Monotonicity testing, routing, and a theorem of Lehman and Ron

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Thanks to my teachers

Manindra Agrawal

Bernard Chazelle

Mike Saks

Tamara Kolda

Ali Pinar

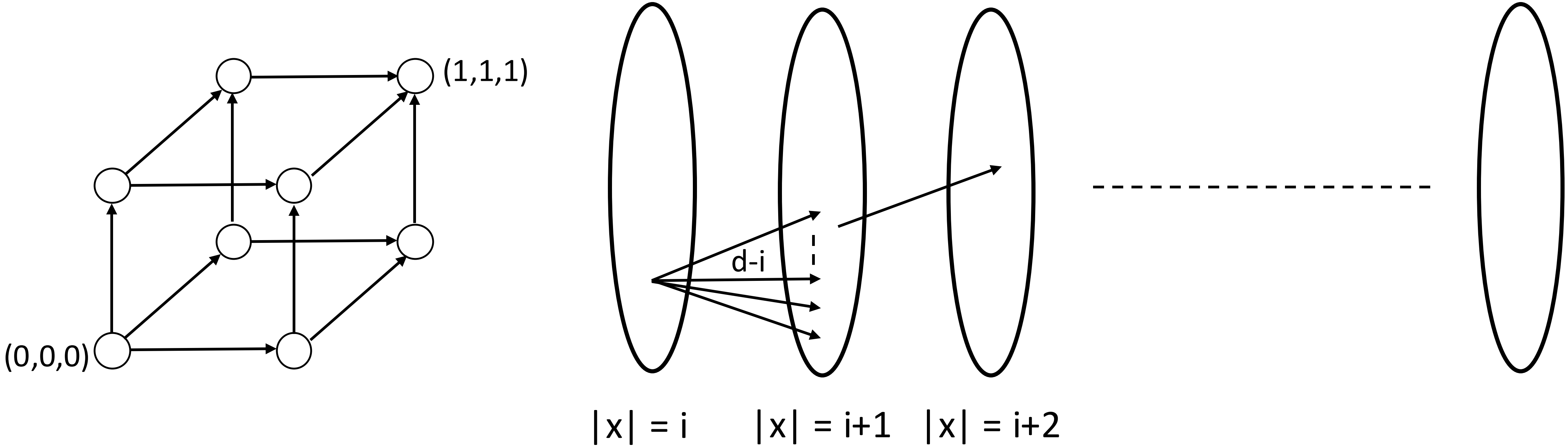
Big (biggest?) influence on my work

Dana Ron

The Lehman-Ron theorem

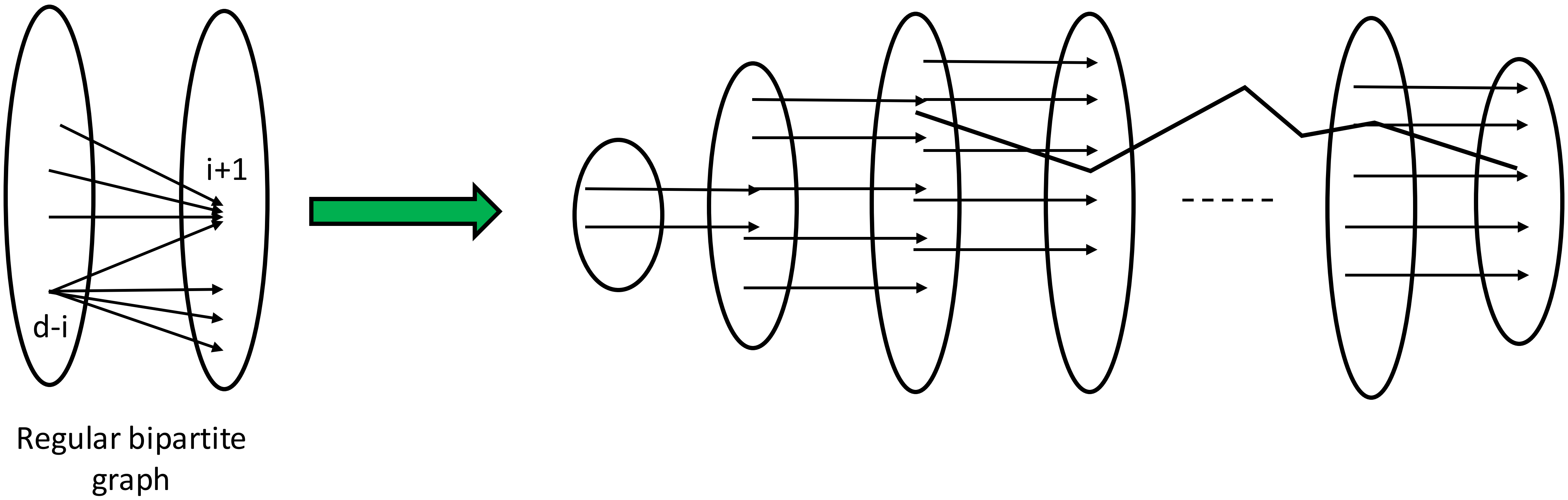
A lesser known result

The (directed) Boolean hypercube



- $\{0,1\}^d$ as a directed graph
- $x \prec y$ if $\forall i$, $x_i \leq y_i$
- A collection of levels with (regular) bipartite graphs

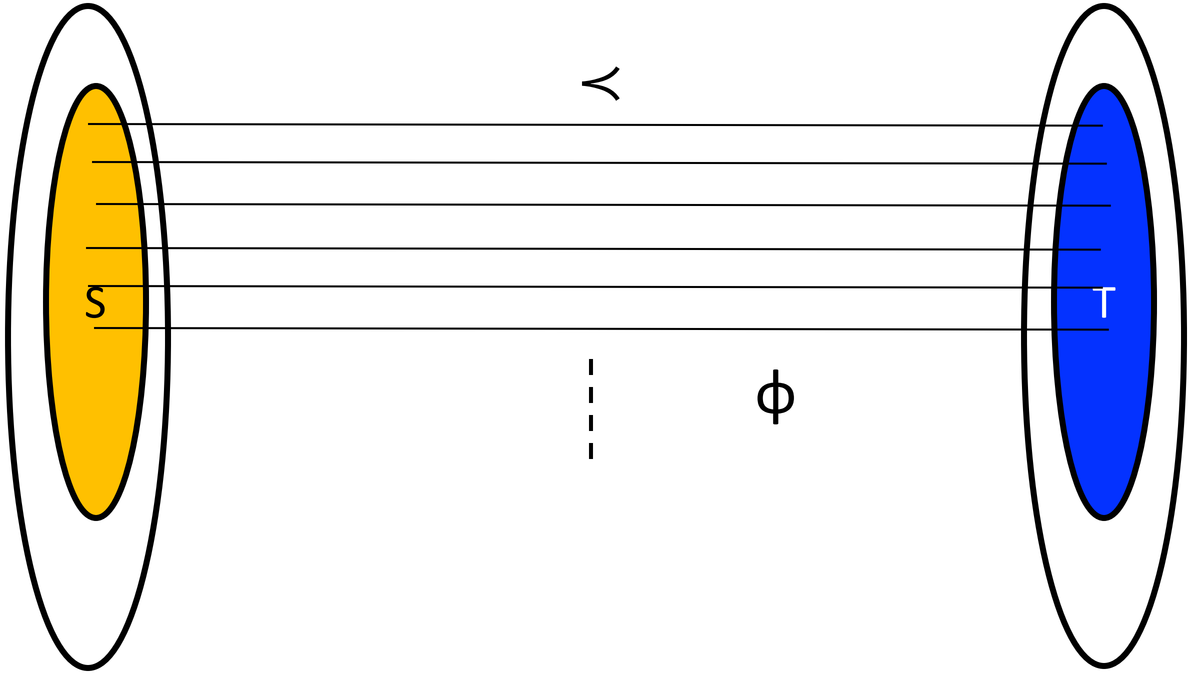
Some facts



By Hall's theorem, graph has matching from smaller side to larger

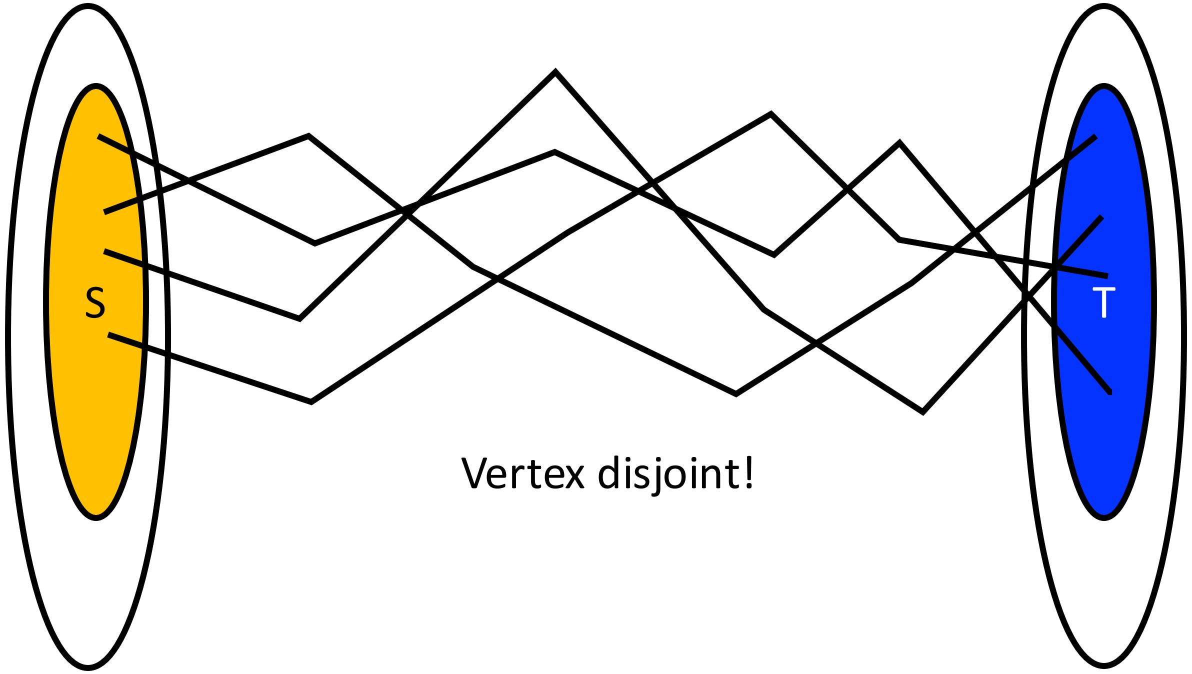
Symmetric Chain Decomposition

The LR theorem



- [Lehman-Ron 01] Consider S, T level subsets with bijection ϕ: S -> T where $s \prec \phi(s)$

