Testing Intersectingness of Uniform Families

By Ishay Haviv, Michal Parnas

Or how Dana and I intersected

Where it all started

Hebrew University, Jerusalem:
B.Sc., M.Sc., Ph.D. 1984 - 1994

Belgium House

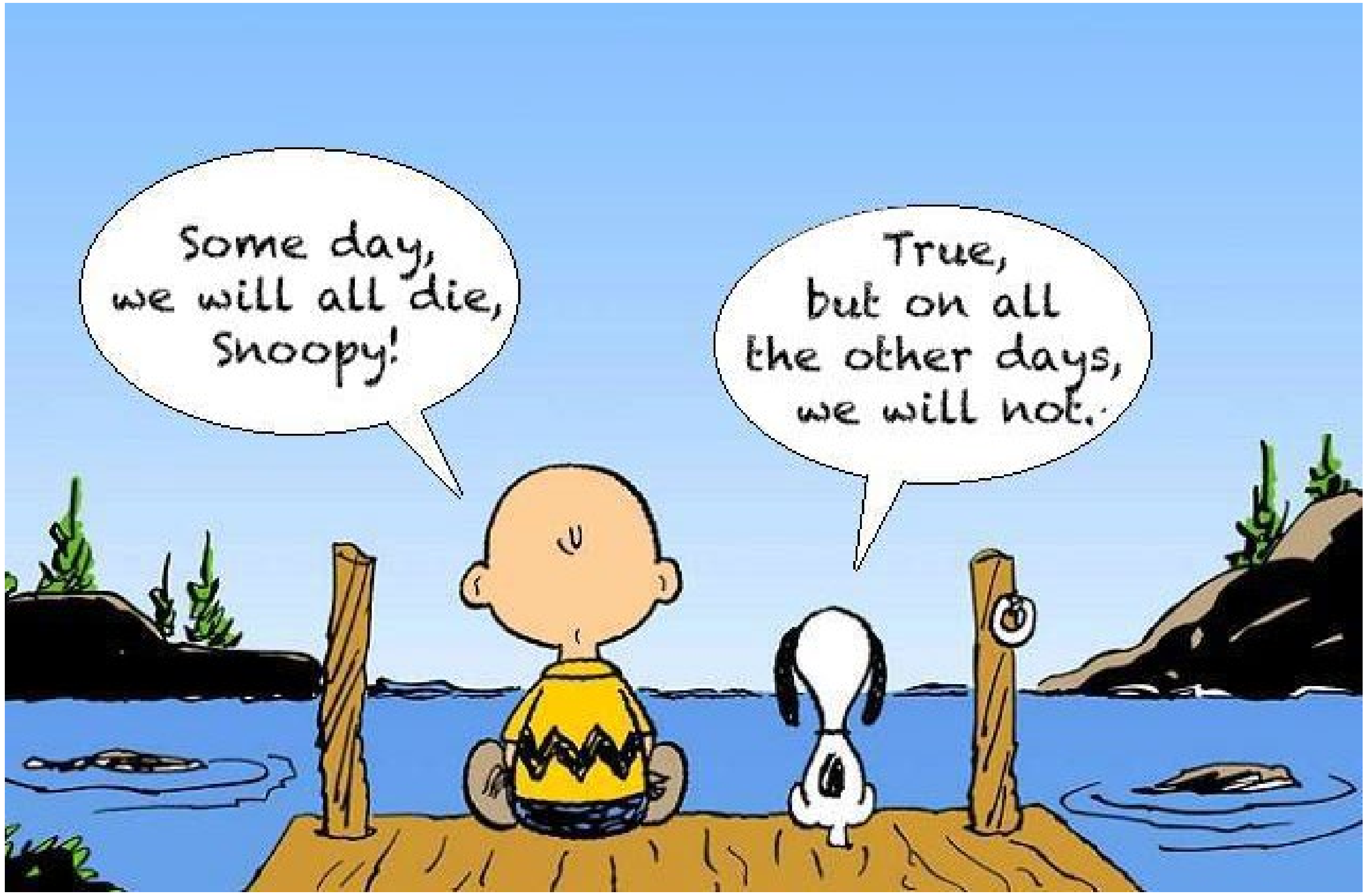
Old CS Building

