

References

- [1] Jay Alammar. The illustrated transformer, 2018. <https://jalammar.github.io/illustrated-transformer/>.
- [2] Dana Angluin, David Chiang, and Andy Yang. Masked hard-attention transformers and boolean rasp recognize exactly the star-free languages. *arXiv preprint arXiv:2310.13897*, 2023.
- [3] Tom Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared D Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, Sandhini Agarwal, Ariel Herbert-Voss, Gretchen Krueger, Tom Henighan, Rewon Child, Aditya Ramesh, Daniel Ziegler, Jeffrey Wu, Clemens Winter, Chris Hesse, Mark Chen, Eric Sigler, Mateusz Litwin, Scott Gray, Benjamin Chess, Jack Clark, Christopher Berner, Sam McCandlish, Alec Radford, Ilya Sutskever, and Dario Amodei. Language models are few-shot learners. In *Advances in Neural Information Processing Systems*, 2020.
- [4] Sitan Chen and Yuanzhi Li. Provably learning a multi-head attention layer. *arXiv preprint arXiv:2402.04084*, 2024.
- [5] Mete Demircigil, Judith Heusel, Matthias Löwe, Sven Upgang, and Franck Vermet. On a model of associative memory with huge storage capacity. *Journal of Statistical Physics*, 168:288–299, 2017.
- [6] Zeev Dvir, Anup Rao, Avi Wigderson, and Amir Yehudayoff. Restriction access. In *Innovations in Theoretical Computer Science Conference*, 2012.
- [7] Benjamin L Edelman, Ezra Edelman, Surbhi Goel, Eran Malach, and Nikolaos Tsilivis. The evolution of statistical induction heads: In-context learning markov chains. *arXiv preprint arXiv:2402.11004*, 2024.
- [8] Benjamin L Edelman, Surbhi Goel, Sham Kakade, and Cyril Zhang. Inductive biases and variable creation in self-attention mechanisms. In *International Conference on Machine Learning*, 2022.
- [9] Nelson Elhage, Neel Nanda, Catherine Olsson, Tom Henighan, Nicholas Joseph, Ben Mann, Amanda Askell, Yuntao Bai, Anna Chen, Tom Conerly, Nova DasSarma, Dawn Drain, Deep Ganguli, Zac Hatfield-Dodds, Danny Hernandez, Andy Jones, Jackson Kernion, Liane Lovitt, Kamal Ndousse, Dario Amodei, Tom Brown, Jack Clark, Jared Kaplan, Sam McCandlish, and Chris Olah. A mathematical framework for transformer circuits. *Transformer Circuits Thread*, 2021. <https://transformer-circuits.pub/2021/framework/index.html>.
- [10] Borjan Geshkovski, Cyril Letrouit, Yury Polyanskiy, and Philippe Rigollet. A mathematical perspective on transformers. *arXiv preprint arXiv:2312.10794*, 2023.
- [11] Michael Hahn. Theoretical limitations of self-attention in neural sequence models. *Transactions of the Association for Computational Linguistics*, 8:156–171, 2020.
- [12] Yiding Hao, Dana Angluin, and Robert Frank. Formal language recognition by hard attention transformers: Perspectives from circuit complexity. *Transactions of the Association for Computational Linguistics*, 10:800–810, 2022.

