Almost Optimal Sublinear Additive Spanners

Zihan Tan

Based on joint work with Tianyi Zhang (Tel Aviv University)

size - accuracy tradeoff

Spanners are (almost)-optimal distance oracles

by $n^{o(1)}$ factor

given G, pre-process
given u,v, report dist(u,v)

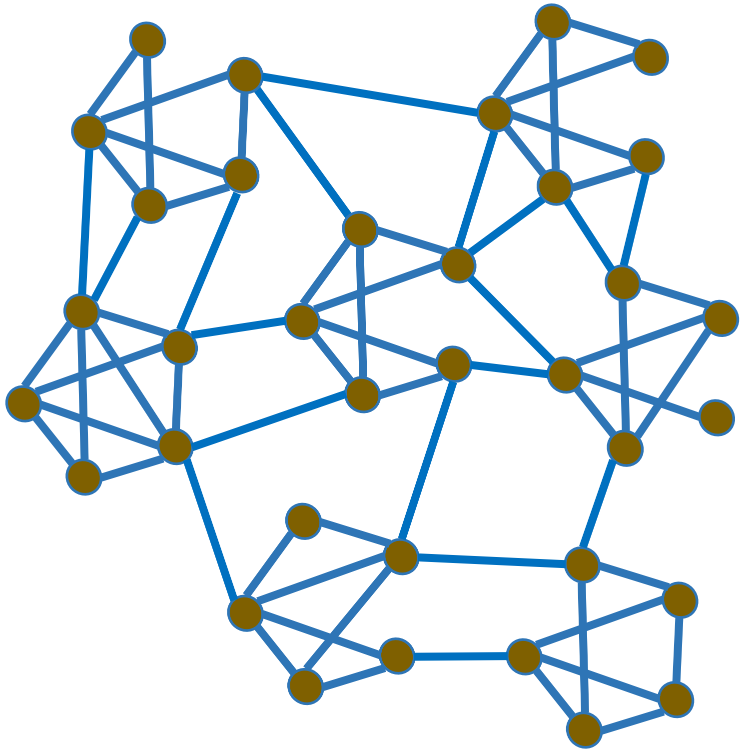
Spanners

Given G, a spanner H is a subgraph of G, s.t. for every pair u,v in G:

$$\text{dist}_G(u,v) \leq \text{dist}_H(u,v) \leq f(\text{dist}_G(u,v))$$

$f(d) = 5 \cdot d$ (multiplicative)

$f(d) = d + 2$ (additive)



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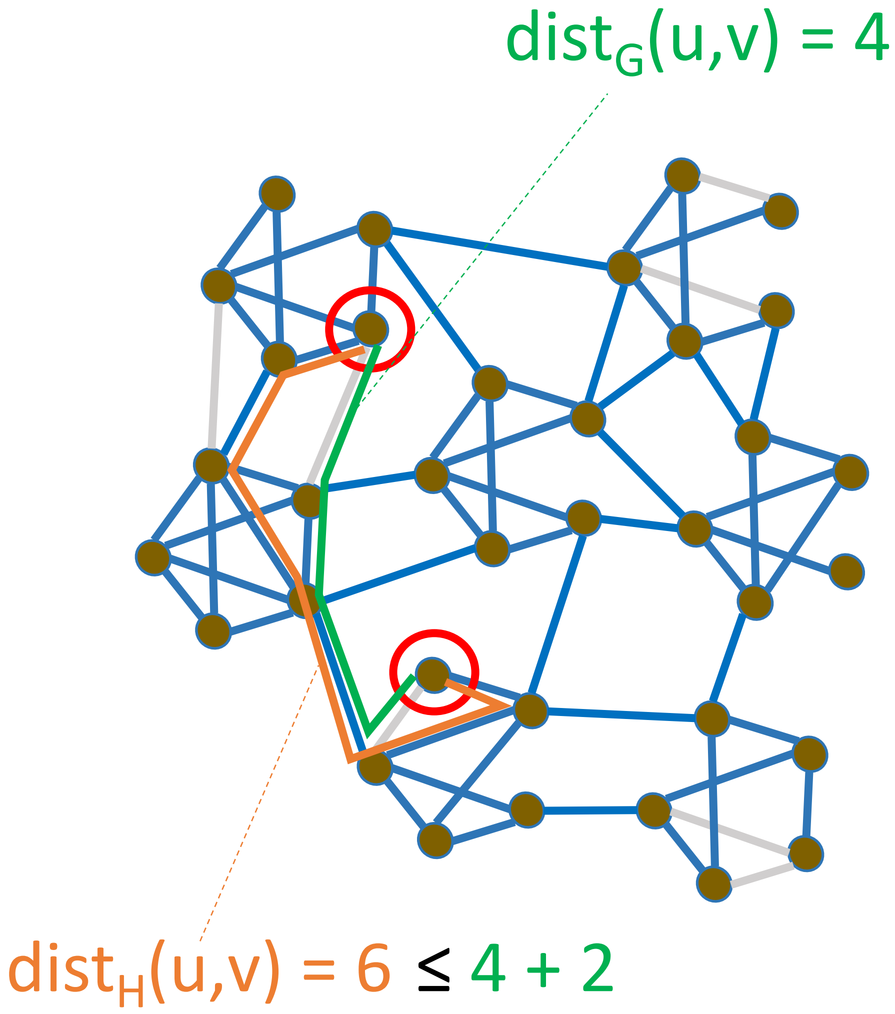

