Online Algorithms for Spectral Hypergraph Sparsification

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Joint work with
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Slides mostly by Soma. Mistakes mostly by Yoshida.

Graph Laplacian

$G = (V, E, w)$: (undirected) graph with edge weights $w \in \mathbb{R}_+^E$

Graph Laplacian Matrix

$$L_G = \sum_{e=\{i,j\} \in E} w_e (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top$$

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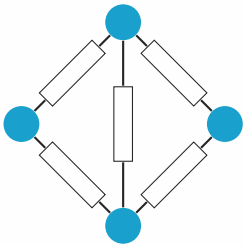
Quadratic form

$$Q_G(x) := \boldsymbol{x}^\top L_G \boldsymbol{x} = \sum_{e=\{i,j\} \in E} w_e (x_i - x_j)^2$$



Energy of G (as a circuit) with potential $\boldsymbol{x}$

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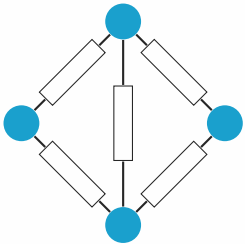
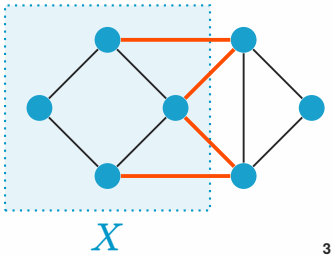


Cut function

For a vertex set $X \subseteq V$,

$$\kappa_G(X) = \sum_{\{i,j\} \in E : |\{i,j\} \cap X| = 1} w_e = Q_G(\mathbf{1}_X).$$

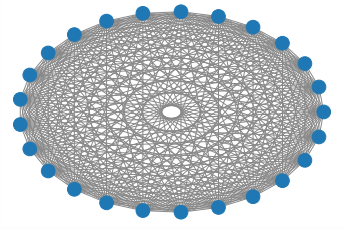


Spectral Graph Sparsification [Spielman, Teng 2011]

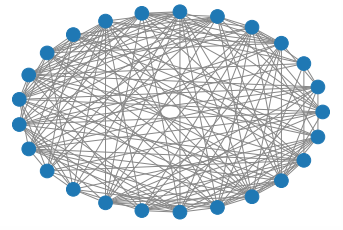
edge subset $\tilde{E} \subseteq E$ ····· ····· edge weight can be modified

A weighted subgraph $\tilde{G} = (V, \tilde{E}, \tilde{w})$ of G is an **ε-spectral sparsifier**

$$\overset{\triangle}{\iff} \quad (1-\varepsilon)Q_G(\boldsymbol{x}) \leq Q_{\tilde{G}}(\boldsymbol{x}) \leq (1+\varepsilon)Q_G(\boldsymbol{x}) \qquad (\forall \boldsymbol{x} \in \mathbb{R}^V)$$



G



$\tilde{G}$

- Spectral sparsifier $\implies$ all cuts are preserved (i.e., **cut sparsifier**)
- $O(n/\varepsilon^2)$ edges is sufficient [Batson, Spielman, Srivastava 2014; Y. T. Lee, Sun 2015]

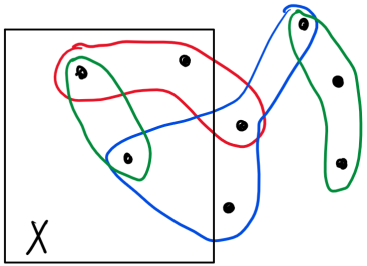
Hypergraph Cut

$H = (V, \mathcal{E}, w)$: hypergraph with edge weight $w \in \mathbb{R}_+^{\mathcal{E}}$

Hypergraph cut function

For vertex subset $X \subseteq V$,

$$\kappa_H(X) = \sum_{e \in \mathcal{E} : 0 < |e \cap X| < |e|} w_e$$



Hypergraph cut function is **NOT** a quadratic form in the usual sense!