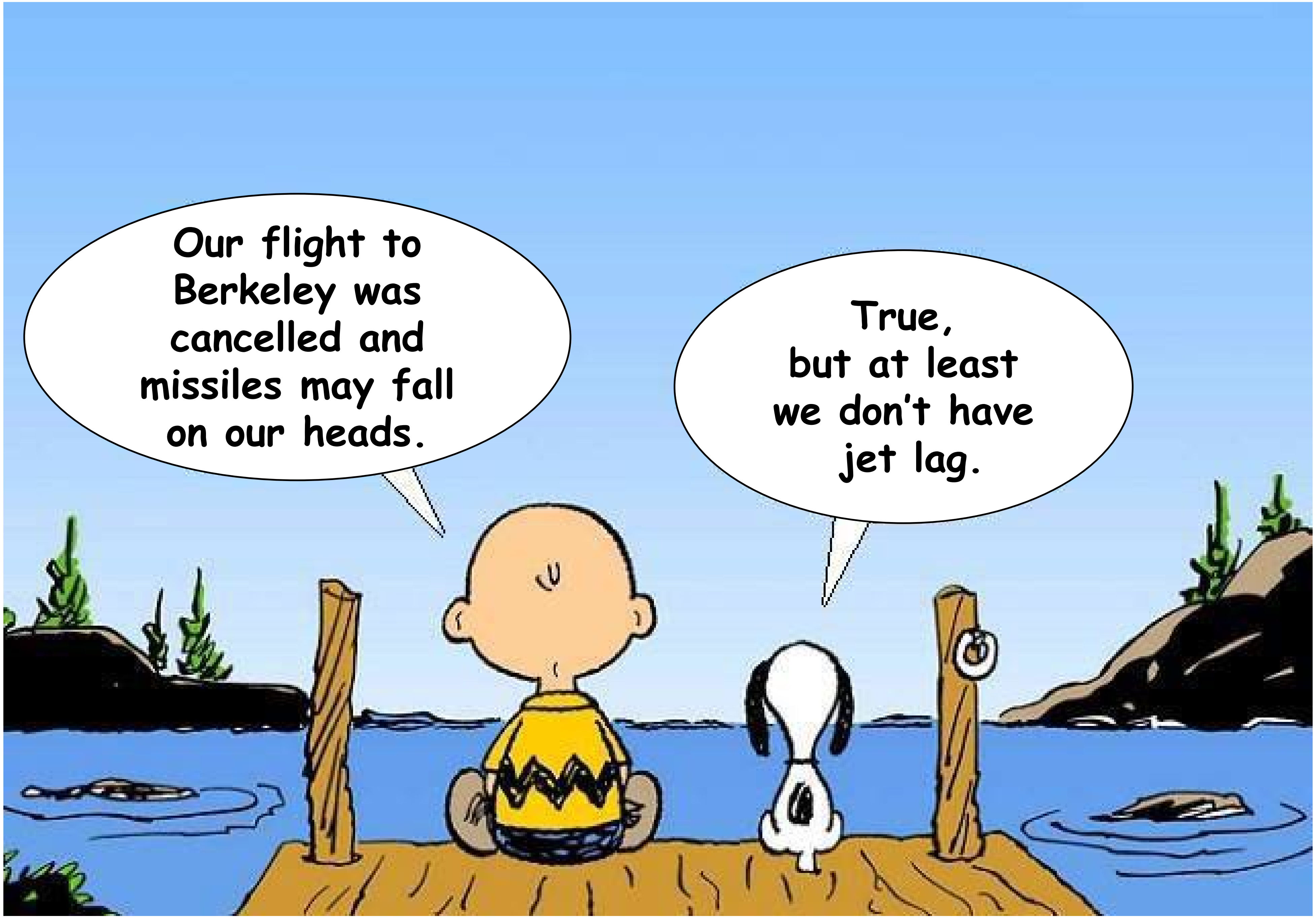
I fixed the no photos problem…

Acknowledgements in Ph.D. Thesis

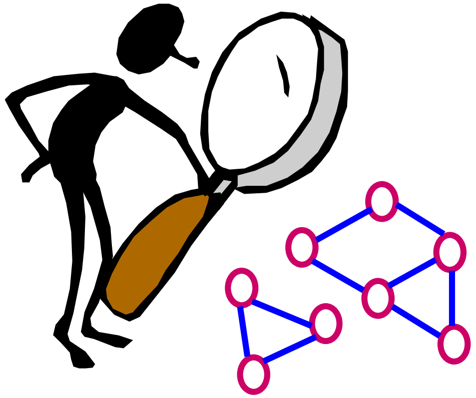
Dana: I had great fun working with Michal (despite all her teasing), and perhaps “our robots" can once come back to life.