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$f(d) = (2k - 1) \cdot d$

$E(H) = O(n^{(1+1/k)})$ [Althofer-Das-Dobkin-Joseph-Soares 93]

tight under Erdos Girth Conjecture

$f(d) = d + c$

$c = 2$ $E(H) = O(n^{1.5})$ [ADDJS 93]

$c = 4$ $E(H) = \tilde{O}(n^{1.4})$ [Chechik 13]

$c = 6$ $E(H) = O(n^{4/3})$ [Baswana-Kavitha-Mehlhorn-Pettie 05]

$c = n^{\Omega(1)}$ $E(H) = O(n^{4/3-\varepsilon})$ [Abboud-Bodwin 16]

Sublinear Additive Spanners

Given G, a spanner H is a subgraph of G, s.t. for every pair u,v in G:

$\text{dist}_G(u,v) \le \text{dist}_H(u,v) \le f(\text{dist}_G(u,v))$

$f(d) = 5 \cdot d$	(multiplicative)
$f(d) = d + 2$	(additive)
$f(d) = d + O(d^{0.5})$	(sublinear)
$f(d) = (1 + \varepsilon)\, d + \beta$	(mixed)

$f(d) = d + O(d^{1-1/k})$

$E(H) = O(n^{1+1/k})$

[Thorup-Zwick 06]

$E(H) = O\left(n^{1+\frac{(3/4)^{k-2}}{7-2\cdot(3/4)^{k-2}}}\right)$

[Pettie 09]

$E(H) = \Omega\left(n^{1+\frac{1}{2^{k+1}-1}-o(1)}\right)$

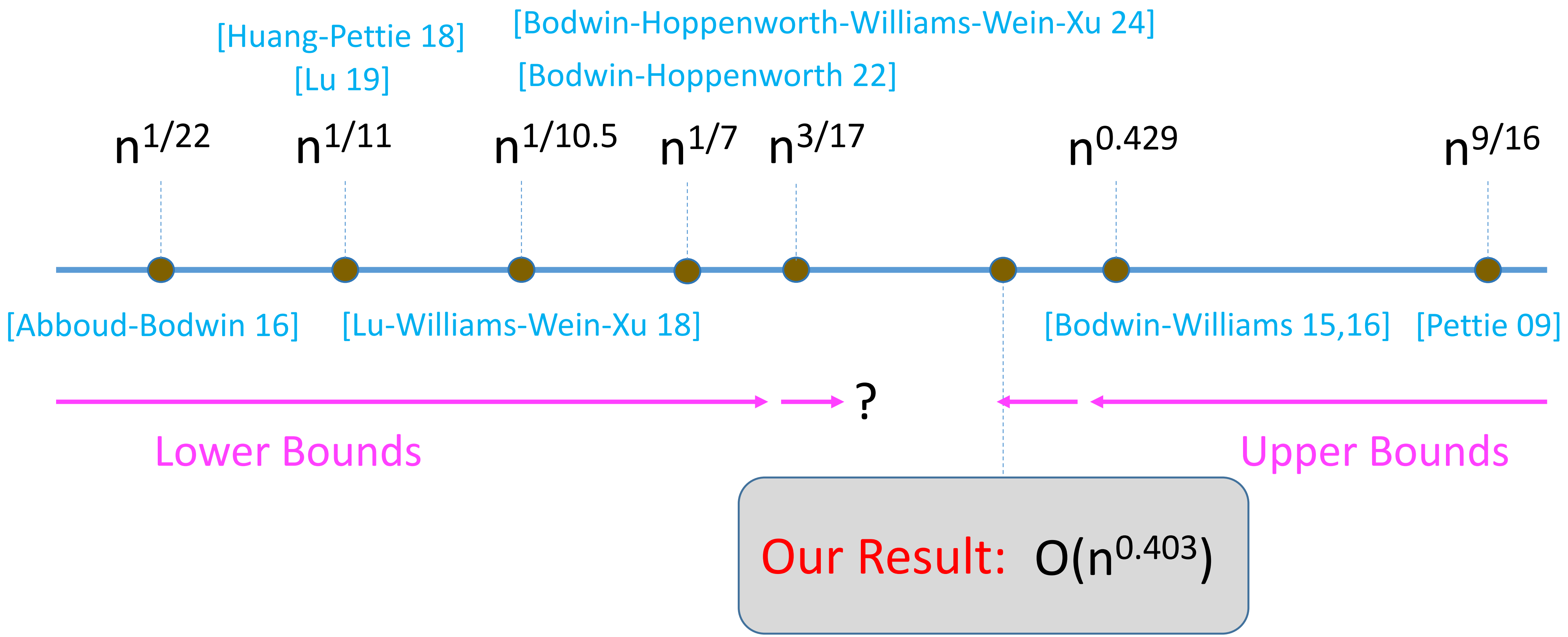
[Abboud-Bodwin-Pettie 18]