Spectral Hypergraph Sparsification [Soma, Yoshida 2019]

Energy function

$$Q_H(\boldsymbol{x}) := \sum_{e \in \mathcal{E}} w_e \max_{i,j \in e} (x_i - x_j)^2$$

In particular, $Q_H(\mathbf{1}_X) = \kappa_H(X)$ for $X \subseteq V$.

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A weighted subhypergraph $\tilde{H}$ of H is an **ε-spectral sparsifier**

$$\overset{\triangle}{\iff} \quad (1-\varepsilon)Q_H(\boldsymbol{x}) \leq Q_{\tilde{H}}(\boldsymbol{x}) \leq (1+\varepsilon)Q_H(\boldsymbol{x}) \qquad (\forall \boldsymbol{x} \in \mathbb{R}^V)$$

spectral sparsifier $\implies$ cut sparsifier [Kogan, Krauthgamer 2015]

Bounds for offline hypergraph sparsification

reference	cut	spectral
Kogan, Krauthgamer (2015)	$O(\frac{n(r+\log n)}{\varepsilon^2})$	
Soma, Yoshida (2019)		$O(\frac{n^3}{\varepsilon^2}\log n)$
Bansal, Svensson, Trevisan (2019)		$O(\frac{nr^3}{\varepsilon^2}\log n)$
Chen, Khanna, Nagda (2020a)	$O(\frac{n}{\varepsilon^2}\log n)$	
Kapralov, Krauthgamer, Tardos, Yoshida (2021)		$\tilde{O}(\frac{nr}{\varepsilon^{O(1)}})$
Kapralov, Krauthgamer, Tardos, Yoshida (2022)		$O(\frac{n}{\varepsilon^4}\log^3 n)$
J. R. Lee (2023) Jambulapati, Liu, Sidford (2023)		$O(\frac{n}{\varepsilon^2}\log n \log r)$

$n = |V|, r = \max_{e\in\mathcal{E}} |e|$

Drawbacks of offline sparsification algorithms

To store the input hypergraph, we may need **exponential** space in n

But, the output sparsifier has only $O(\varepsilon^{-2} n \log n \log r)$ hyperedges, which is **nearly linear!**

Q. Can we construct a spectral sparsifier of a hypergraph using only polynomial space?

Online hypergraph spectral sparsification

- Hyperedges and weights $(e_1, w_1), (e_2, w_2), \ldots, (e_m, w_m)$ arrive in stream.
- When (e_i, w_i) arrives, we **irrevocably** decide whether to include e_i to $\tilde{H}$ along with its weight.
- Only $\mathrm{poly}(n)$ working space is available.

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A weighted subhypergraph $\tilde{H}$ of H is an (ε, δ)**-spectral sparsifier**

$$\overset{\triangle}{\Longleftrightarrow} \quad (1-\varepsilon)Q_H(\boldsymbol{x}) - \delta\|\boldsymbol{x}\|_2^2 \le Q_{\tilde{H}}(\boldsymbol{x}) \le (1+\varepsilon)Q_H(\boldsymbol{x}) + \delta\|\boldsymbol{x}\|_2^2 \qquad (\forall \boldsymbol{x} \in \mathbb{R}^V)$$

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Theorem (Soma, Tung, and Yoshida '24)

There is an online algorithm that outputs an (ε, δ)-spectral sparsifier with $O(\varepsilon^{-2} n \log n \log r \log \frac{\varepsilon W}{\delta n})$ many hyperedges using $O(n^2)$ space. (Here, $W = \sum_i w_i$)

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Theorem (Soma, Tung, and Yoshida '24)

For unweighted r-uniform hypergraphs, there is an online algorithm that outputs an ε-spectral sparsifier with $O(\varepsilon^{-2} nr \log^2 n \log r)$ many hyperedges using $O(n^2)$ space.