The LR theorem

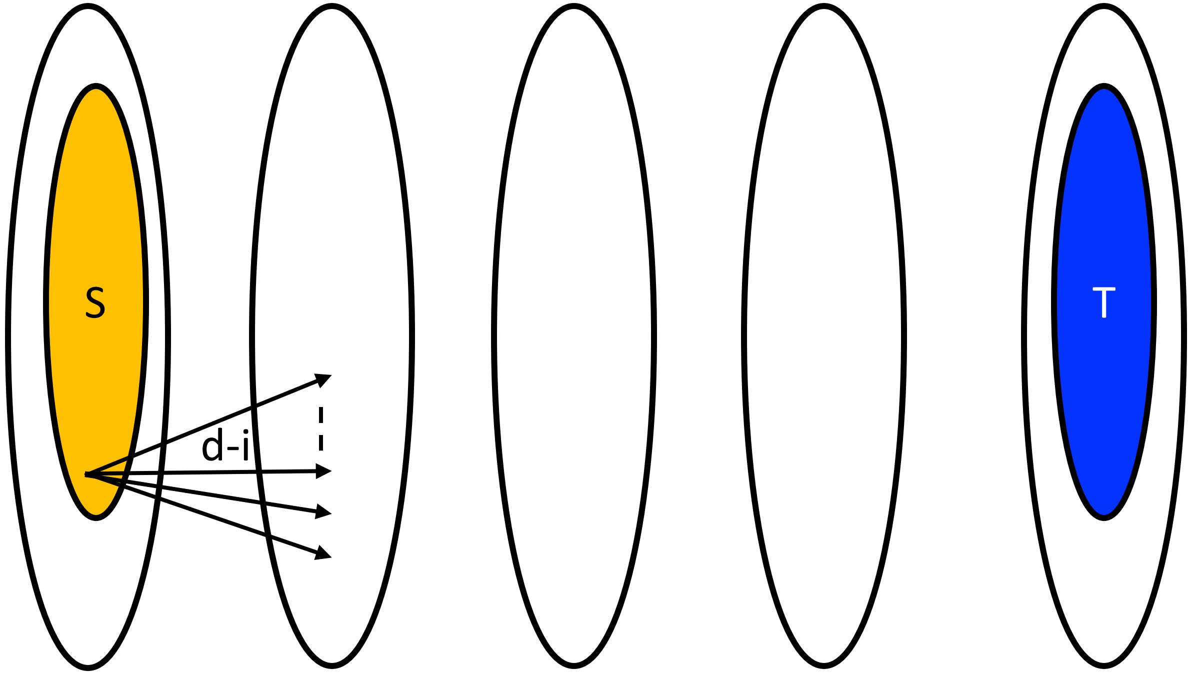


- [Lehman-Ron 01] Consider S, T level subsets with bijection ϕ: S -> T where $s \prec \phi(s)$

There exists |S| vertex disjoint paths from S to T

Fundamental discovery about hypercube

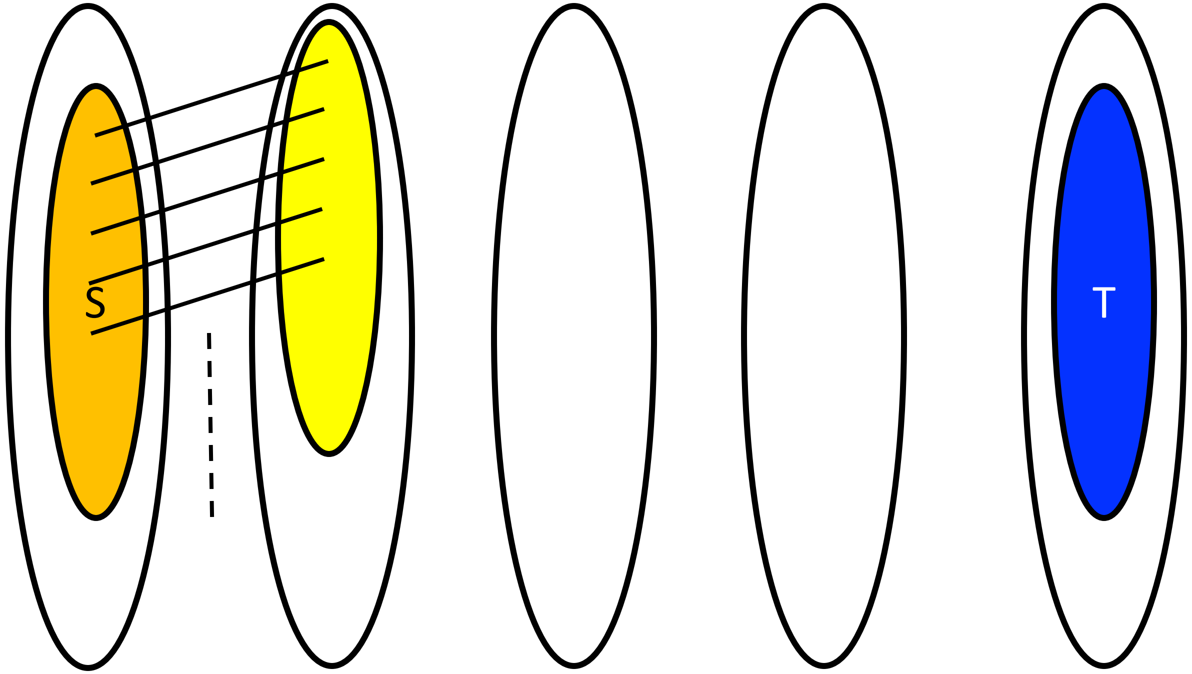
By Hall's theorem, graph is disjoint union of matchings



Regular bipartite graph

Fundamental discovery about hypercube

By Hall's theorem, graph is disjoint union of matchings



Regular bipartite graph

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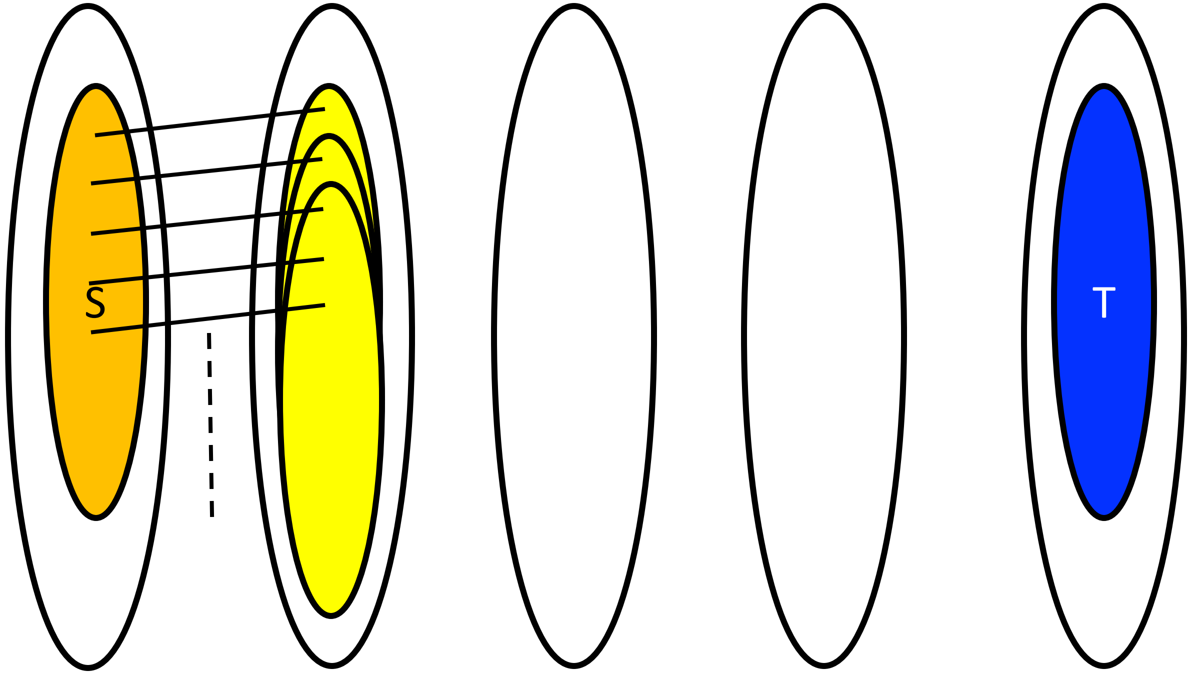
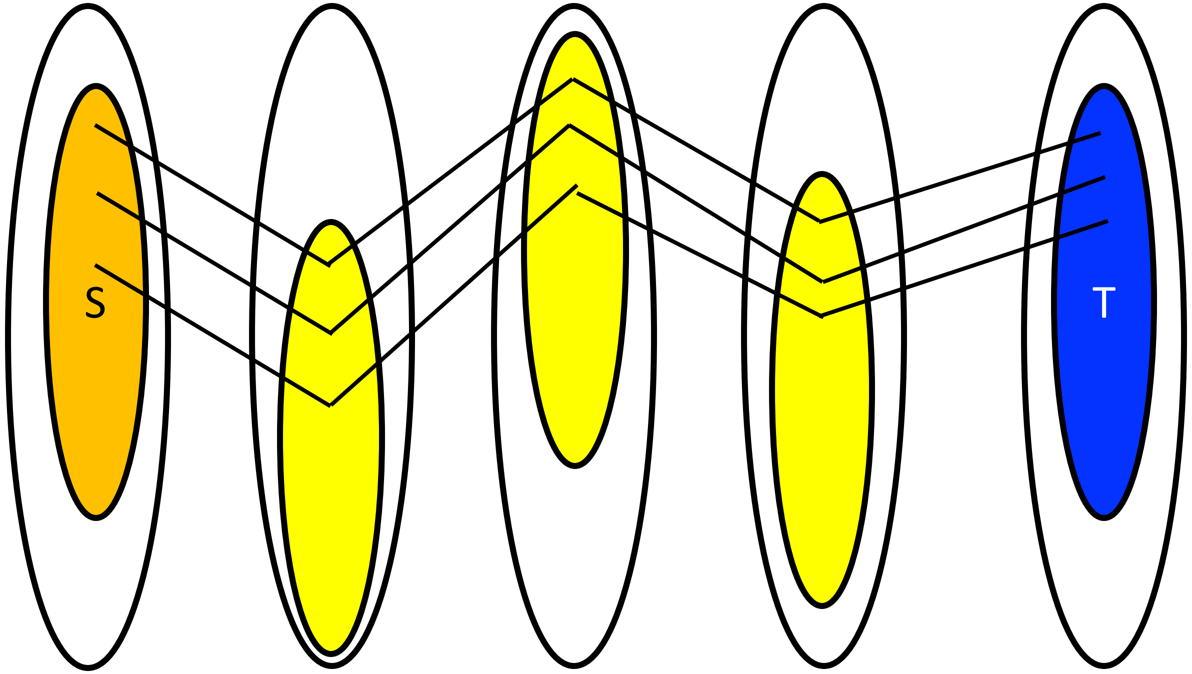


Fundamental discovery about hypercube

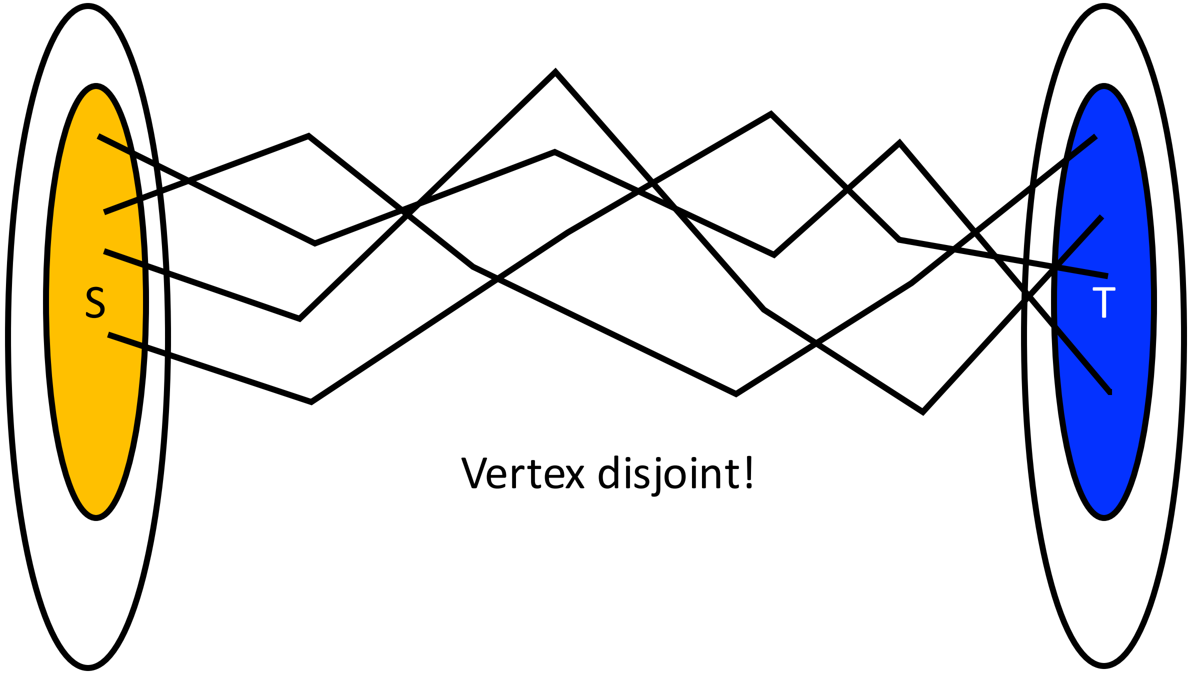
By Hall's theorem, graph is disjoint union of matchings