holds for any data structure

Our Result: $E(H) = O\left(n^{1+\frac{1}{2^{k+1}-1}+o(1)}\right)$

Linear-size Additive Spanners

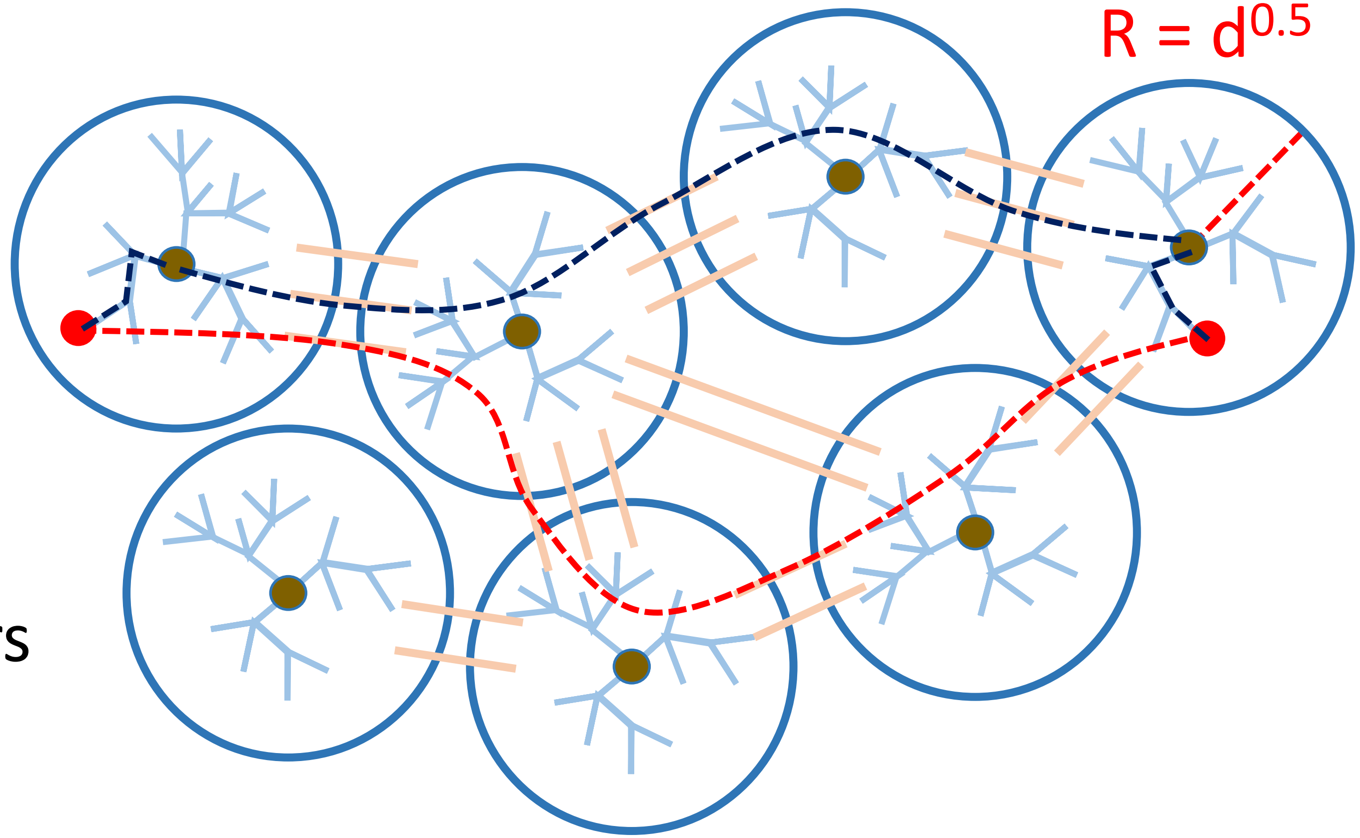
Question: $E(H) = O(n)$, $f(d) = d + g(n)$, how small can $g(n)$ be?