Edge sampling algorithm

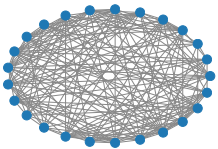
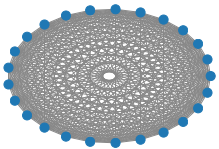
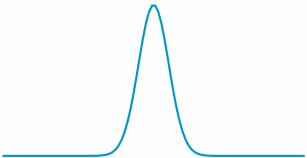
Algorithm

1: Compute edge sampling probability $p_e \in [0, 1]$ for every hyperedge $e \in \mathcal{E}$.
2: **for** each hyperedge e :
3: Add a copy of e to $\tilde{H}$ with weight w_e/p_e w.p. p_e.

Properties

- $\mathbf{E}_{\tilde{H}}[Q_{\tilde{H}}(\boldsymbol{x})] = Q_H(\boldsymbol{x})$ (i.e., unbiased)
- Expected total # of hyperedges $= \sum_{e \in \mathcal{E}} p_e$

Challenge

	small p_e	large p_e
# of edges	 sparse	 dense
Variance of cut weight	 large	 small

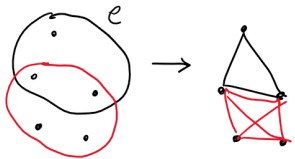
Need to carefully choose p_e!

Reweighting

[Kapralov, Krauthgamer, Tardos, Yoshida 2022; J. R. Lee 2023; Jambulapati, Liu, Sidford 2023]

Clique expansion:

$G = (V, F)$, each hyperedge being replaced with clique



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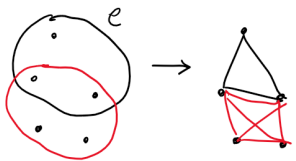

Lemma ([Kapralov, Krauthgamer, Tardos, Yoshida 2022; J. R. Lee 2023])

There are edge weights $c_{e,u,v} \geq 0$ on F such that

- $\sum_{u,v \in e} c_{e,u,v} = w_e \quad (e \in E)$
- $c_{e,u,v} > 0 \implies r_{u,v} = \max_{u',v' \in e} r_{u',v'}$, *where $r_{u,v}$ is the effective resistance between u, v.*

Effective resistance:

$r_{u,v} = (\boldsymbol{e}_u - \boldsymbol{e}_v)^\top L_G^+ (\boldsymbol{e}_u - \boldsymbol{e}_v)$

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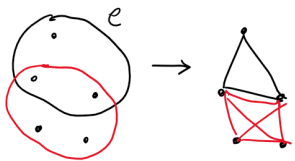

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Effective resistance:
$r_{u,v} = (\boldsymbol{e}_u - \boldsymbol{e}_v)^\top L_G^+ (\boldsymbol{e}_u - \boldsymbol{e}_v)$

Given $\{c_{e,u,v}\}$, $p_e \propto w_e \max_{u,v \in e} r_{u,v}$ yields ε-spectral sparsifier with $O(\varepsilon^{-2} n \log n \log r)$ hyperedges. [J. R. Lee 2023; Jambulapati, Liu, Sidford 2023]

Graph case: effective resistance sampling [Spielman, Srivastava 2011]

Convex optimization for reweighting [J. R. Lee 2023]

The edge weight $\{c_{e,u,v}\}$ can be found by convex optimization:

$$\max \quad \log\det\left(\sum_{e\in E}\sum_{u,v\in V} c_{e,u,v}L_{u,v} + J\right)$$

$$\text{sub. to} \quad \sum_{u,v\in e} c_{e,u,v} = w_e \quad (e \in \mathcal{E})$$

$$c_{e,u,v} \geq 0 \quad (e \in E, u, v \in e)$$

$L_{u,v} := (\boldsymbol{e}_u - \boldsymbol{e}_v)(\boldsymbol{e}_u - \boldsymbol{e}_v)^\top$
J: all-one matrix

KKT condition $\Longrightarrow$ the required conditions for reweighting.