Regular bipartite graph

- One can choose and "chain" matchings to get vertex disjoint paths

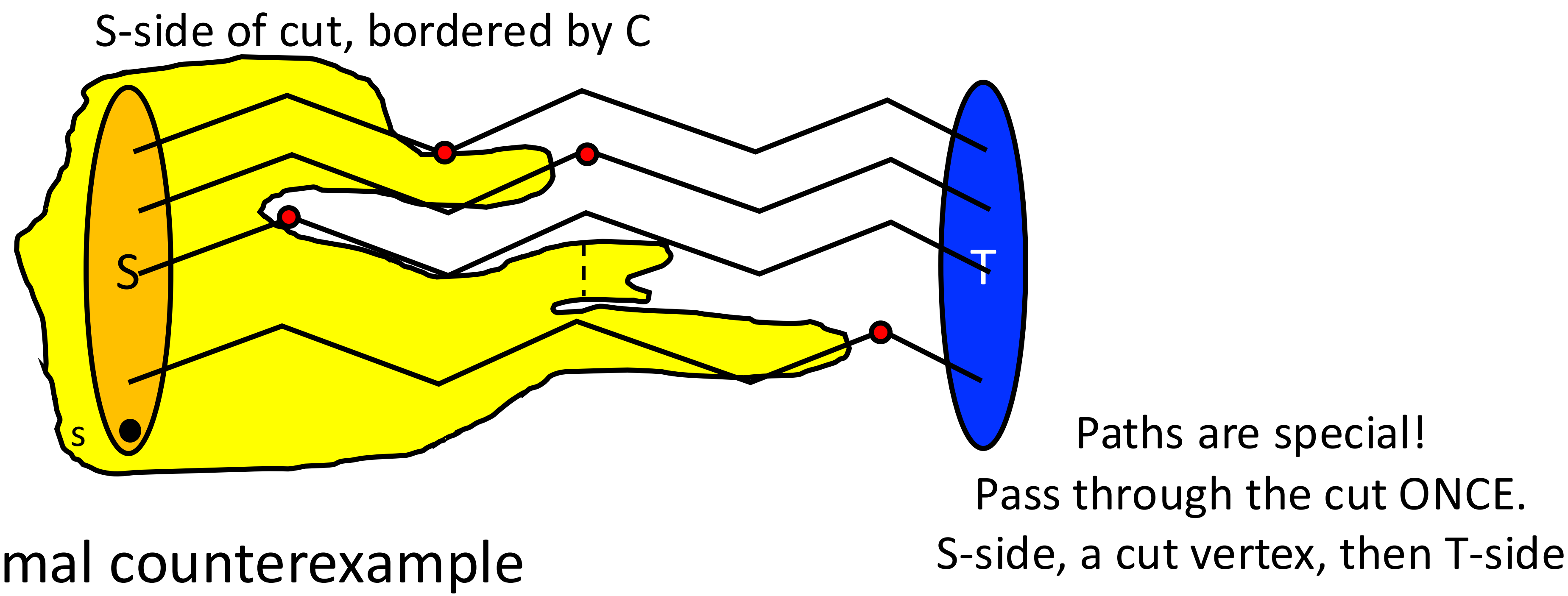
The LR theorem



- [Lehman-Ron 01] Consider S, T level subsets with bijection ϕ: S -> T where s $\prec$ ϕ(s)

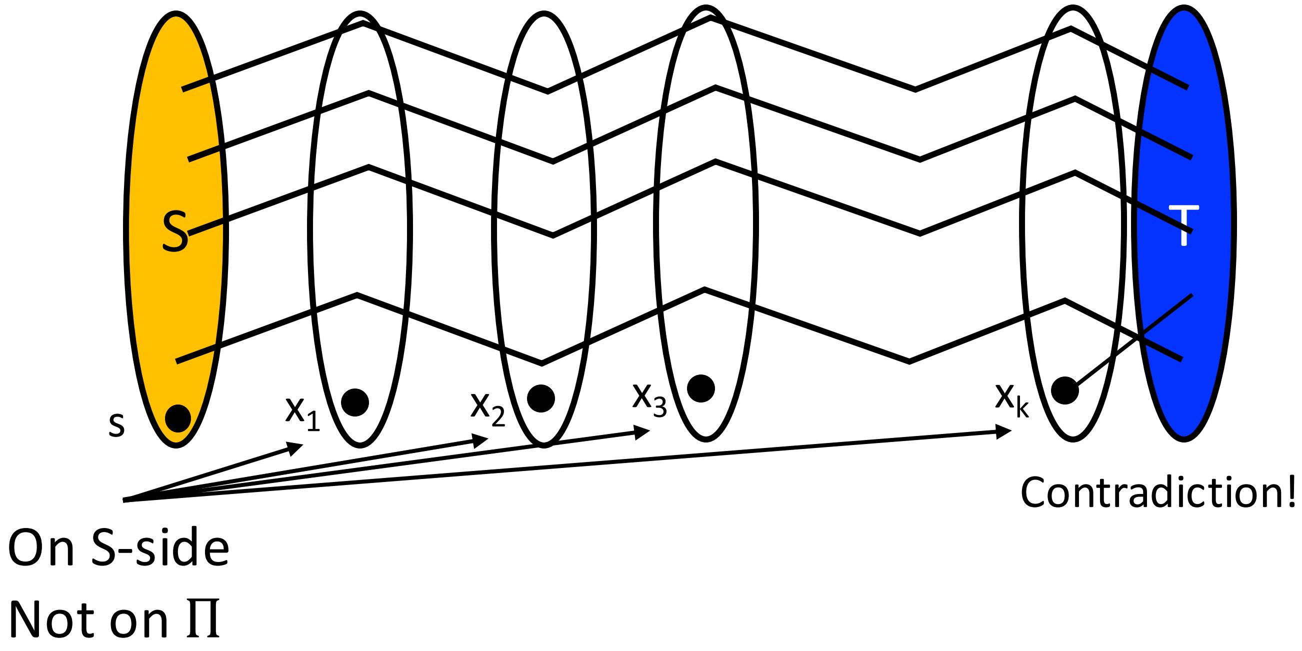
There exists |S| vertex disjoint paths from S to T

A new proof



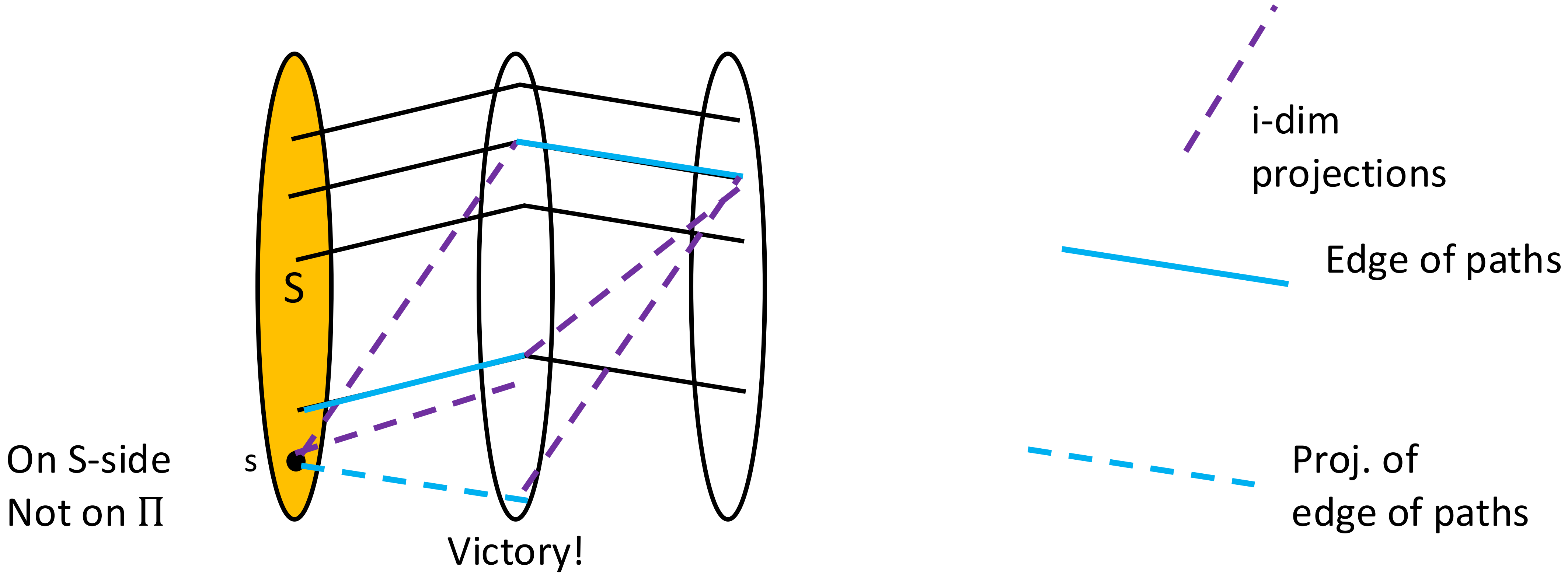
- Consider minimal counterexample
- By MaxFlow-MinCut (Menger), S-T cut C has size |S|-1
- By duality and complementary slackness, there exist |S|-1 vertex disjoint paths from S to T saturating C. Call it Π
  - A single s does not participate in Π

The inchworm plan



- Focus on graph G formed by union of all S-T paths

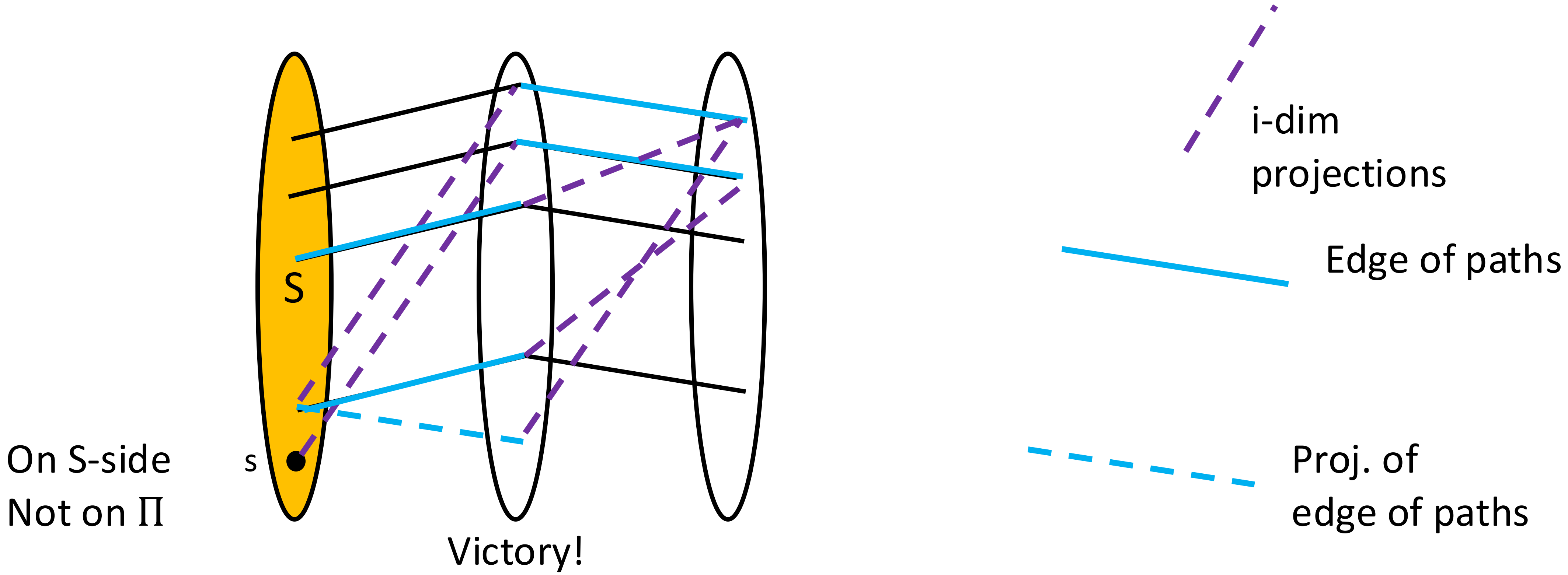
The inchworm plan



What did the adversary tell s?

