, Lin, Zhang, 2024]:

$$\mathrm{Gap}\left(\mathscr{L}_{\mathrm{local}}\Big|_{\mathrm{syndrome}}\right) \geq \max\left\{\exp(-\Theta(\beta)), \mathrm{poly}(1/n)\right\}$$

Recall: Fast low temperature mixing might be helpful for passively protected quantum memory

Q: Is $\mathscr{L}_{\mathrm{local}}$ a good candidate for passive protected quantum memory?

Implication 3: Quantum memory

Thm [Ding, Li, Lin, Zhang, 2024]:

$$\mathrm{Gap}\left(\mathscr{L}_{\mathrm{local}}\Big|_{\mathrm{syndrome}}\right) \geq \max\left\{\exp(-\Theta(\beta)), \mathrm{poly}(1/n)\right\}$$

Q: Is $\mathscr{L}_{\mathrm{local}}$ a good candidate for passive protected quantum memory?

$$\mathscr{L}(\rho) = \mathscr{L}_e(\rho) + \mathscr{L}_{\mathrm{local},\,\beta=\infty}(\rho)$$

$$\left\| \mathrm{Tr}_{\mathrm{syndrome}}\left(\exp(\mathscr{L}t)\rho(0)\right) - \rho(0) \right\| \leq \exp(-\Theta(n)), \quad \forall t > 0$$

Intuition: $\mathscr{L}_{\mathrm{local},\,\beta=\infty}$ kills the excitation fast and then preserves the logic (??).

Not necessary

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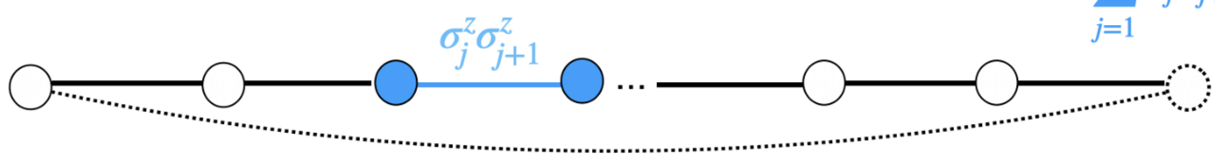
Implication 3: Quantum memory

$$\mathscr{L}(\rho) = \mathscr{L}_e(\rho) + {\color{red}\mathscr{L}_{\mathrm{local},\,\beta=\infty}(\rho)}$$

Think about 1D Ising:

$$H^{\mathrm{Ising}} = -J\sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z$$



$$|\uparrow\uparrow\uparrow\uparrow\rangle$$

Logic

Implication 3: Quantum memory

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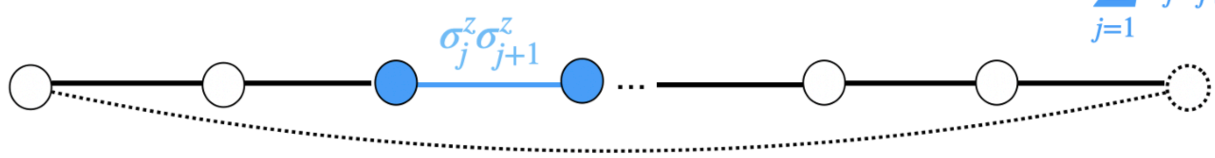

$$|\uparrow\uparrow\uparrow\uparrow\rangle \xrightarrow{\mathscr{L}_e} |\uparrow\downarrow\uparrow\uparrow\rangle \xrightarrow{\mathscr{L}_{\beta=\infty}} |\uparrow\uparrow\uparrow\uparrow\rangle$$

Logic

Implication 3: Quantum memory

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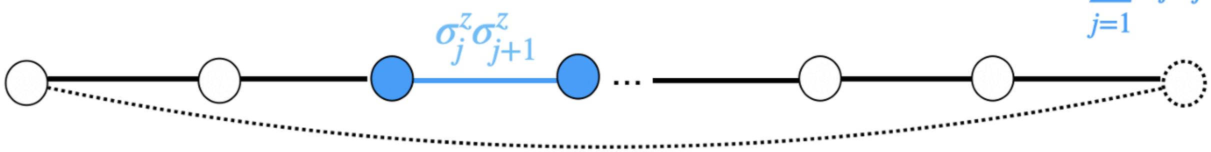

$$|\uparrow\uparrow\uparrow\uparrow\rangle \xrightarrow{\mathscr{L}_e} |\uparrow\downarrow\uparrow\uparrow\rangle \xrightarrow{\mathscr{L}_{\beta=\infty}} |\uparrow\uparrow\uparrow\uparrow\rangle$$