Warm-up: $f(d) = d + O(d^{0.5})$, $E(H) = O(n^{8/7})$

Separately deal with pairs at distance d=1,2,4,…

Simplifying Assumption 1:
disjoint diameter-$d^{0.5}$ clusters



Step 1: BFS trees in clusters

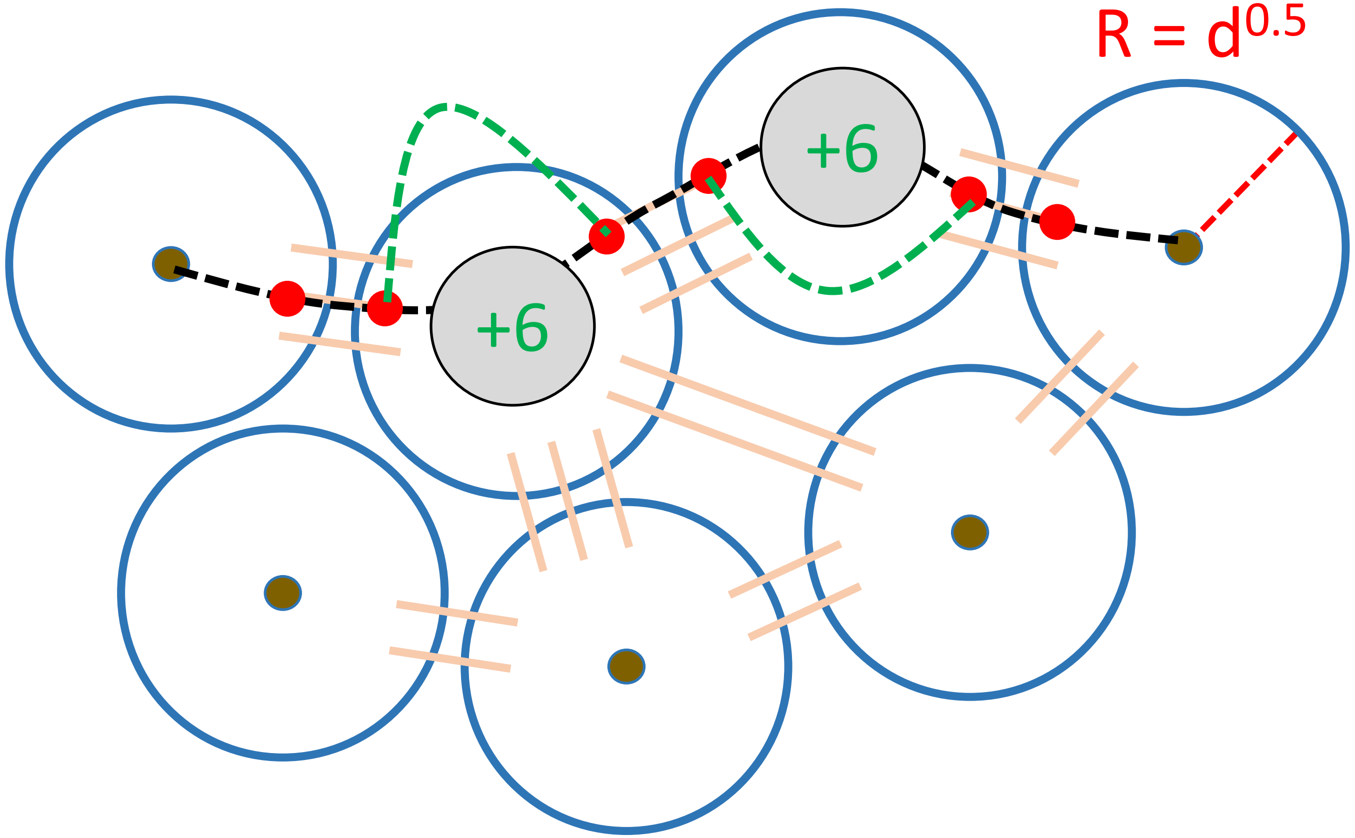
⇒ only need to settle center pairs

for C,C’: sufficient to add any “almost shortest” between any pair v∈C, v’∈C’

Warm-up: $f(d) = d + O(d^{0.5})$, $E(H) = O(n^{8/7})$

Step 1: BFS trees ⇒ only need to settle center pairs (at distance $\simeq d$)

Simplifying Assumption 2:
each length-d shortest path
goes through $\simeq d^{0.5}$ clusters.