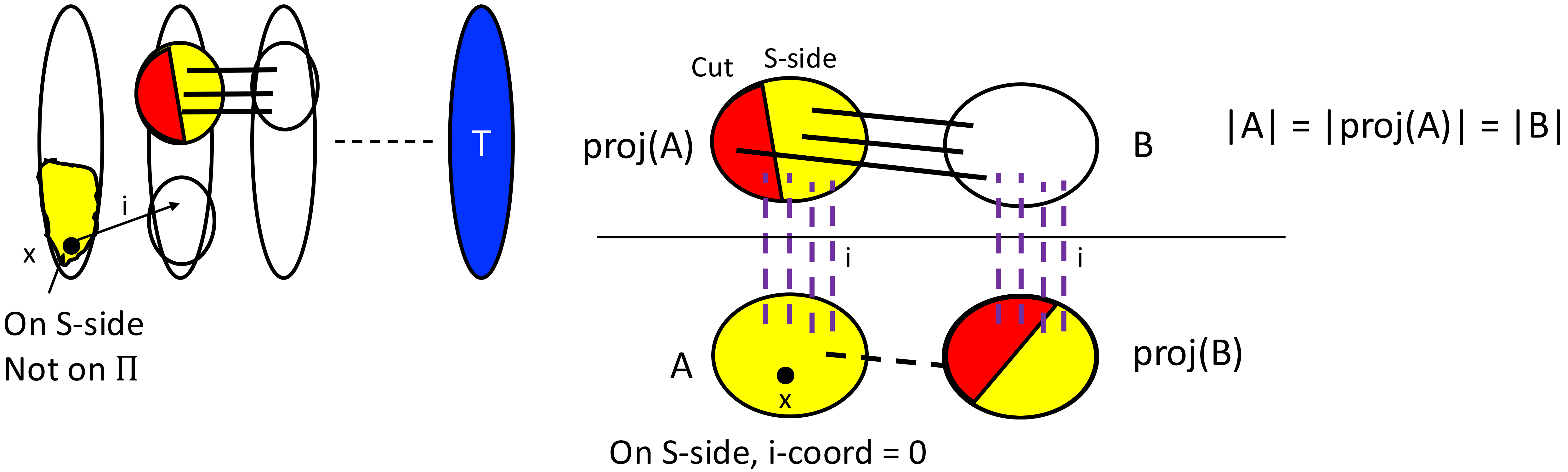
Despair will lead you nowhere.

The inchworm plan



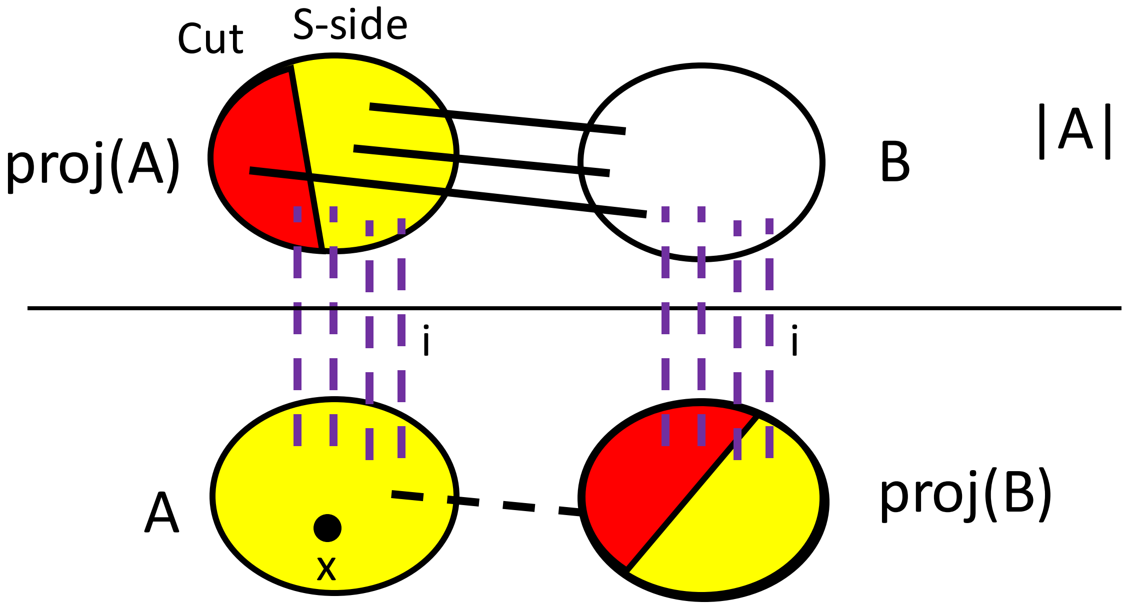
- Always taking matchings, so same vertex never visited twice
  - We start from s, never see it again
- The process has to complete. Victory!

One step



- For contradiction, all “further vertices” on S-side are on Π
  - Whether in the cut or on S-side, must participate in Π

Completing the path…er…argument

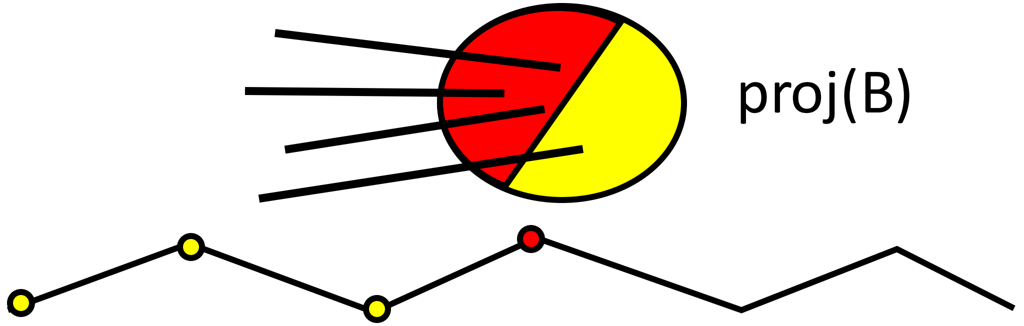


On S-side, i-coord = 0

|A| = |proj(A)| = |B|

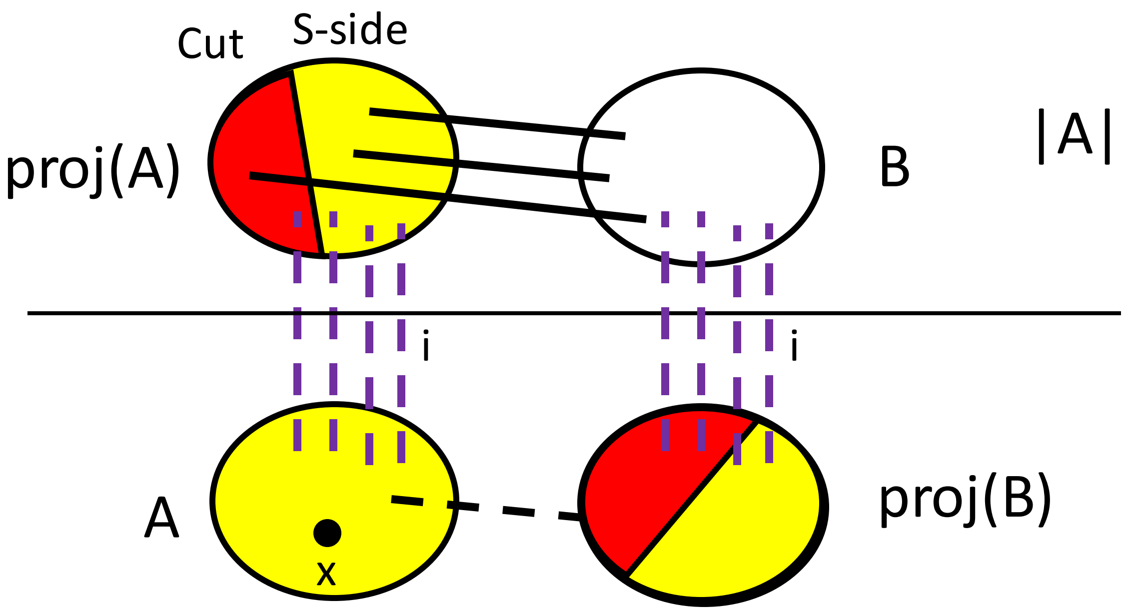
There are |proj(B)| = |A|
Π-paths through proj(B)

Follow backwards.



Must go to S-side, so goes to A!
|A| paths going into A

Completing the path…er…argument



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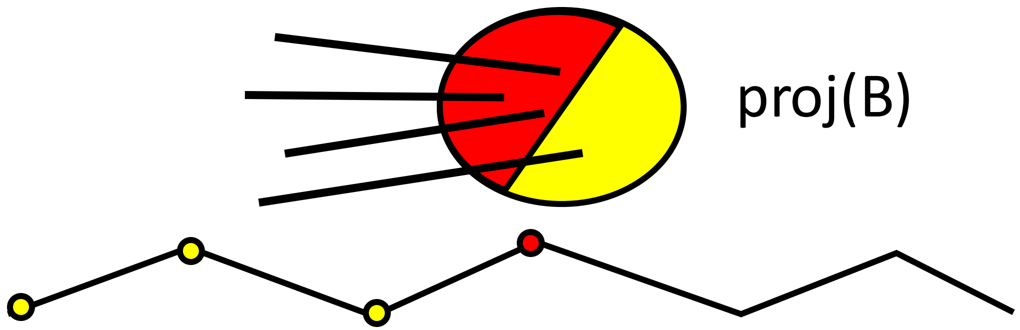
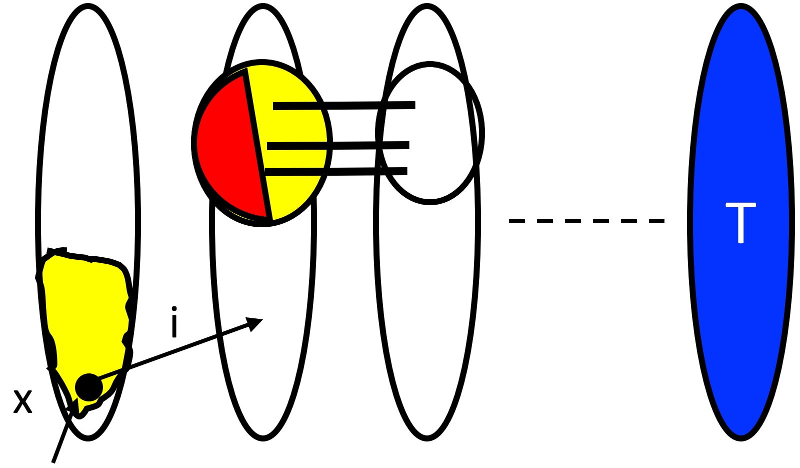