Michal: To Dana who always saw the bright side of everything.



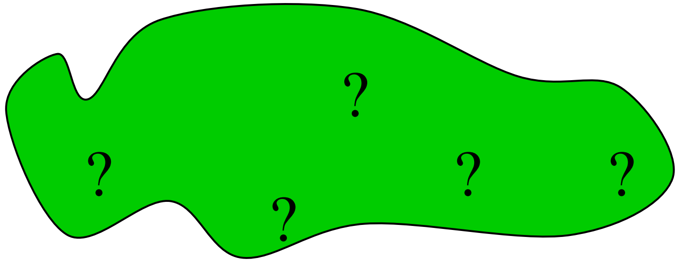

Our flight to Berkeley was cancelled and missiles may fall on our heads.
True, but at least we don't have jet lag.

Our first paper together and the evolution of property testing art.

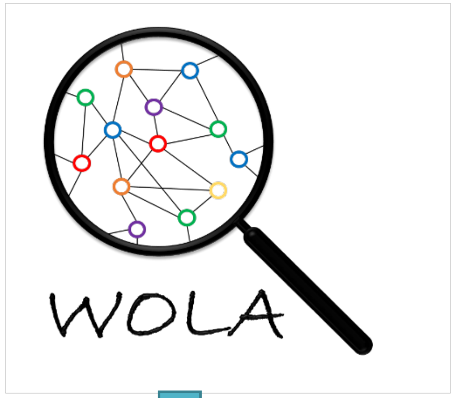


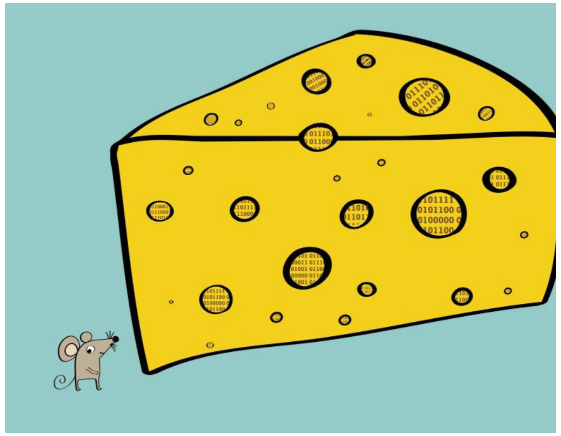
Parnas & Ron 1999:
Testing the diameter of graphs.

Introducing general model for graph testing.



Later: testing clustering, metrics,
Dictators, tolerant testing,
Sublinear algorithms and more.





WOLA 2024

Testing Intersecting Families

A family of sets F over $[n]$ is **intersecting** if $\forall S_1, S_2 \in F$ it holds that $S_1 \cap S_2 \neq \emptyset$

Chen, De, Li, Nadimpalli, Servedio, 2024:

"Inspired by the classic problem of monotonicity testing…"

(Goldreich, Goldwasser, Lehman, Ron 1998)

$F \subseteq 2^{[n]}$ is ε-far from intersecting if at least $\varepsilon 2^n$ of its sets should be removed to make it intersecting.

One sided error tester should accept if F is intersecting and reject with probability $\geq 2/3$ if F is ε-far from intersecting.

Results for Intersecting Families

Chen, De, Li, Nadimpalli, Servedio, 2024:

Upper bound:

- Non-adaptive one sided tester with $poly\left(n^{\sqrt{nlog(1/\varepsilon)}}, \frac{1}{\varepsilon}\right)$ queries.

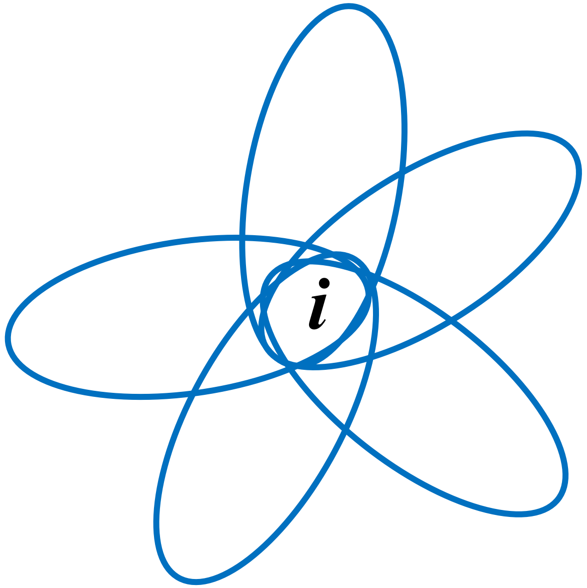
Lower bound:

- Non-adaptive one sided tester requires $2^{\Omega(\sqrt{nlog(1/\varepsilon)})}$ queries.
- Non-adaptive two sided tester requires $2^{\Omega(n^{1/4}/\sqrt{\varepsilon})}$ queries.

