Are there graphs whose shortest path structure requires large edge weights?

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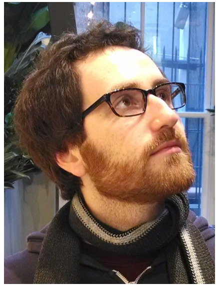


Joint work with: Aaron Bernstein Greg Bodwin

Structural graph problems where you want to **minimize** one thing and **preserve** another

	Minimize	**Preserve**
Spanners	# edges	approximate distances
Shortcut sets	diameter	reachability
Flow/Cut-Sparsifiers	# edges	flow/cut
New Problem	aspect ratio	shortest paths structure

Purpose: Work with a simpler graph but still learn something about the original graph

Problem Definition

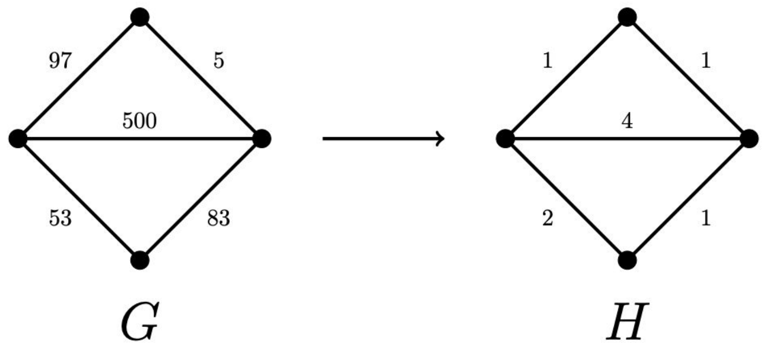
Aspect ratio = (largest edge weight in graph) / (smallest edge weight in graph)

Input: Positive weighted graph G (directed or undirected) with arbitrary aspect ratio

Goal: Reweight the edges of G to form a graph H so that:

1. the aspect ratio is minimized
2. for all vertices s,t: **P is a shortest path in G ⇔ P is a shortest path in H**

$s \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow t$



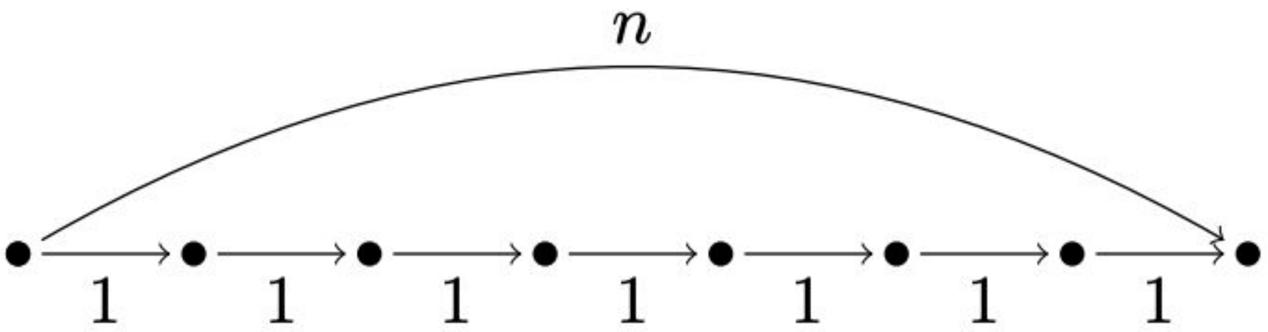
What to do with the answer (hypothetically)

Is it always possible to reweight a graph to bounded aspect ratio (say polynomial) while preserving the structure of shortest paths?

YES → Use the reweighting as a preprocessing step to **get rid of the dependence on aspect ratio** in the running time of algorithms (e.g. log(aspect ratio))

NO → Use the fact that bounded aspect ratio graphs do not capture all possible shortest path structures to get **better algorithms for graphs with bounded aspect ratio**

Simple lower bound: Ω(n)



Input: Weighted graph G with arbitrary aspect ratio
Goal: Reweight the edges of G to form a graph H so that:
1. the aspect ratio is minimized
2. **P is a shortest path in G ⇔ P is a shortest path in H**

Is linear (or polynomial) aspect ratio always possible?

Our results:

	TODAY:		
DAGs	YES	1.	O(n)
Directed Graphs	NO	2.	$2^{\Omega(n)}$
Undirected Graphs	NO		$2^{\Omega(n)}$

We also study the approximate version

DAGs with integer weights in [1,W] ?