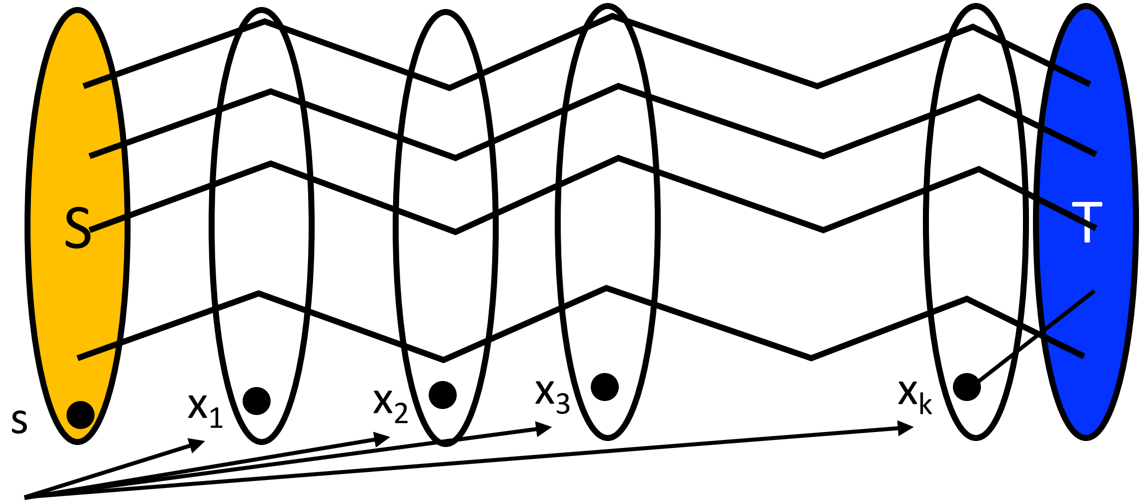
Must go to S-side, so goes to A!
|A| paths going into A

- But x has no paths through it, so at most |A|-1 paths through A
- Contradiction!

And so…



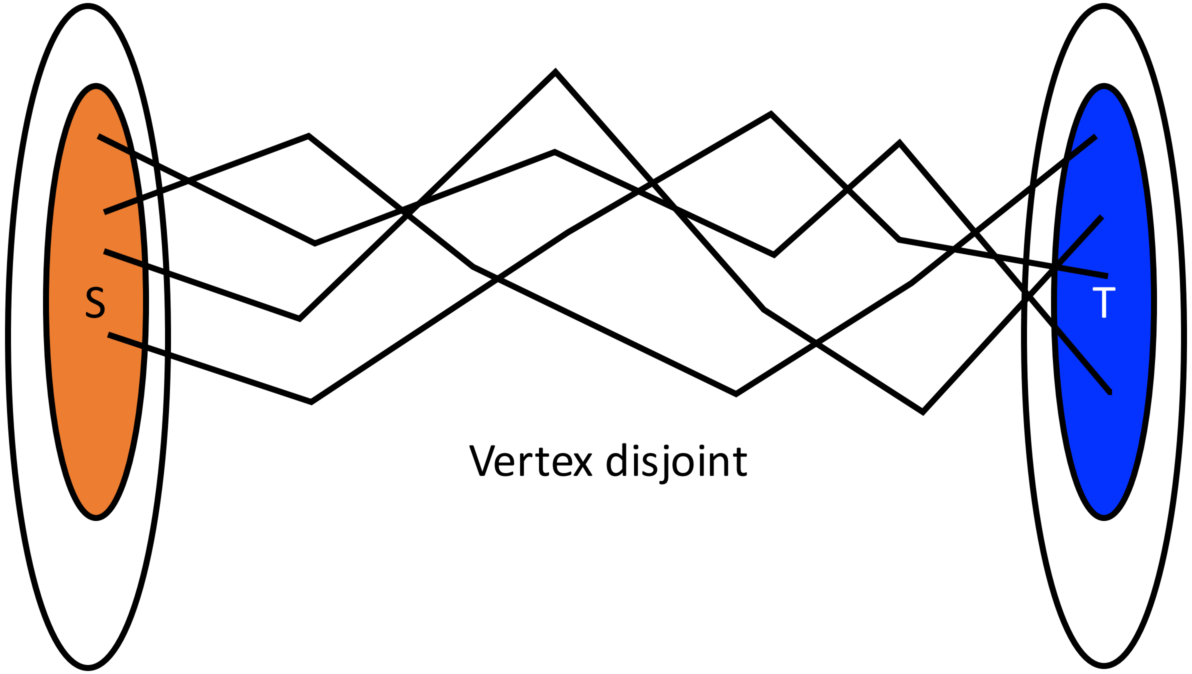
On S-side
Not on Π

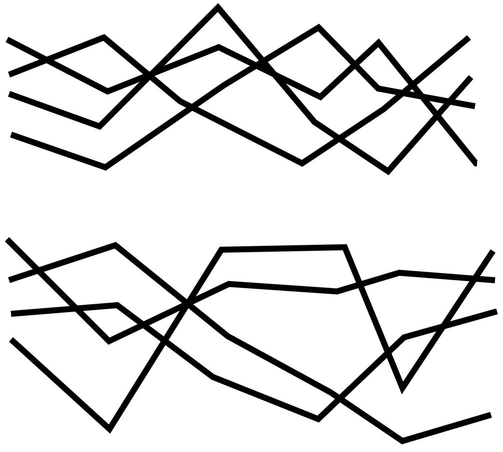


On S-side
Not on Π

- ~~For contradiction, all "further vertices" on S-side are on Π~~
- There is a further vertex on S-side, not on Π

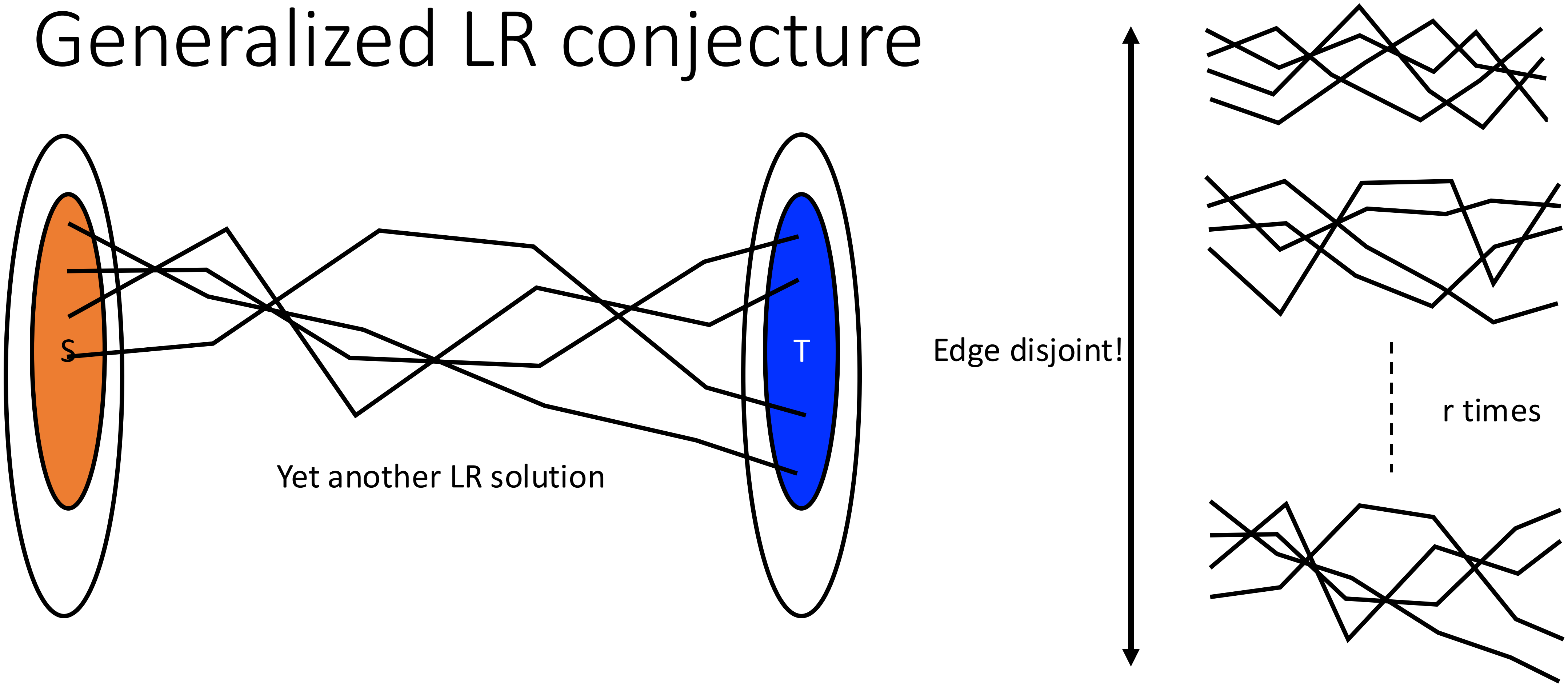
The birthday gift





- [Chakrabarty-S 24] (S, T; ϕ) as before. Suppose S, T are distance r > 1 apart
- LR solution is a set of |S| vertex-disjoint paths
- There are TWO edge disjoint LR solutions

Generalized LR conjecture

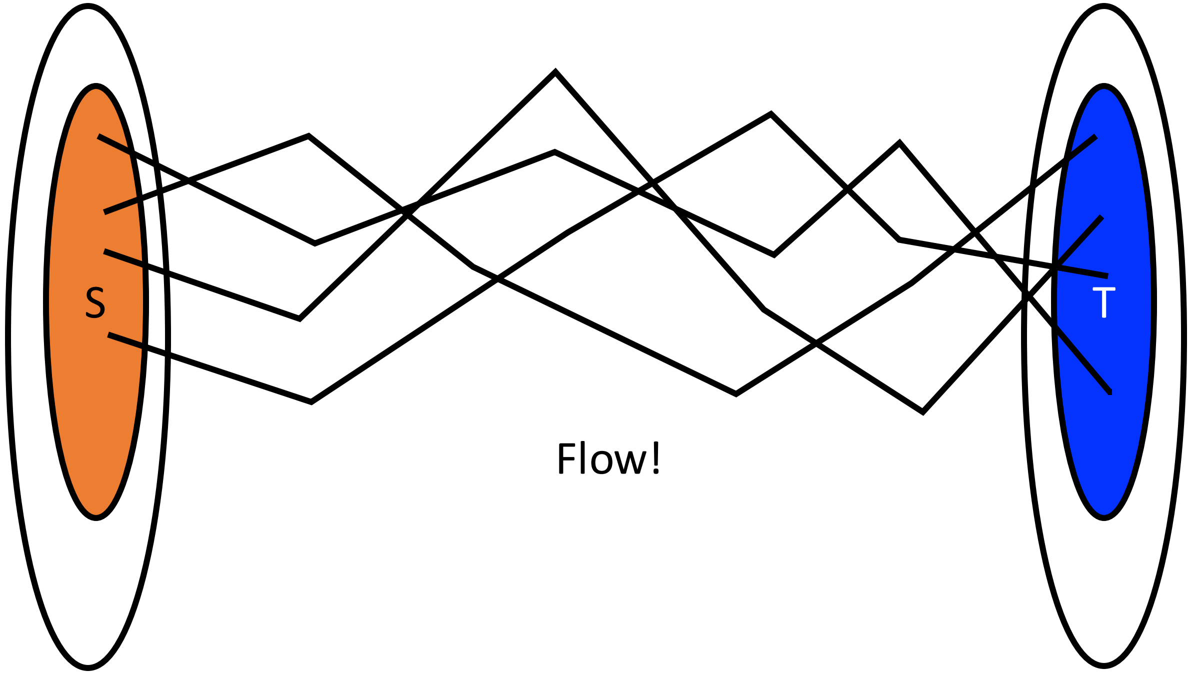


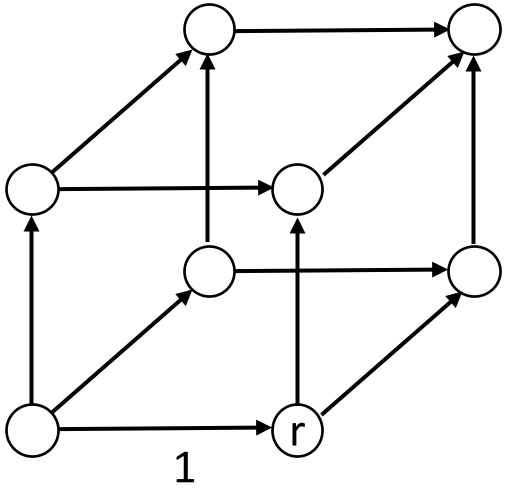
(S, T; ϕ) as before. Suppose S, T are distance r apart