Testing Intersecting Uniform Families

F is ***k*-uniform** if $F \subseteq \binom{[n]}{k}$

Erdős–Ko–Rado theorem:
Let $\boldsymbol{F} \subseteq \binom{[n]}{k}$ be **intersecting.** Then $|\boldsymbol{F}| \leq \binom{n-1}{k-1}$.
$|\boldsymbol{F}|$ is maximized when it is a **1-junta**: ***F*** includes all sets with some i.



- $\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$
- $k = 2$: $|F| \leq n-1$
- $k = n/2$: $\binom{n}{n/2} \approx \frac{1}{\sqrt{n}} 2^n$

- For what ε is testing k-uniform families interesting?
- Over which universe is the problem defined?

Universe size matters

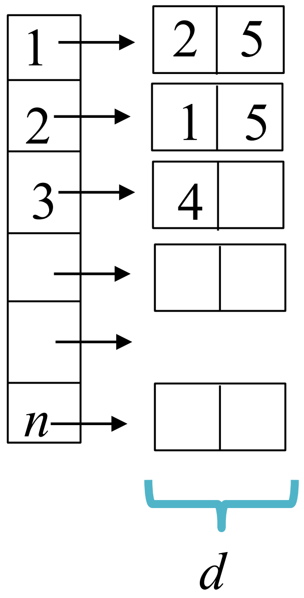
Let $\boldsymbol{G}$ be a graph with $\boldsymbol{n}$ vertices and $\boldsymbol{m}$ edges.

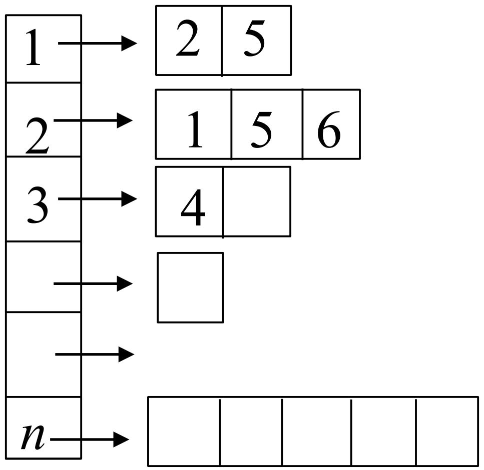
> G is ε-far from property if #edges that should be modified is:

- Goldreich, Goldwasser, Ron, 1996: εn^2 edges for dense graphs.
- Goldreich, Ron, 1997: εdn edges for bounded degree graphs.
- Parnas, Ron, 1999: εm edges for general graphs.

G =

	a	b	c	d	e
a	0	1	0	1	0
b	1	0	1	0	0
c	0	1	0	1	0
d	1	0	1	0	1
e	0	0	0	1	0





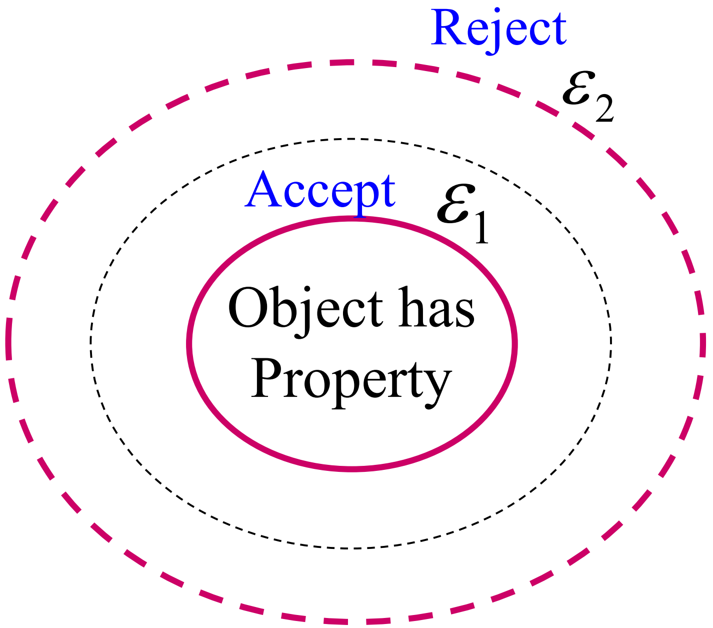
Definitions

$F \subseteq \binom{[n]}{k}$ is ε-far from intersecting if at least $\varepsilon \binom{n}{k}$ of its sets should be removed to make it intersecting .