Step 2: +6 spanner in clusters
(with size $|C|^{4/3}$)

If all $|C| \leq n^{3/7}$ (small),
total +6 spanner size $\leq n^{4/7} \cdot (n^{3/7})^{4/3} = n^{8/7}$ ⇒ only need to handle large clusters

Warm-up: $f(d) = d + O(d^{0.5})$, $E(H) = O(n^{8/7})$

Step 1: BFS trees in clusters

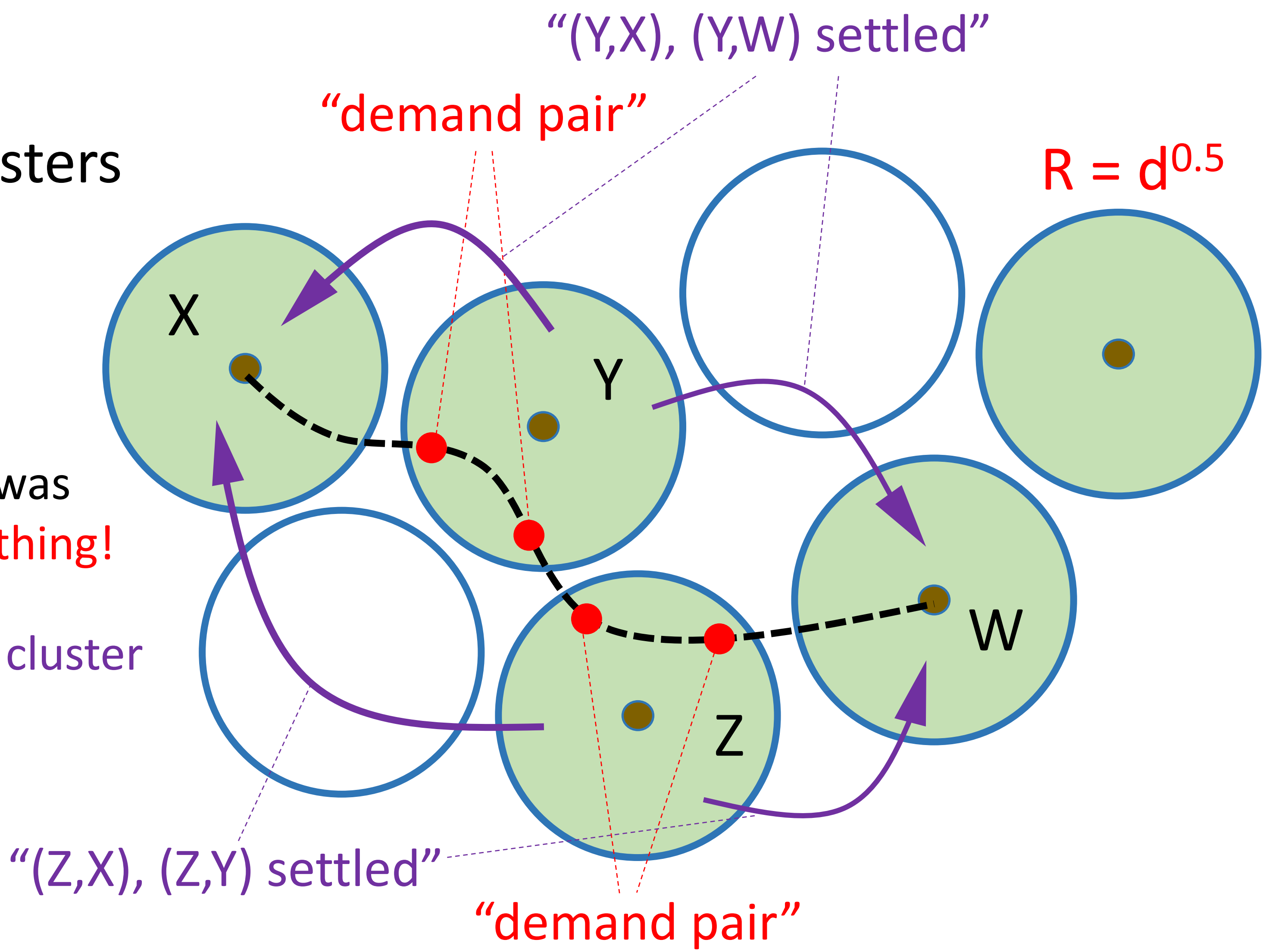
Step 2: +6 spanner in small clusters

$\leq n^{4/7}$ large clusters (size $> n^{3/7}$)

If some cluster on the X-W shortest path was already settled with both X and W: do nothing!