- [13] John J Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the National Academy of Sciences*, 79(8):2554–2558, 1982.
- [14] Adam Tauman Kalai and Santosh S Vempala. Calibrated language models must hallucinate. In *Symposium on Theory of Computing*, 2024.
- [15] Takeshi Kojima, Shixiang Shane Gu, Machel Reid, Yutaka Matsuo, and Yusuke Iwasawa. Large language models are zero-shot reasoners. In *Advances in Neural Information Processing Systems*, 2022.
- [16] Dmitry Krotov and John J Hopfield. Dense associative memory for pattern recognition. In *Advances in Neural Information Processing Systems 29*, 2016.
- [17] Dmitry Krotov and John J Hopfield. Large associative memory problem in neurobiology and machine learning. In *International Conference on Learning Representations*, 2021.
- [18] Bingbin Liu, Jordan T Ash, Surbhi Goel, Akshay Krishnamurthy, and Cyril Zhang. Transformers learn shortcuts to automata. In *International Conference on Learning Representations*, 2023.
- [19] Eran Malach. Auto-regressive next-token predictors are universal learners. *arXiv preprint arXiv:2309.06979*, 2023.
- [20] William Merrill and Ashish Sabharwal. The parallelism tradeoff: Limitations of log-precision transformers. *Transactions of the Association for Computational Linguistics*, 11:531–545, 2023.
- [21] William Merrill and Ashish Sabharwal. The expressive power of transformers with chain of thought. In *International Conference on Learning Representations*, 2024.
- [22] Eshaan Nichani, Alex Damian, and Jason D Lee. How transformers learn causal structure with gradient descent. In *International Conference on Machine Learning*, 2024.
- [23] Catherine Olsson, Nelson Elhage, Neel Nanda, Nicholas Joseph, Nova DasSarma, Tom Henighan, Ben Mann, Amanda Askell, Yuntao Bai, Anna Chen, Tom Conerly, Dawn Drain, Deep Ganguli, Zac Hatfield-Dodds, Danny Hernandez, Scott Johnston, Andy Jones, Jackson Kernion, Liane Lovitt, Kamal Ndousse, Dario Amodei, Tom Brown, Jack Clark, Jared Kaplan, Sam McCandlish, and Chris Olah. In-context learning and induction heads. *Transformer Circuits Thread*, 2022. <https://transformer-circuits.pub/2022/in-context-learning-and-induction-heads/index.html>.
- [24] Samet Oymak, Ankit Singh Rawat, Mahdi Soltanolkotabi, and Christos Thrampoulidis. On the role of attention in prompt-tuning. In *International Conference on Machine Learning*, pages 26724–26768, 2023.
- [25] Binghui Peng, Sriniv Narayanan, and Christos Papadimitriou. On limitations of the transformer architecture. *arXiv preprint arXiv:2402.08164*, 2024.
- [26] Mary Phuong and Marcus Hutter. Formal algorithms for transformers. *arXiv preprint arXiv:2207.09238*, 2022.

- [27] Hubert Ramsauer, Bernhard Schäfl, Johannes Lehner, Philipp Seidl, Michael Widrich, Lukas Gruber, Markus Holzleitner, Thomas Adler, David Kreil, Michael K Kopp, Günter Klambauer, Johannes Brandstetter, and Sepp Hochreiter. Hopfield networks is all you need. In *International Conference on Learning Representations*, 2021.
- [28] Ronald L Rivest and Robert Sloan. Learning complicated concepts reliably and usefully. In *Workshop on Computational Learning Theory*, 1988.
- [29] Clayton Sanford, Daniel Hsu, and Matus Telgarsky. One-layer transformers fail to solve the induction heads task. *arXiv preprint arXiv:2408.14332*, 2024.
- [30] Clayton Sanford, Daniel Hsu, and Matus Telgarsky. Transformers, parallel computation, and logarithmic depth. In *International Conference on Machine Learning*, 2024.
- [31] Claude E Shannon. Prediction and entropy of printed english. *Bell System Technical Journal*, 30(1):50–64, 1951.
- [32] Stuart M Shieber. Evidence against the context-freeness of natural language. In *The Formal complexity of natural language*, pages 320–334. 1985.
- [33] Lena Strobl. Average-hard attention transformers are constant-depth uniform threshold circuits. *arXiv preprint arXiv:2308.03212*, 2023.
- [34] Lena Strobl, William Merrill, Gail Weiss, David Chiang, and Dana Angluin. What formal languages can transformers express? a survey. *Transactions of the Association for Computational Linguistics*, 12:543–561, 2024.
- [35] Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Lukasz Kaiser, and Illia Polosukhin. Attention is all you need. In *Advances in Neural Information Processing Systems 30*, 2017.
- [36] Kevin Ro Wang, Alexandre Variengien, Arthur Conmy, Buck Shlegeris, and Jacob Steinhardt. Interpretability in the wild: a circuit for indirect object identification in GPT-2 small. In *International Conference on Learning Representations*, 2023.
- [37] Jason Wei, Xuezhi Wang, Dale Schuurmans, Maarten Bosma, Fei Xia, Ed Chi, Quoc V Le, and Denny Zhou. Chain-of-thought prompting elicits reasoning in large language models. In *Advances in Neural Information Processing Systems 35*, 2022.