Tolerant testing algorithm (Parnas, Ron, Rubinfeld, 2004):

Accept *object* with probability ≥ 2/3 if it is ε_1-close to property

and reject with probability ≥ 2/3 if it is ε_2-far from property.



Our Results

For every fixed integer r, for all $n \geq 2k$, there exist non-adaptive testers:

Tester	Condition	Query Complexity
Two sided error Tolerant	$\varepsilon_2 \geq \Omega\left(\varepsilon_1 + \frac{k}{n}\right)$	$O\left(\frac{1}{\varepsilon_2}\right)$
	$\varepsilon_2 \geq \Omega\left(\varepsilon_1 + \left(\frac{k}{n}\right)^r\right), \quad r \geq 2$	$O\left(\frac{\ln(n)}{\varepsilon_2}\right)$
One sided error	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^2\right)$	$O\left(\frac{1}{\varepsilon}\right)$
	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^r\right), \quad r \geq 3$	$O\left(\frac{\ln(k)}{\varepsilon}\right)$

Lower bound: $\Omega\left(\frac{1}{\varepsilon}\right)$ queries for $\binom{n}{k}^{-1} \leq \varepsilon < \frac{1}{2}$

Our Results

For every fixed integer r, for all

Approximate size of family

Tester	C	Query Complexity
Two sided error Tolerant	$\varepsilon_2 \geq \Omega\left(\varepsilon_1 + \frac{k}{n}\right)$	$O\left(\frac{1}{\varepsilon_2}\right)$
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	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^r\right), \quad r \geq 3$	$O\left(\frac{\ln(k)}{\varepsilon}\right)$

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Our Results

For every fixed integer r, for all $n \geq 2k$, there exist non-adaptive testers:

By sampling check if family is almost contained in a junta (Dinur, Friedgut 2009)

Tester		
Two sided error Tolerant		
	$\varepsilon_2 \geq \Omega\left(\varepsilon_1 + \left(\frac{k}{n}\right)^r\right), \quad r \geq 2$	$O\left(\frac{\ln(n)}{\varepsilon_2}\right)$
One sided error	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^2\right)$	$O\left(\frac{1}{\varepsilon}\right)$
	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^r\right), \quad r \geq 3$	$O\left(\frac{\ln(k)}{\varepsilon}\right)$

Lower bound: $\Omega\left(\frac{1}{\varepsilon}\right)$ queries for $\binom{n}{k}^{-1} \leq \varepsilon < \frac{1}{2}$

One sided error tester

Theorem: One sided tester for $\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^r\right)$ with $O\left(\frac{\ln(k)}{\varepsilon}\right)$ queries.

Canonical Tester $(Family\ F)$:

1. Choose m sets $S_1, \dots, S_m \subseteq \binom{[n]}{k}$ uniformly at random.
2. If $\exists i, j \in [m]$ such that $S_i, S_j \in F$ and $S_i \cap S_j = \emptyset$

   then reject, otherwise accept.

Tester always accepts intersecting families.

Proof Idea

Let $F \subseteq \binom{[n]}{k}$ be ε-far from intersecting ⟹ $|F| > \varepsilon\binom{n}{k}$.   Recall: $\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^r\right)$

Assumption: for every $A \subseteq [n], |A| < r$, we sampled $S_A \in F$, such that $S_A \cap A = \emptyset$

> Lemma: Number of sets $S \in \binom{[n]}{k}$ that intersect all sets S_A is $\leq \left(\frac{k^2}{n}\right)^r \binom{n}{k}$

But $|F| > \varepsilon\binom{n}{k} \geq \Omega\left(\left(\frac{k^2}{n}\right)^r \binom{n}{k}\right)$ ⟹ After a few more samples we get a set $S \in F$

that doesn't intersect at least one of the sets S_A ⟹ Algorithm rejects.