a new demand pair ⇒ settled with a new cluster

demand pair $\leq n^{4/7}$



Warm-up: $f(d) = d + O(d^{0.5})$, $E(H) = O(n^{8/7})$

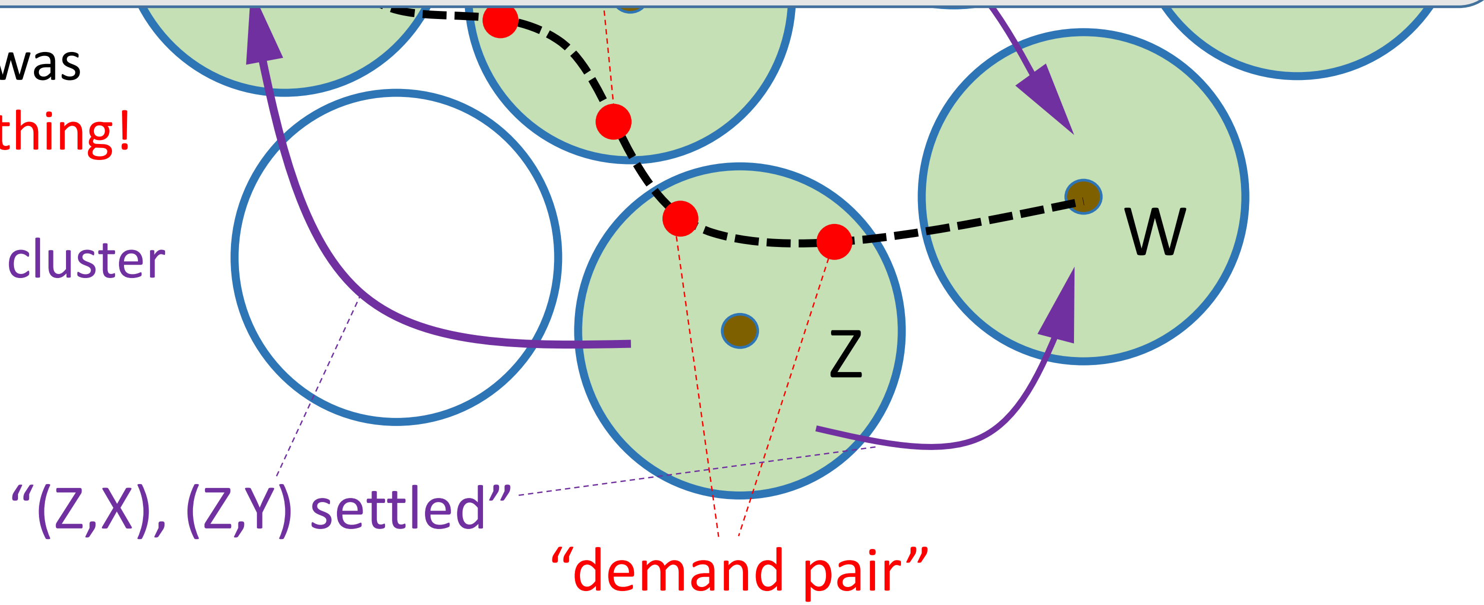
[Kavitha 17] Graph G, pairs $\mathcal{P}$, +6 pairwise spanner of size $n \cdot |\mathcal{P}|^{1/4}$

$\Rightarrow$ total size of all +6 pairwise spanners: $\sum |C| \cdot (n^{4/7})^{1/4} \leq n \cdot (n^{4/7})^{1/4} = n^{8/7}$

If some cluster on the X-W shortest path was already settled with both X and W: do nothing!