Applications?

Input: Weighted graph G with arbitrary aspect ratio
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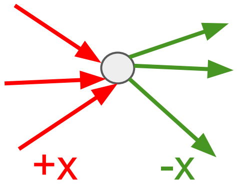
O(n) upper bound for DAGs

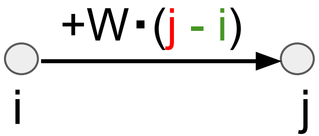
Observation: 

transformation does not change any shortest paths.

Assume edge weights between 1 and W.

Algorithm: For all i, apply transformation with $x = W \cdot i$ to the i^{th} vertex in topological order

What happens to a single edge?



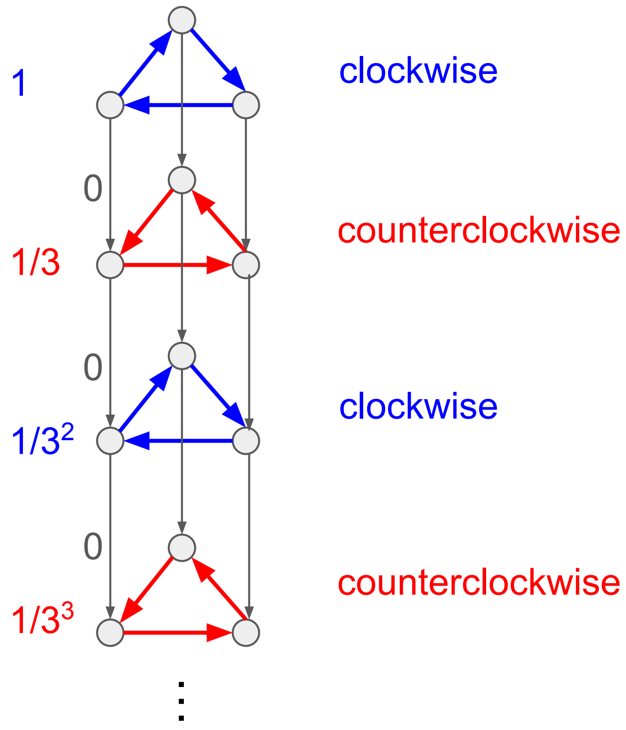
⇒ Minimum edge weight ≥ W

⇒ Maximum edge weight ≤ W + W(n-1)

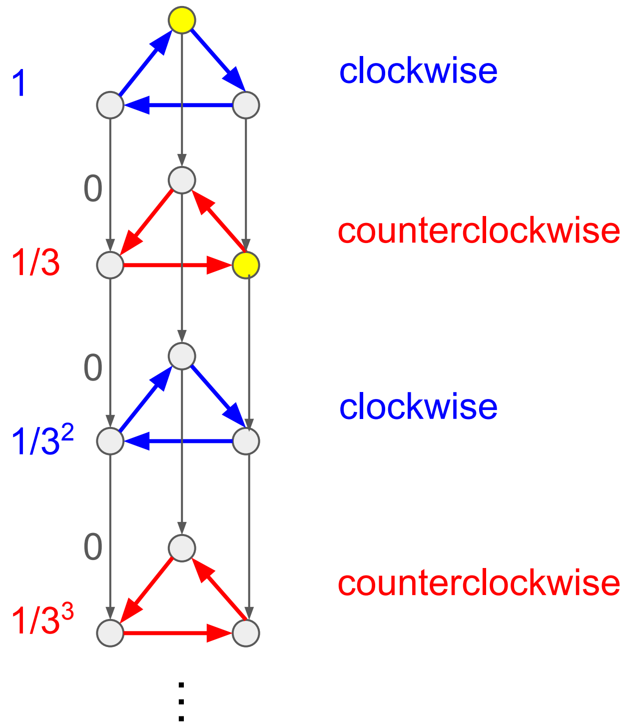
Input: Weighted graph G with arbitrary aspect ratio
Goal: Reweight the edges of G to form a graph H so that:

1. the aspect ratio is minimized
2. **P is a shortest path in G ⇔ P is a shortest path in H**