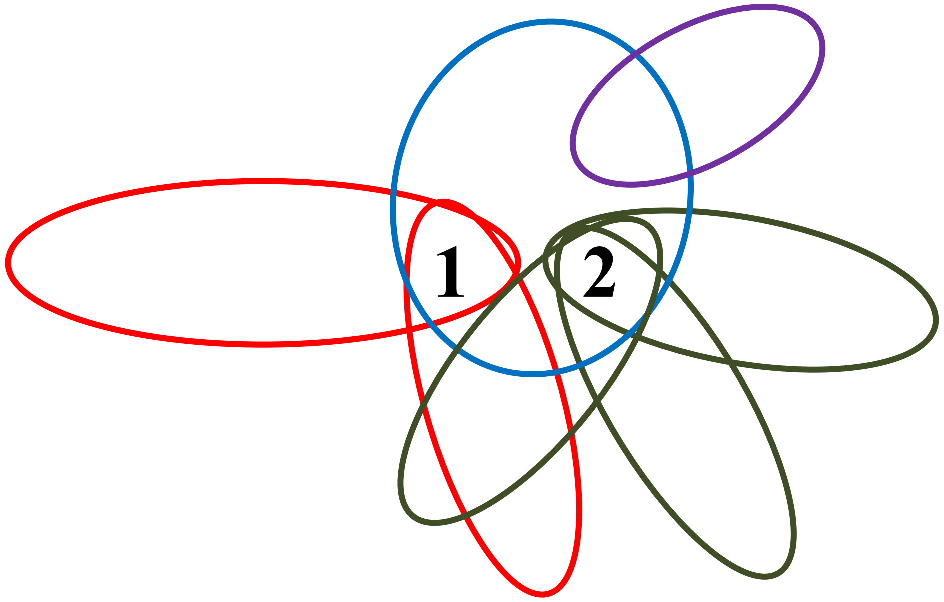
However…

Assumption doesn't always hold: There may be a subset $A \subseteq [n], |A| < r$,

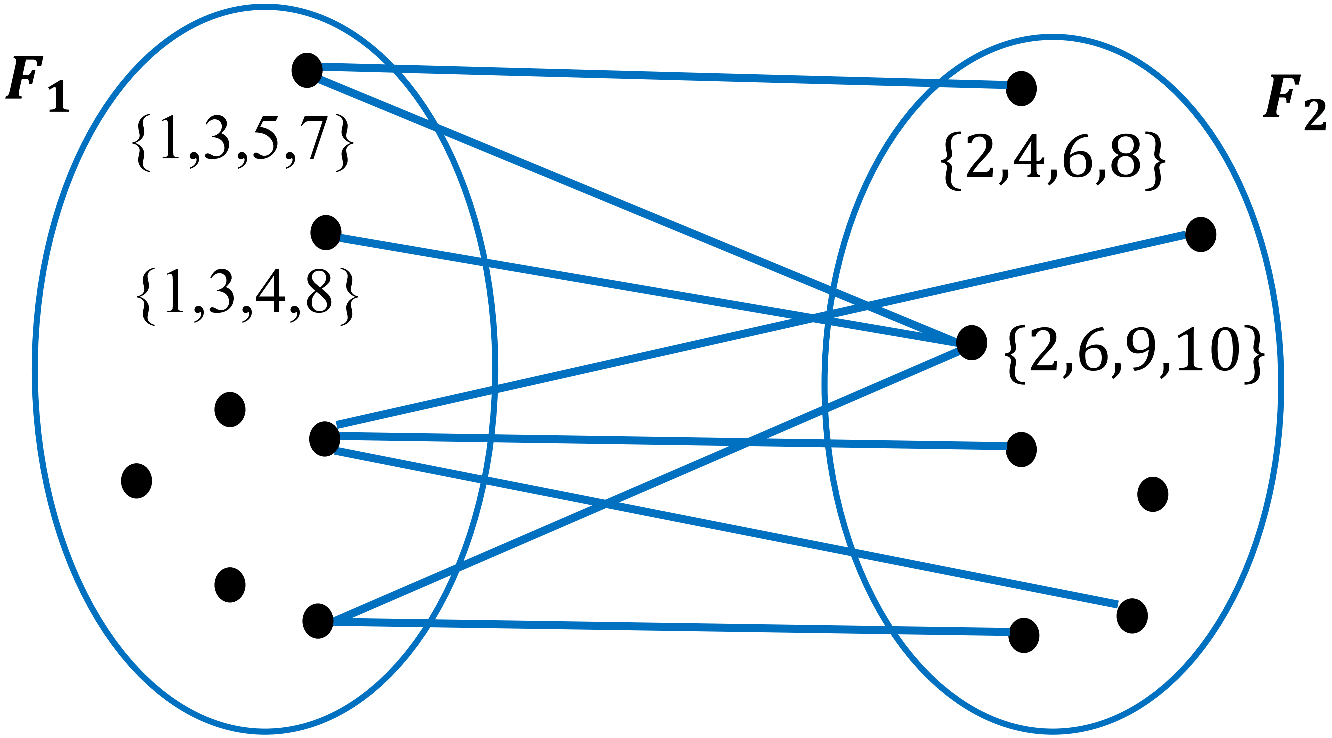
such that for all sampled $S \in F$, $S \cap A \neq \emptyset$.