a new demand pair $\Rightarrow$ settled with a new cluster

demand pair $\leq n^{4/7}$



Warm-up: $f(d) = d + O(d^{0.5})$, $E(H) = O(n^{8/7})$

Step 1: BFS trees in clusters

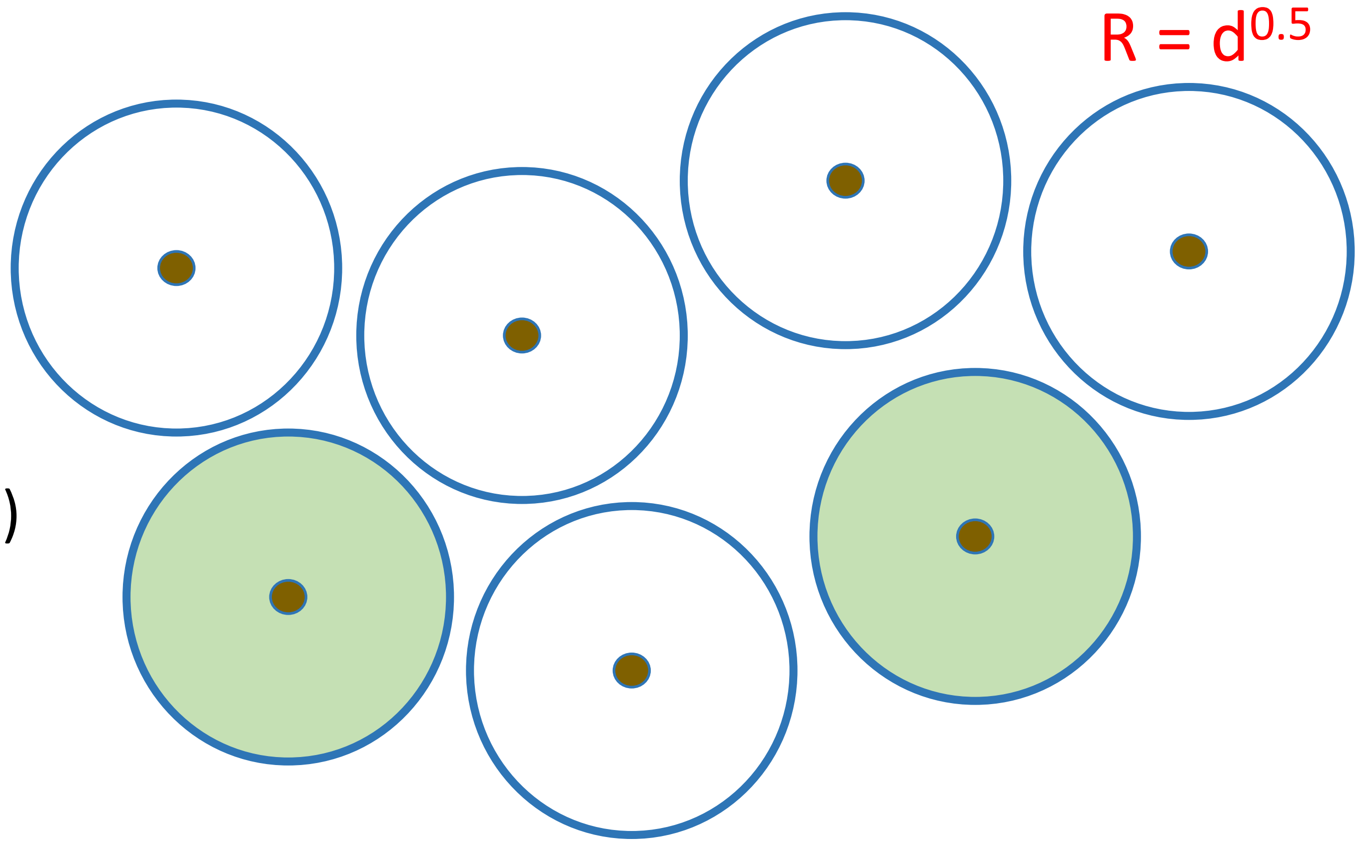
Step 2: +6 spanner in small clusters

Step 3: handle large centers by

Path Buying

- going over large center pairs
- adding “demand pairs”
- marking “settled”
- building +6 pairwise spanner

(w.r.t “demand pairs”, using [Kavitha 17])



General Case: $f(d)=d+O(d^{1-1/k})$, $E(H)= O\left(n^{1+\frac{1}{2^{k+1}-1}+o(1)}\right)$

Step 1: BFS trees in clusters

Step 2,3: +6 pairwise spanner in clusters

For $d + O(d^{1-1/k})$ stretch, we need a $d + O(d^{1-1/(k-1)})$ pairwise spanner

For an s-t shortest path of length $\simeq$ d:

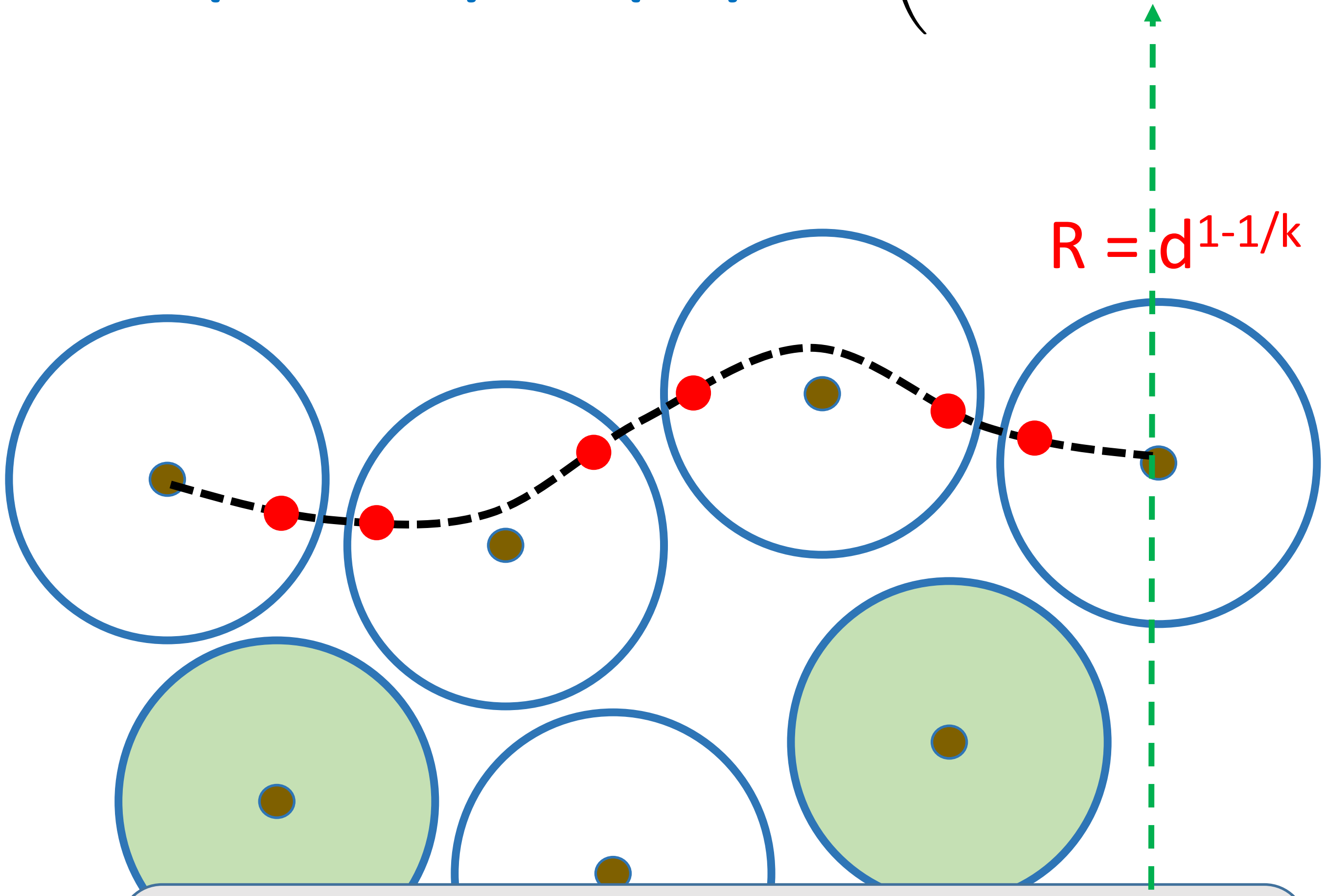
# of clusters it goes through: $d^{1/k}$