There exist r edge disjoint LR solutions

(We can only show r=2)

Generalized LR conjecture in flows





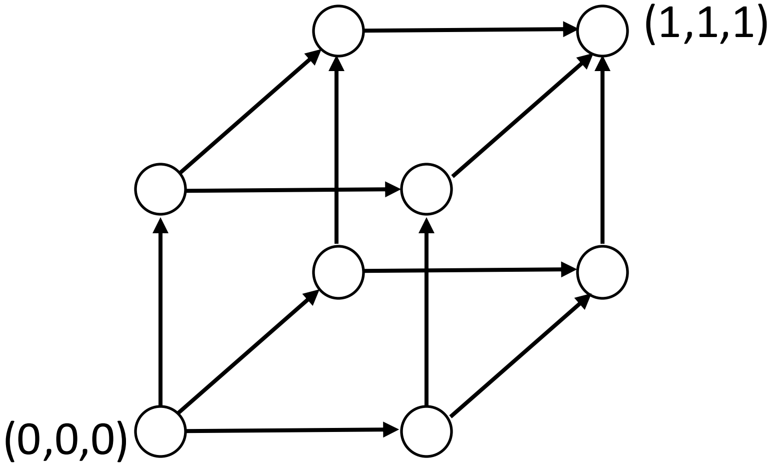
- (S, T; ϕ) as before. Suppose S, T are distance at least r apart
- Edge capacity is 1, vertex capacity is r

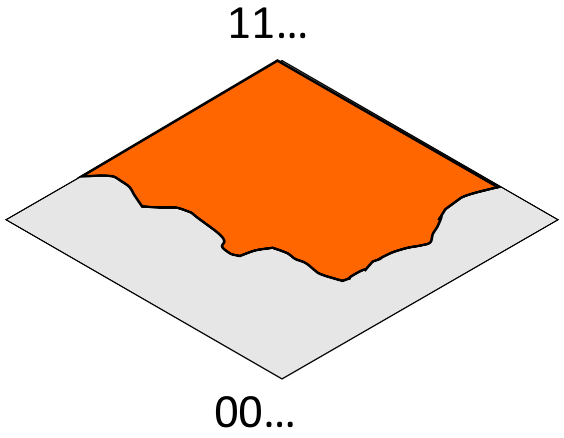
r|S| units of S-T flow can be routed

Why?

A story of success, as well as failure

Monotone functions

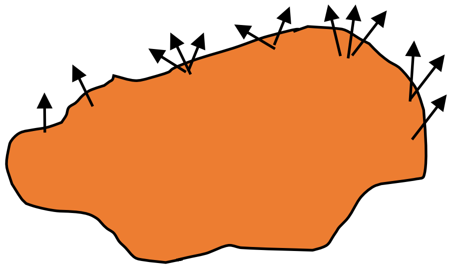




- [Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky 00]
- $f:\{0,1\}^d \mapsto \{0,1\}$
  - $x \prec y$ if $\forall i, x_i \leq y_i$
- f is monotone: if $x \prec y$, $f(x) \leq f(y)$
- Given ε: distinguish monotone ($\varepsilon_f = 0$) vs far from monotone ($\varepsilon_f > \varepsilon$)
  - What is the (non-adaptive) complexity of monotonicity testing?

The history

Measures of directed surface area = Directed Isoperimetric Theorem



Paper	Directed Isoperimetric Theorem	Tester Query Complexity	Core Proof Complexity
GGLRS00	Poincare	d	2 pages
CS13, DST14	Margulis	$d^{5/6}$	8 pages
KMS15	Talagrand	$\sqrt{d}$	30 pages

- LR theorem is central to proving directed Margulis
  - First step to o(d) testers
  - Margulis has (elegant?) combinatorial proof
- Could generalized LR theorems prove directed Talagrand?
  - KMS15 proof is "analytic"

The flow connection

Paper	Directed Isoperimetric Theorem	Tester Query Complexity	Core Proof Complexity
GGLRS00	Poincare	d	2 pages
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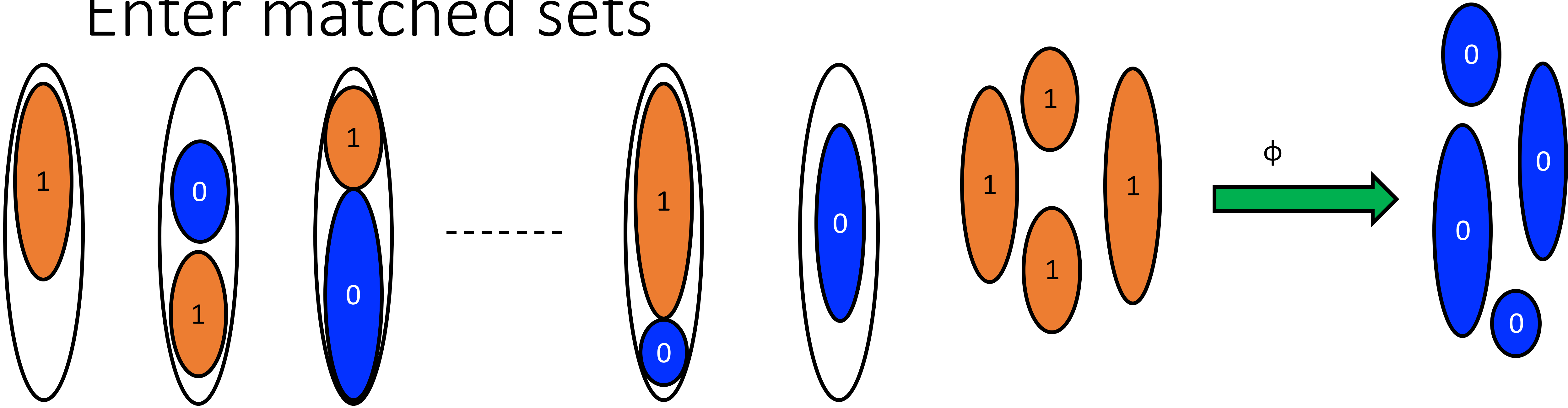
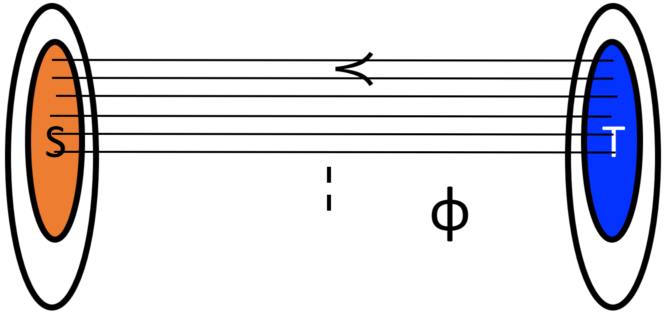
- Each directed isoperimetric theorem “equivalent” to a directed hypercube flow
- Poincare ≈ Only edge capacities
- Margulis ≈ Stronger Poincare + Only Vertex capacities
- Talagrand ≈ Simultaneous edge and vertex capacities

Enter matched sets

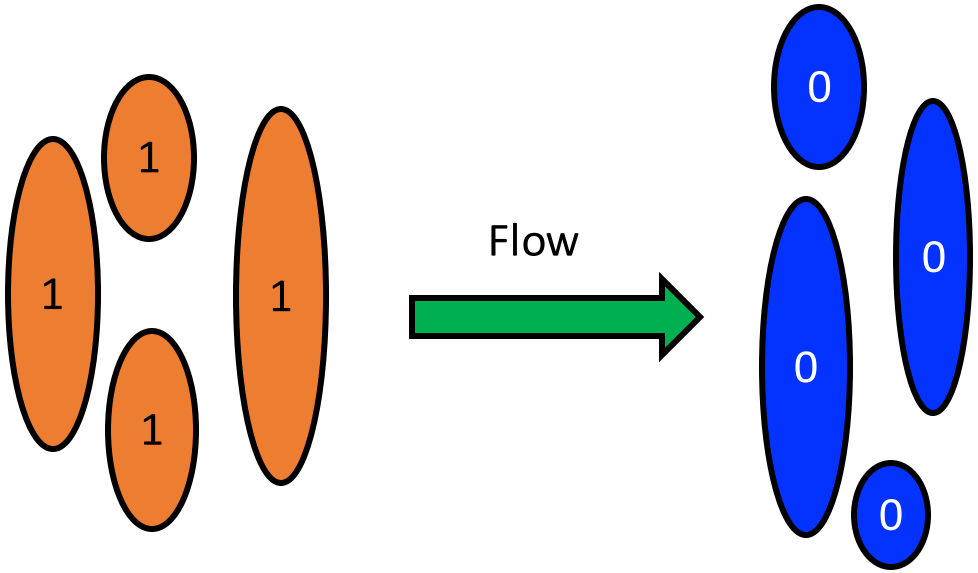


- [GGLRS00, FLN+02] f is Ω(1)-far from monotone. Then there exist sets S, T, and comparison bijection ϕ: S -> T such that:
  1. f(s) = 1, f(t) = 0
  2. |S| = |T| = $\Omega(2^d)$ (S, T are large)
  3. s ≺ ϕ(s)



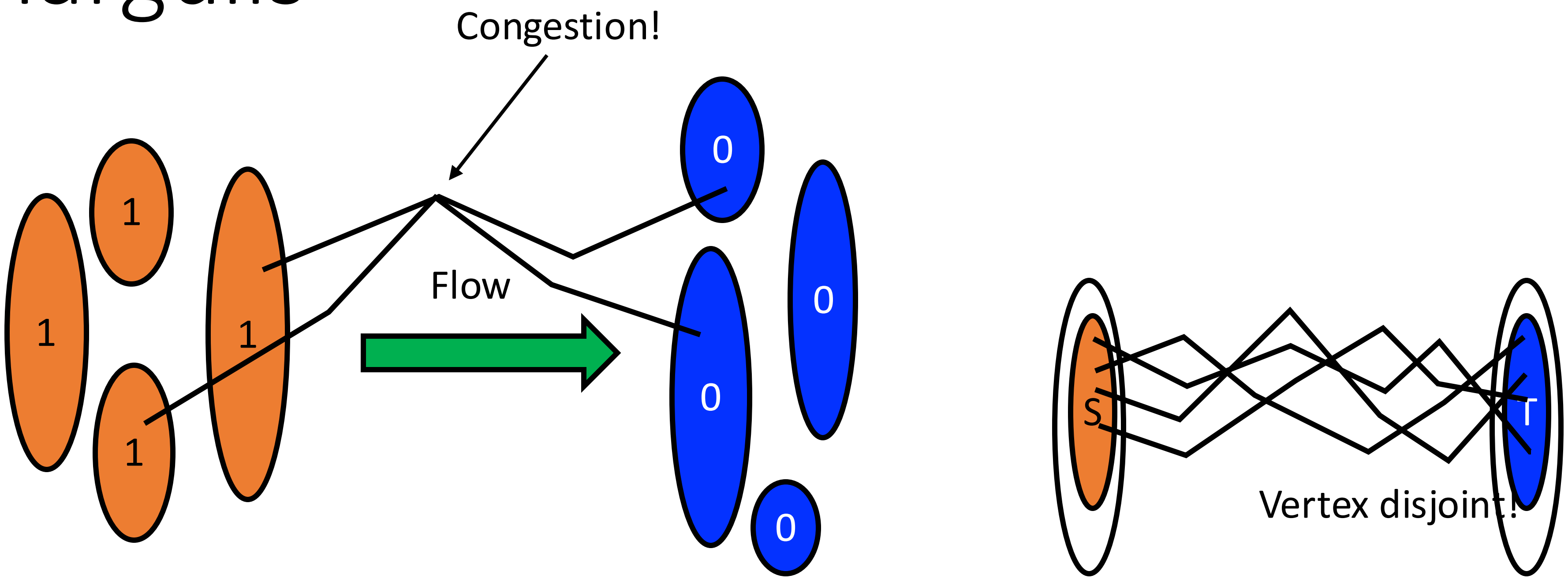
- Let r = “avg distance” = avg of |ϕ(s)| - |s|
  - Pick S, T that minimize r