A subset A ε-captures F if the number of sets $S \in F$, for which $S \cap A = \emptyset$ is $< \varepsilon \binom{n}{k}$

A = {1,2}



Lemma: If F is ε-far from intersecting and A ε-captures F then $\exists B, C \subseteq A$ s.t. $B \cap C = \emptyset$ and $F_1 = \{S \in F | S \cap A = B\}$, $F_2 = \{S \in F | S \cap A = C\}$ are ε'-far from cross-intersecting.

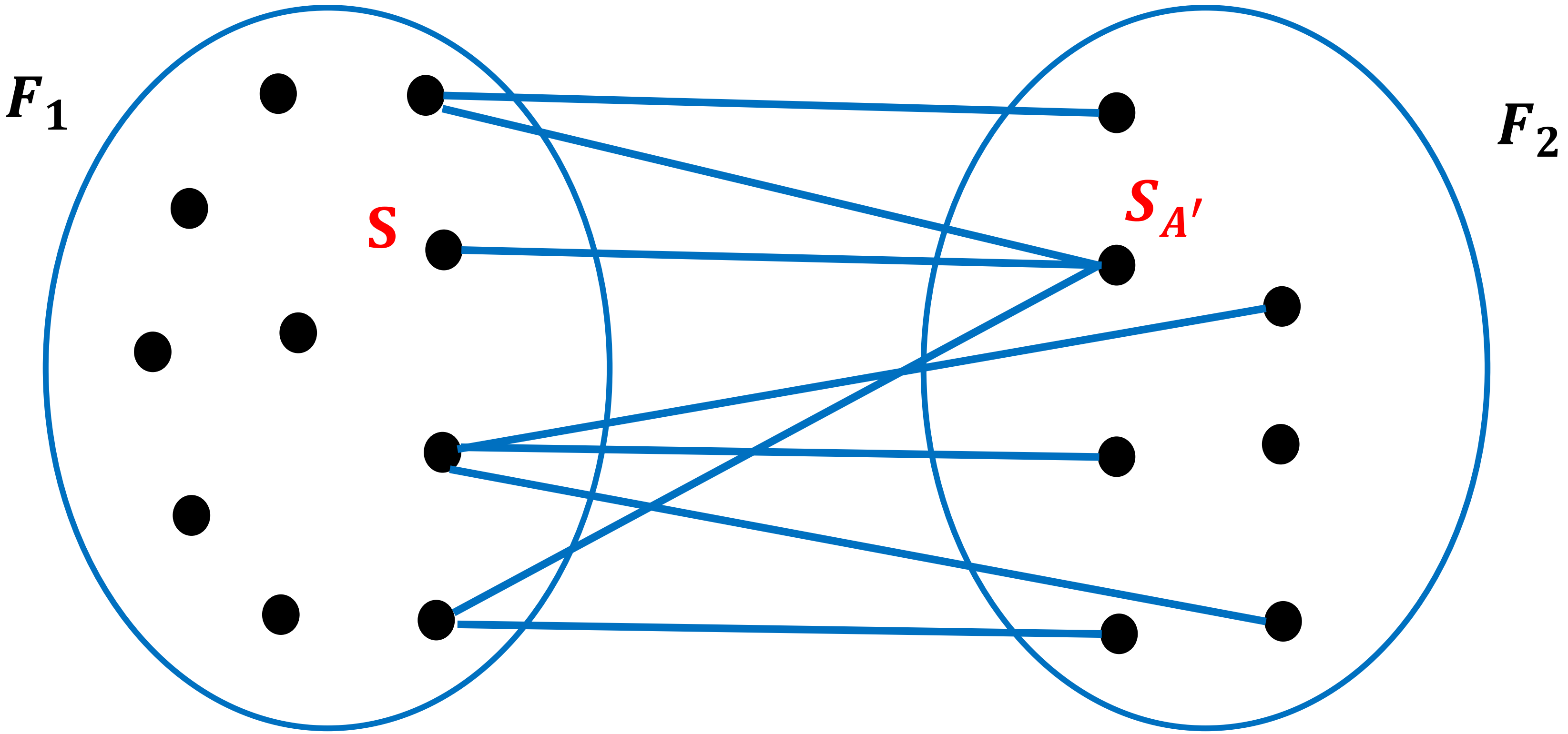


$A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$

Edges: sets don't intersect!

If F is ε-far from intersecting then many edges cross between F_1 and F_2 → Show that algorithm will sample a pair of disjoint sets from F_1 and F_2.

$F_1 = \{S \in F | S \cap A = B\}$ $F_2 = \{S \in F | S \cap A = C\}$ $B \cap C = \emptyset$



Assume, $\forall A' \subseteq [n] \setminus A,\ |A'| < r$, we sampled $S_{A'} \in F_2$, such that $S_{A'} \cap A' = \emptyset$

But number of sets $S \in F_1$ that intersect all sets $S_{A'}$ is small.

⇨ After a few more samples we get a set $S \in F_1$ that doesn't intersect one of the sets $S_{A'}$

⇨ Algorithm rejects. Otherwise, exists A' that captures F_2…

Proof of Lemma

Lemma: Number of sets S in $\binom{[n]}{k}$ that intersect all sets S_A is $\leq k^r \binom{n-r}{k-r} \leq \left(\frac{k^2}{n}\right)^r \binom{n}{k}$

Proof:

$S_\emptyset = \{j_1, \dots\}$, $A = \emptyset$

$j_1 \notin S_{j_1} = \{j_2, \dots\}$, $A = \{j_1\}$

$j_1, j_2 \notin S_{j_1, j_2} = \{j_3, \dots\}$, $A = \{j_1, j_2\}$

$S_{j_1, j_2, \dots, j_{r-1}} = \{j_r, \dots\}$, $A = \{j_1, j_2, \dots, j_{r-1}\}$

To intersect all sets S_A, a set S must contain at least one of the k^r possible subsets $\{j_1, j_2, \dots, j_{r-1}, j_r\}$.