Our online algorithm

(cf. online spectral sparsification of graphs [Cohen, Musco, Pachocki 2020])

Let $\eta = \delta/\varepsilon > 0$. (regularization parameter)

Define matrix L_i ($i = 0, 1, \dots$) as follows:

- $L_0 = O_n$,
- $L_i = L_{i-1} + \sum_{u,v \in e_i} w_i c_{i,u,v} L_{u,v}$, where $c_{i,u,v}$ is the solution of the following convex optimization:

$$\max_{c_i \in \Delta_{e_i}} \log\det\left(L_{i-1} + \sum_{u,v \in e_i} w_i c_{i,u,v} L_{u,v} + \eta I_n \right)$$

Laplacian matrix up to $i-1$ .... ridge regularizer

Δ_e: simplex in $\mathbb{R}^{\binom{e}{2}}$

Our online algorithm

Let $\eta = \delta/\varepsilon > 0$.

1: $\tilde{H}_0 := \emptyset$, $L_0 := O_n$ ····· Laplacian matrix for reweighted clique expansion

2: **for** $i = 1, 2, \dots$:

3: Solve the following convex optimization:

$$\max_{c_i \in \Delta_{e_i}} \log\det\left(L_{i-1} + \sum_{u,v \in e_i} w_i c_{i,u,v} L_{u,v} + \eta I_n \right)$$

····· ridge regularizer

Δ_e: simplex in $\mathbb{R}^{\binom{e}{2}}$

$L_{u,v} := (\boldsymbol{e}_u - \boldsymbol{e}_v)(\boldsymbol{e}_u - \boldsymbol{e}_v)^\top$

4: $L_i \leftarrow L_{i-1} + \sum_{u,v \in e_i} w_i c_{i,u,v} L_{u,v}$

5: $p_i \propto w_i \max_{u,v \in e_i} \|(L_i + \eta I)^{-1/2}(\boldsymbol{e}_u - \boldsymbol{e}_v)\|_2^2$. ····· ridged effective resistance

6: Add e_i with weight w_i/p_i to $\tilde{H}_{i-1}$ with probability p_i and obtain $\tilde{H}_i$.

Analysis

H_i: input hypergraph up to time i

- L_i is deterministically determined by H_i.
  $\longrightarrow$ Sampling of each e_i is independent
- Maintains only sparsifier $\tilde{H}_i$ and Laplacian matrix L_i.
  $\longrightarrow O(n^2)$ space
- $\tilde{H}_i$ is an (ε, δ)-spectral sparsifier of H_i w.h.p.
  Based on the **chaining** analysis by [J. R. Lee 2023].

Analysis

The expected # of hyperedges $= \tilde{O}(\varepsilon^{-2} n \log \frac{\varepsilon W}{\delta n})$

Consider the potential function $\Phi_i := \log\det L_i$.

- $\Phi_0 = \log\det \eta I = n \log \eta$.
- $\Phi_i - \Phi_{i-1} = \Omega(p_i)$ at every step i.
- $\Phi_m = \log\det(L_m + \eta I) \leq n \log\left(\frac{\mathrm{tr} L_m}{n} + \eta\right) = n \log\left(\frac{2W}{n} + \eta\right)$.

$\longrightarrow$ Expected # of hyperedges

$$\approx \sum_{i=1}^{m} p_i \leq \sum_{i=1}^{m} (\Phi_i - \Phi_{i-1}) = \Phi_m - \Phi_0 = n \log\left(\frac{2W}{\eta n} + 1\right)$$

Summary

- **Online algorithm:** $\tilde{O}(\epsilon^{-2} n \log \frac{\varepsilon W}{\delta n})$ hyperedges with $O(n^2)$ space
  - We need $O(\epsilon^{-2} n \log \frac{\varepsilon W}{\delta n})$ for graphs [Cohen, Musco, Pachocki 2020].
- Can we reduce the space complexity from $O(n^2)$ to, say, $O(nr)$?

- For the fully dynamic setting, $\tilde{O}(\epsilon^{-2} nr \log m)$-sized sketch exists for cut sparsification $\Rightarrow$ Putterman's talk on Wednesday!

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Thanks! Questions?