Directed Poincare



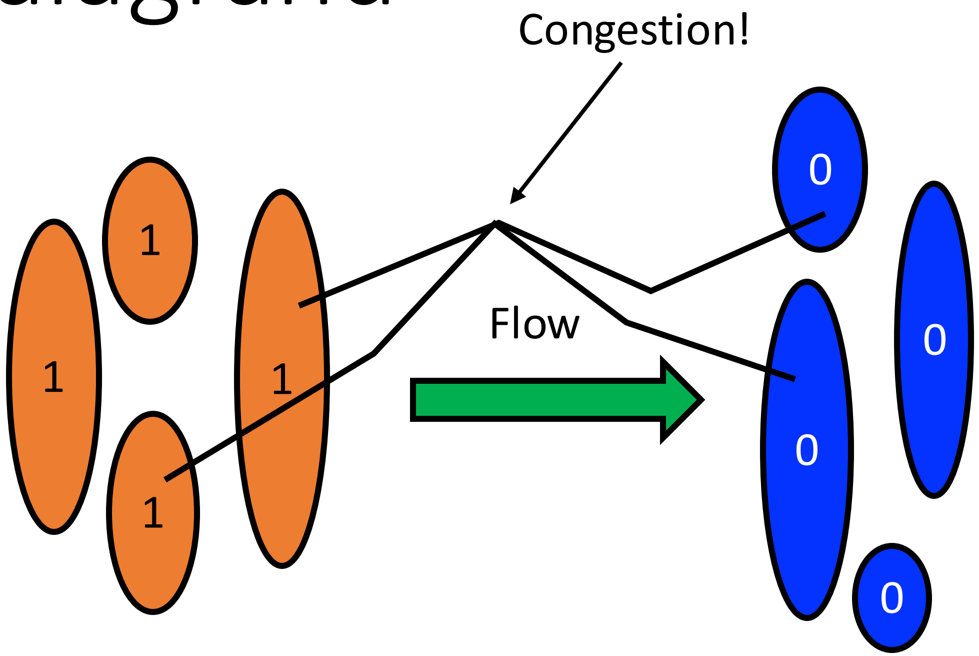
- Edge capacities = 1
- [GGLRS00] One can send |S| units of flow

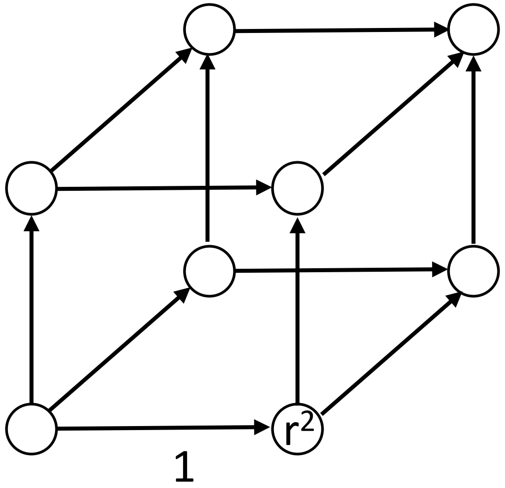
Directed Margulis



- [CS13] Avg distance = r
- With edge capacities 1, one can send r|S| units of flow AND
- With vertex capacities 1, one can send |S|/r units of flow
  - Based on LR theorem

Directed Talagrand



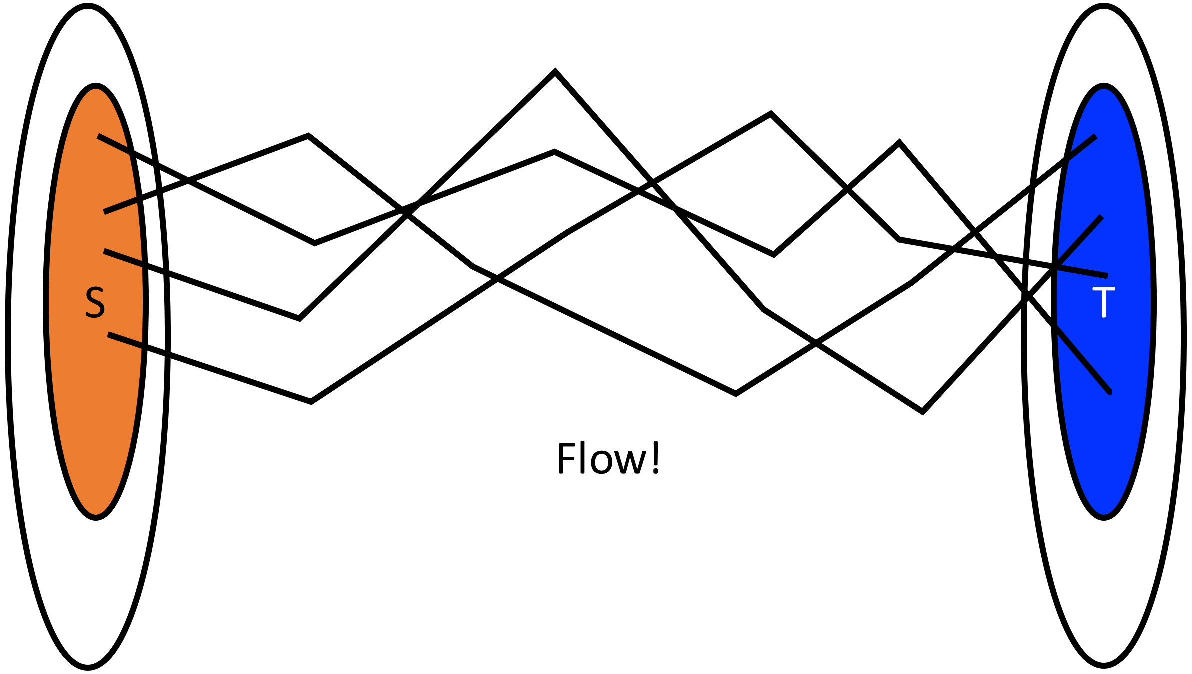


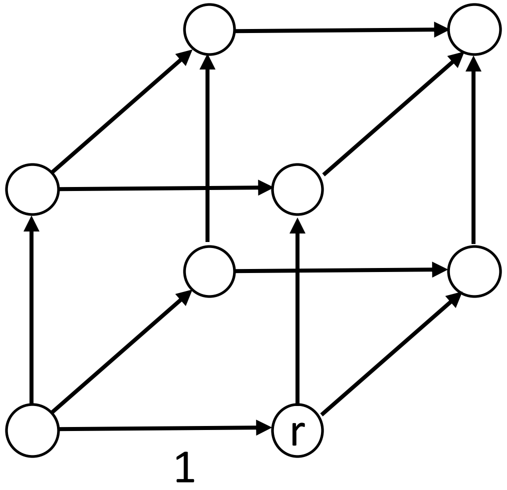
- Avg distance = r
- Edge capacities 1, vertex capacities r^2 ← From the sqrt in Talagrand
- One can send r|S| units of flow
  - Captures simultaneous edge/vertex constraints

Failure

- We thought generalized LR could prove directed Talagrand
- We thought we could prove generalize LR
- Failed on both counts…
- LR statements generalize to other product structures (hypergrids), so more flexible than KMS15 techniques
  - LR was key ingredient for o(d) Boolean monotonicity testing

I leave you with…





The Generalized Lehman-Ron conjecture

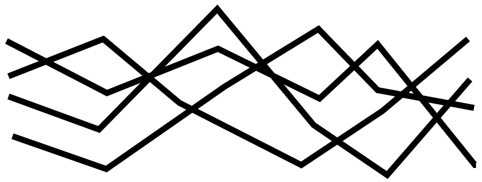
- (S, T; ϕ) as before. Suppose S, T are distance r apart
- Edge capacity is 1, vertex capacity is r

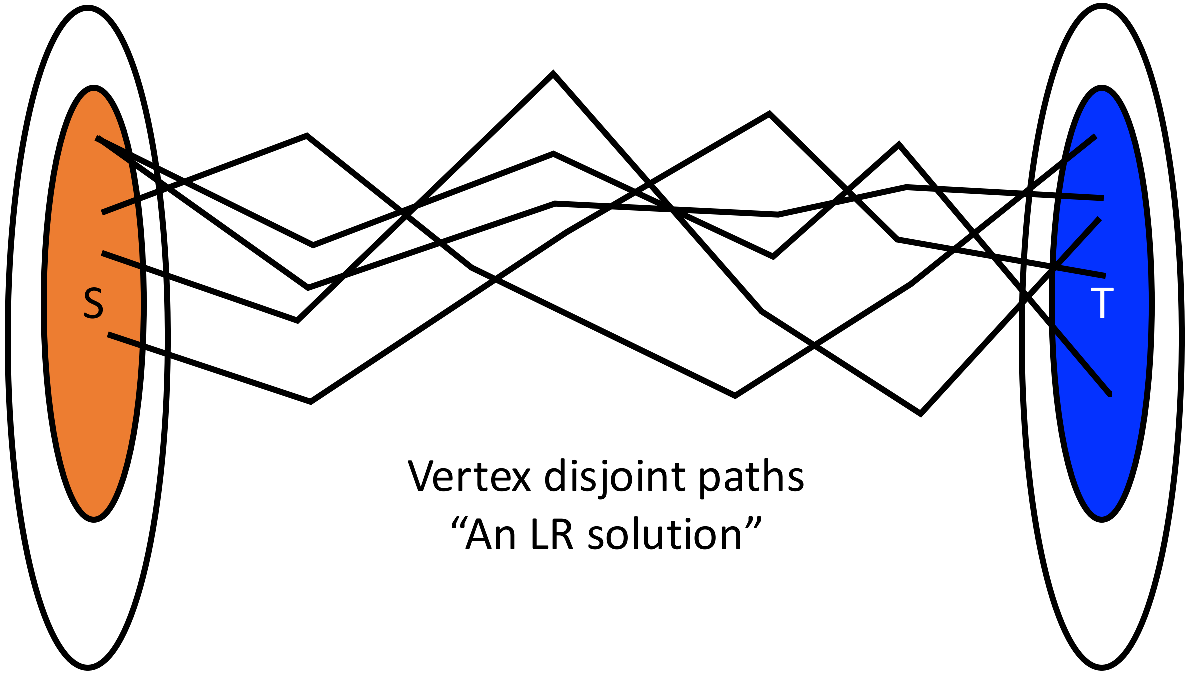
r|S| units of S-T flow can be routed

Thanks for all the intellectual light

Happy birthday!