Logic

$$|\uparrow\downarrow\uparrow\uparrow\rangle \xrightarrow{\mathscr{L}_{\beta=\infty}} |\uparrow\downarrow\downarrow\uparrow\rangle \xrightarrow{\mathscr{L}_{\beta=\infty}} |\uparrow\downarrow\downarrow\downarrow\rangle \xrightarrow{\mathscr{L}_{\beta=\infty}} |\downarrow\downarrow\downarrow\downarrow\rangle$$

Implication 3: Quantum memory

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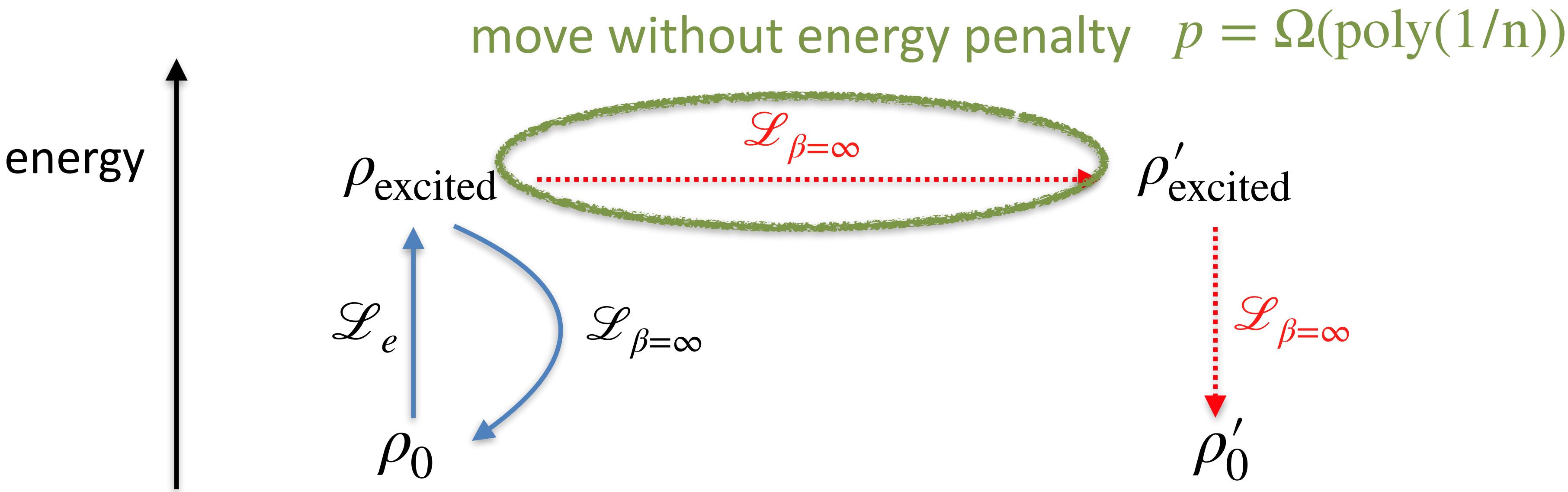
Think about 1D Ising:



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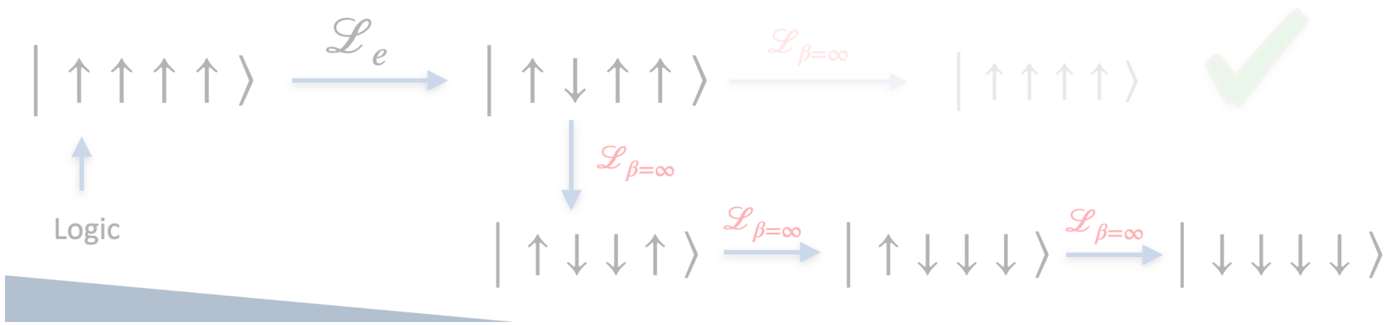
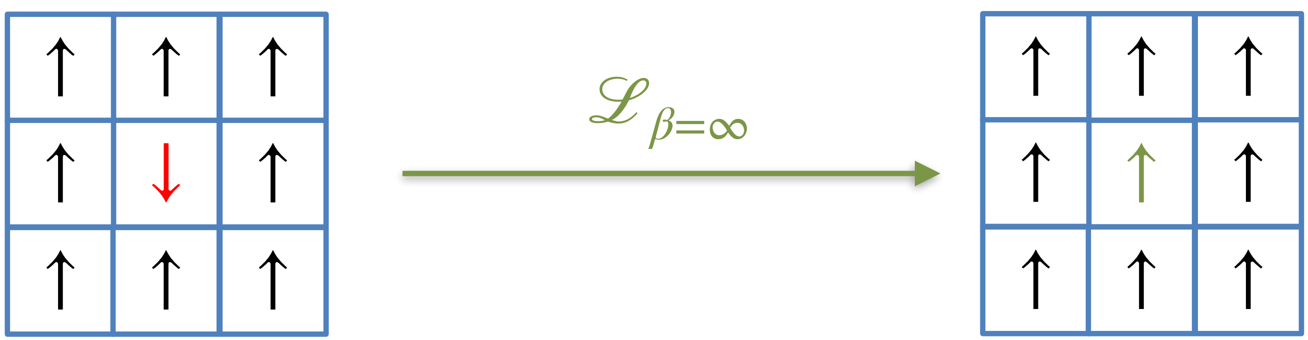
1D Ising:



2D Ising:

[Lieu, Liu et al, PRL, 2024]



Implication 3: Quantum memory

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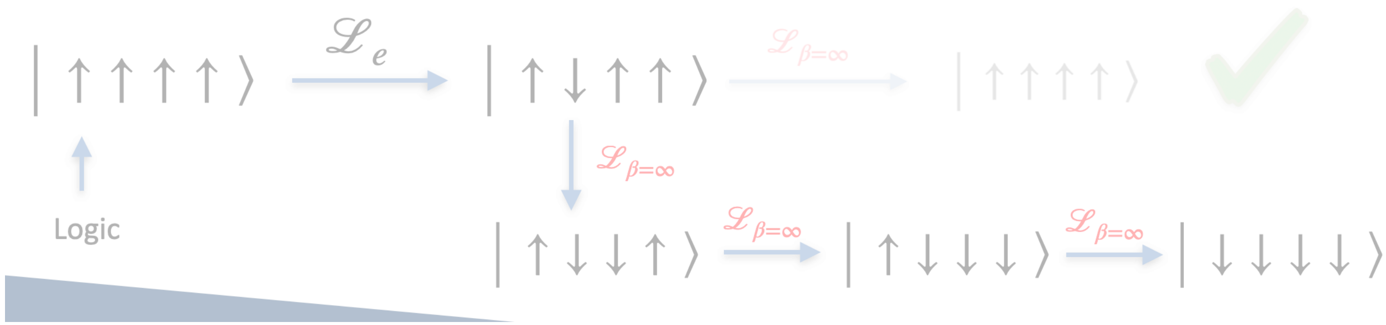
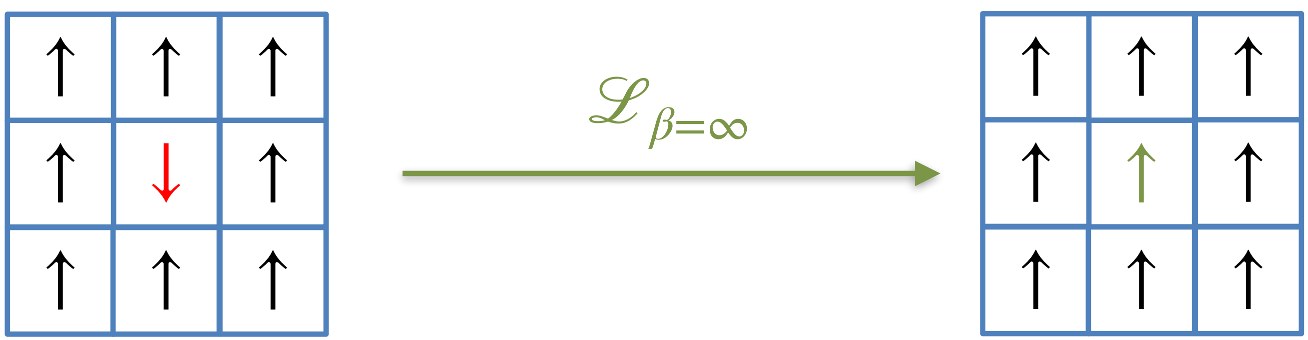
1D Ising:



2D Ising:

[Lieu, Liu et al, PRL, 2024]



Unfortunately, 2D toric code is more like 1D Ising.

Conclusion:

arXiv/2410.01206

- $\mathscr{L}^\dagger_{\mathrm{local}}(\rho) + \mathscr{L}^\dagger_{\mathrm{global}}(\rho)$

Polynomial low temperature thermalization from any initial state

- $\mathscr{L}^\dagger_{\mathrm{local}}(\rho)$

Fast mixing to the ground state

- Passively protected quantum memory?

Still some gaps between fast low temperature mixing and passive quantum memory protection.

Questions?

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