for each cluster, the entrance-exit stretch:

$$\left(d^{\frac{k-1}{k}}\right)^{\frac{k-2}{k-1}} = d^{\frac{k-2}{k}}$$

total stretch:

$$d^{\frac{k-2}{k}} \cdot d^{\frac{1}{k}} = d^{\frac{k-1}{k}}$$



Our Result: Given G, $\mathcal{P}$, $d + O(d^{1-1/(k-1)})$ pairwise spanner of size $O\left(n \cdot |\mathcal{P}|^{\frac{1}{2^k}}\right)$

Roadmap

1. Clustering with diameter R = $d^{1-1/k}$

2. Path Buying

[Kavitha 17] $d+O(d^{1-1/(2-1)})$ pairwise, size $n\cdot|\mathcal{P}|^{1/4}$ —Basic PB→ $d+O(d^{1-1/2})$, size $n^{8/7}$

↓ Advanced PB

$d+O(d^{1-1/(3-1)})$ pairwise, size $n\cdot|\mathcal{P}|^{1/8}$ —Basic PB→ $d+O(d^{1-1/3})$, size $n^{16/15}$

↓ Advanced PB

⋮

↓ Advanced PB

$d+O(d^{1-1/(k-1)})$ pairwise, size $n\cdot|\mathcal{P}|^{1/2\wedge k}$ —Basic PB→ $d+O(d^{1-1/k})$, size $O\left(n^{1+\frac{1}{2^{k+1}-1}+o(1)}\right)$

Clustering (Simplifying Assumptions)

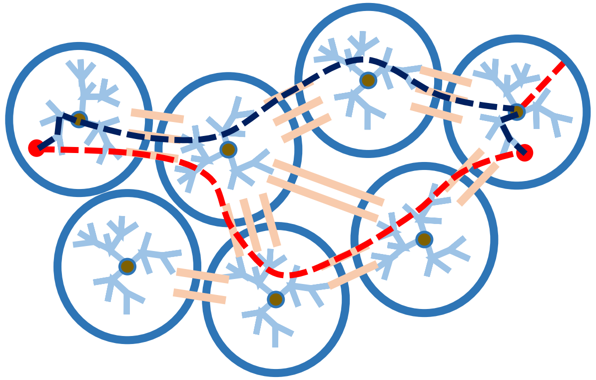
1. For any R, graph = disjoint clusters of diameter R.
2. Each length-d shortest path goes through $\simeq d/R$ clusters.

[Bodwin-Williams 16]* Given G, R, compute a collection of balls in G, s.t. (i) each ball has diameter $\simeq R$;
(ii) balls are almost disjoint (total size is $n^{1+o(1)}$);
(iii) can partition every shortest path into $\simeq d/R$ segments, each contained in a different ball.

Summary & Future Directions

- $f(d) = d + O(d^{1-1/k})$ sublinear additive spanner, size $O\left(n^{1+\frac{1}{2^{k+1}-1}+o(1)}\right)$
- almost optimal: $\simeq$ lower bound in [ABP18] for all data structures
- ***Spanners are (almost)-optimal distance oracles***

- removing $n^{o(1)}$ term? error bound for linear-size additive spanner?



Thanks for Listening!