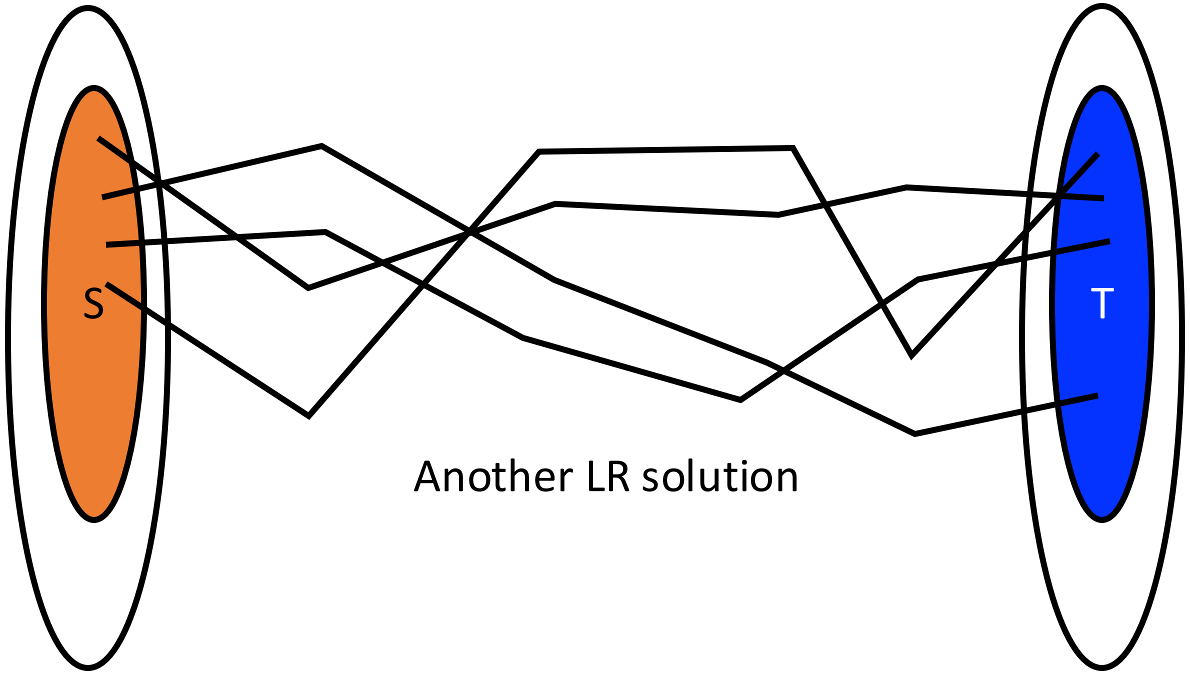
What does this mean?

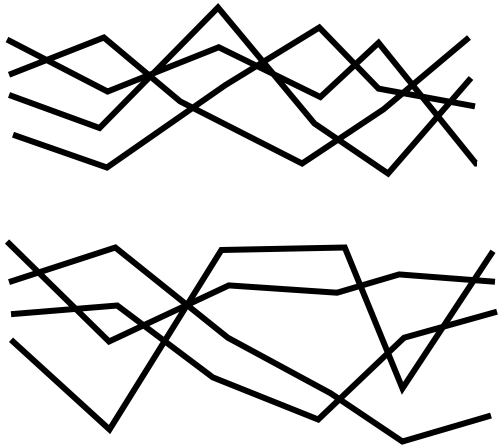




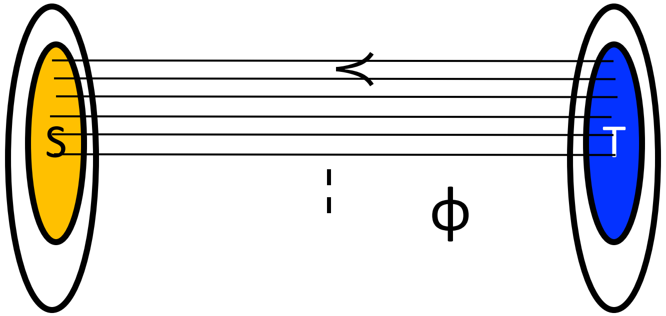
Vertex disjoint paths
“An LR solution”

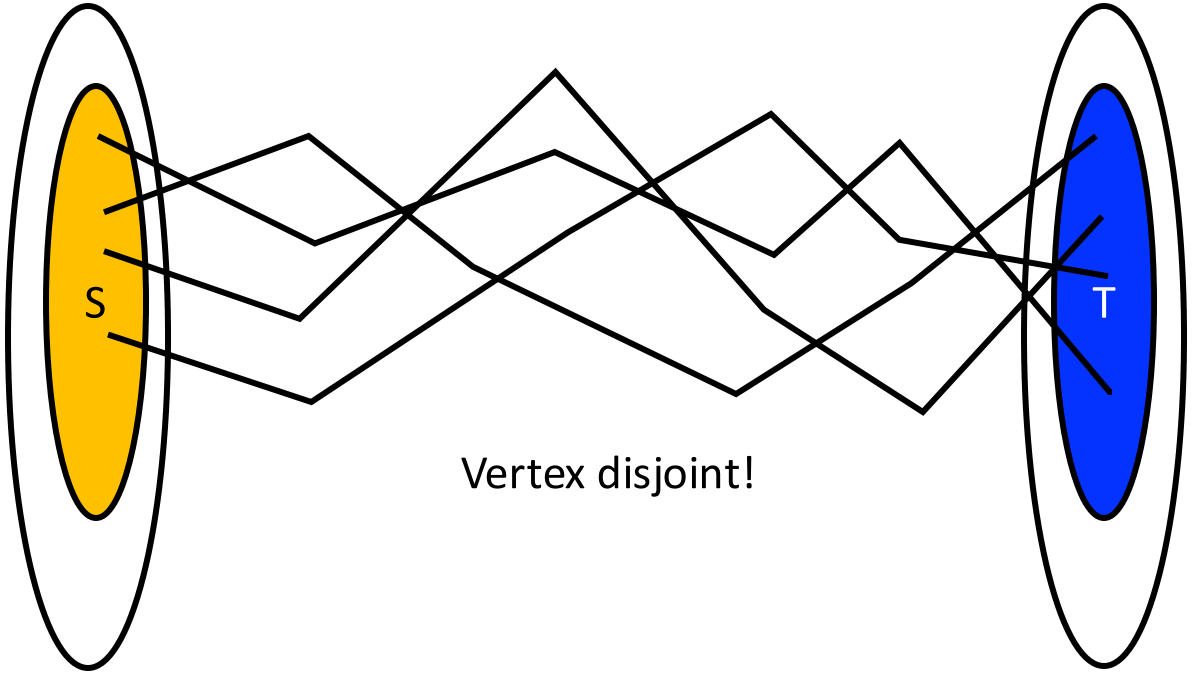
What does this mean?





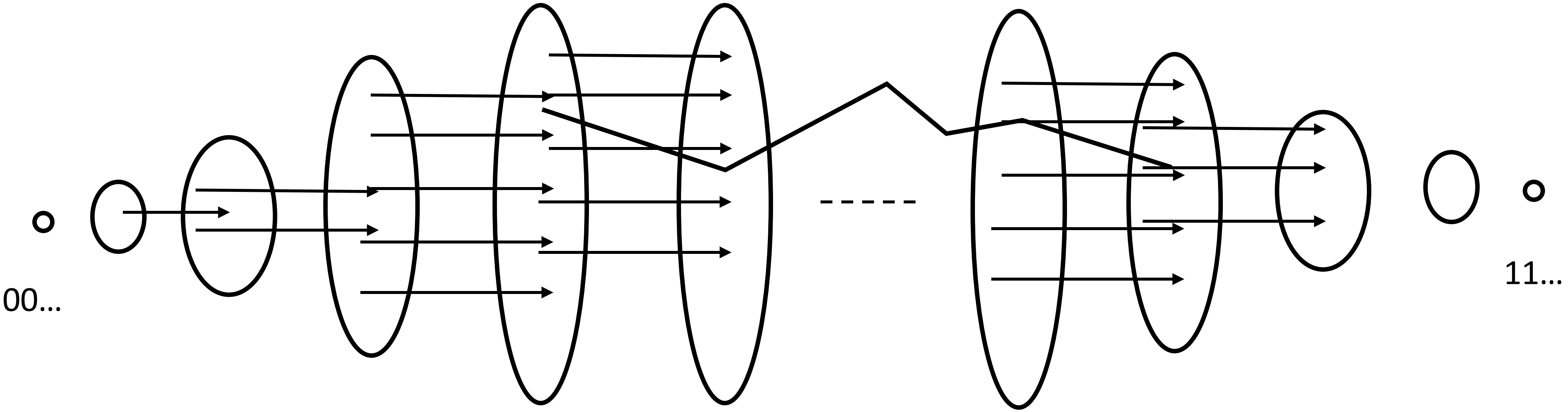
Caveats





- The paths might not "respect" ϕ
    - [Kleitmann, LR01] Counterexample, cannot get paths that route s to ϕ(s)
- [Briet-Chakraborty-GarciaSoriano-Matsliah12] ϕ-respecting not possible even for edge disjoint paths

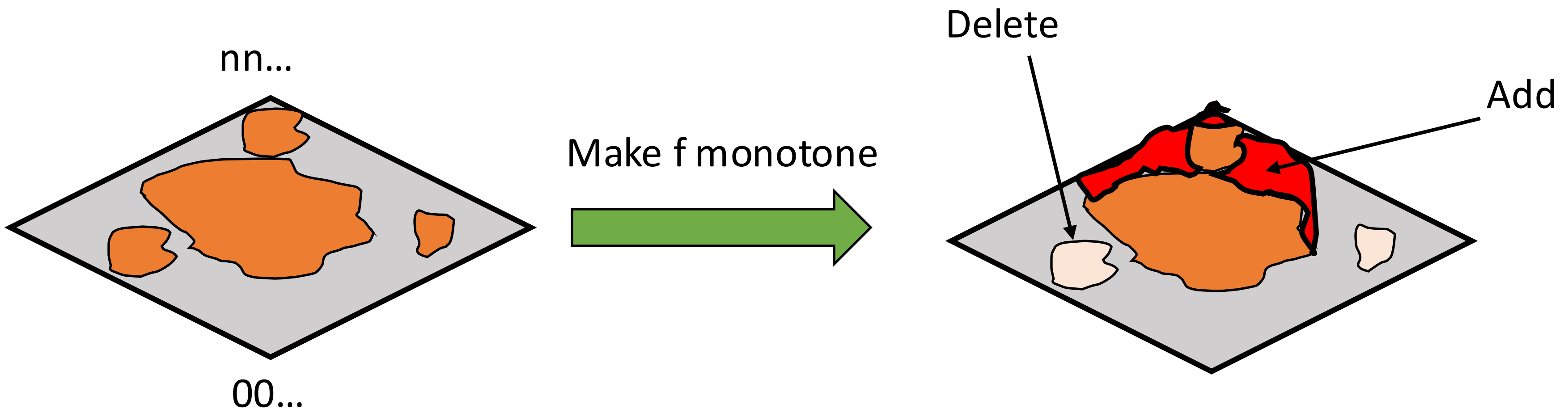
The Symmetric Chain Decomposition



- Symmetric Chain = Directed Path
- Vertices can be partitioned into symmetric chains

- Simply apply the matchings from previous slide

Monotonicity testing



- Distance to monotonicity = (min changes to make set monotone)/2^d
- ε_f in [0,1)
  - Amen
- Given ε: distinguish monotone (ε_f = 0) vs far from monotone (ε_f > ε)
  - What is the complexity of monotonicity testing? Can we get poly(d)?
  - Learning monotonicity needs > $\exp(\sqrt{d})$