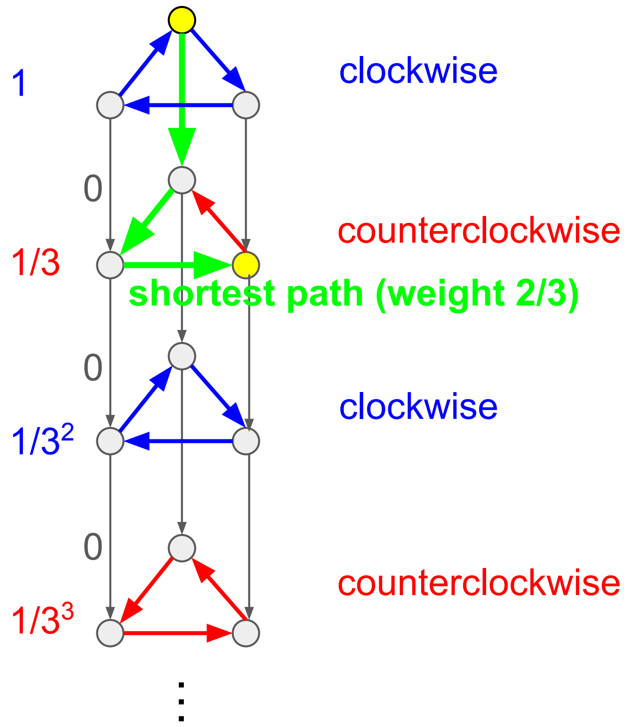
Directed lower bound construction: $2^{\Omega(n)}$



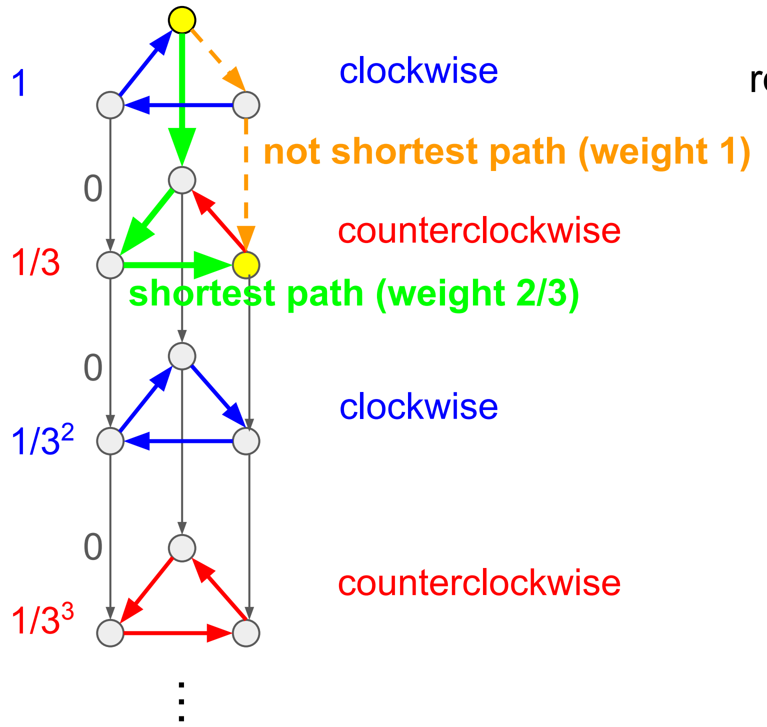
Directed lower bound construction: $2^{\Omega(n)}$



Directed lower bound construction: $2^{\Omega(n)}$



Directed lower bound construction: $2^{\Omega(n)}$



After reweighting:

Δ

$>$

2Δ

Approximate version

h-approximate shortest paths in H ⊆ g-approximate shortest paths in G

Problem is easier for **small h** and **large g**

Results

Easiest version: h=1, g>1

Still $2^{\Omega(n)}$ for **any g>1** (directed)

Still $2^{\Omega(n)}$ for **any 1<g<13/12** (undirected)

open problem

Interesting version for DAGs: h>1, g≥1

$h^{\Omega(\sqrt{n})}$ for **any h>1, g≥1**

Input: Weighted graph G with arbitrary aspect ratio
Goal: Reweight the edges of G to form a graph H so that:

1. the aspect ratio is minimized
2. (see above)

Results

DAGs	**$O(n)$**
Directed Graphs	**$2^{\Omega(n)}$**
Undirected Graphs	**$2^{\Omega(n)}$**
DAGs with integer weights in [1,W]	**?**

Thank you!

Applications?

Input: Weighted graph G with arbitrary aspect ratio
Goal: Reweight the edges of G to form a graph H so that:

1. the aspect ratio is minimized
2. **P is a shortest path in G ⇔ P is a shortest path in H**