Tolerant Property Testing: It’s all in the name

New York, 2003: The apartment of Ronitt and Ran.

Dana and I were visiting Ronitt.

We need a good name like the PCP Theorem!

Since then TPT became famous!

TPT today:

- Transport
- Transactional Privilege Tax
- Third Party Transfer
- Trailer Park Trash
- Throat Punch Thursday (2004).

TPT

TETON PETROLEUM TRANSPORT

TPT

Teachers Pay Teachers (2006)

TPT

stands for

Traffic Parking and Transportation

ABBREVIATIONS

Abbreviations.com

Time Partition Testing

TPT

t|p|t

Retirement Solutions

Tolerant Testing: Who is Tess?

Acknowledgement: We acknowledge the contribution of Tess in our attempts to obtain an improved tolerant testing algorithm for monotonicity in higher dimensions.

Reviewer 2: I do not know who "Tess" is. Please put in her (his?) full name.

Reply: Tess is a dog and therefore does not have a last name...

Open Problems

Tester	Condition	Query Complexity
Two sided error Tolerant	$\varepsilon_2 \geq \Omega\left(\varepsilon_1 + \frac{k}{n}\right)$	$O\left(\frac{1}{\varepsilon_2}\right)$
	$\varepsilon_2 \geq \Omega\left(\varepsilon_1 + \left(\frac{k}{n}\right)^r\right), \quad r \geq 2$	$O\left(\frac{\ln(n)}{\varepsilon_2}\right)$
One sided error	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^2\right)$	$O\left(\frac{1}{\varepsilon}\right)$
	$\varepsilon \geq \Omega\left(\left(\frac{k^2}{n}\right)^r\right), \quad r \geq 3$	$O\left(\frac{\ln(k)}{\varepsilon}\right)$

- Are log factors necessary?
- Find optimal testers for all values of ε.

- Find other interesting properties with a complexity gap between general and uniform case.

Here's to many more years of research and friendship.