Computing small Rainbow Cycle Numbers with SAT modulo Symmetries

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joint work with **Markus Kirchweger** (TU Wien)

FWF Der Wissenschaftsfonds.
WWTF VIENNA SCIENCE AND TECHNOLOGY FUND
ac ALGORITHMS AND COMPLEXITY GROUP
SIMONS INSTITUTE for the Theory of Computing

Outline

Fair allocation of goods

↓

Rainbow Cycle Number

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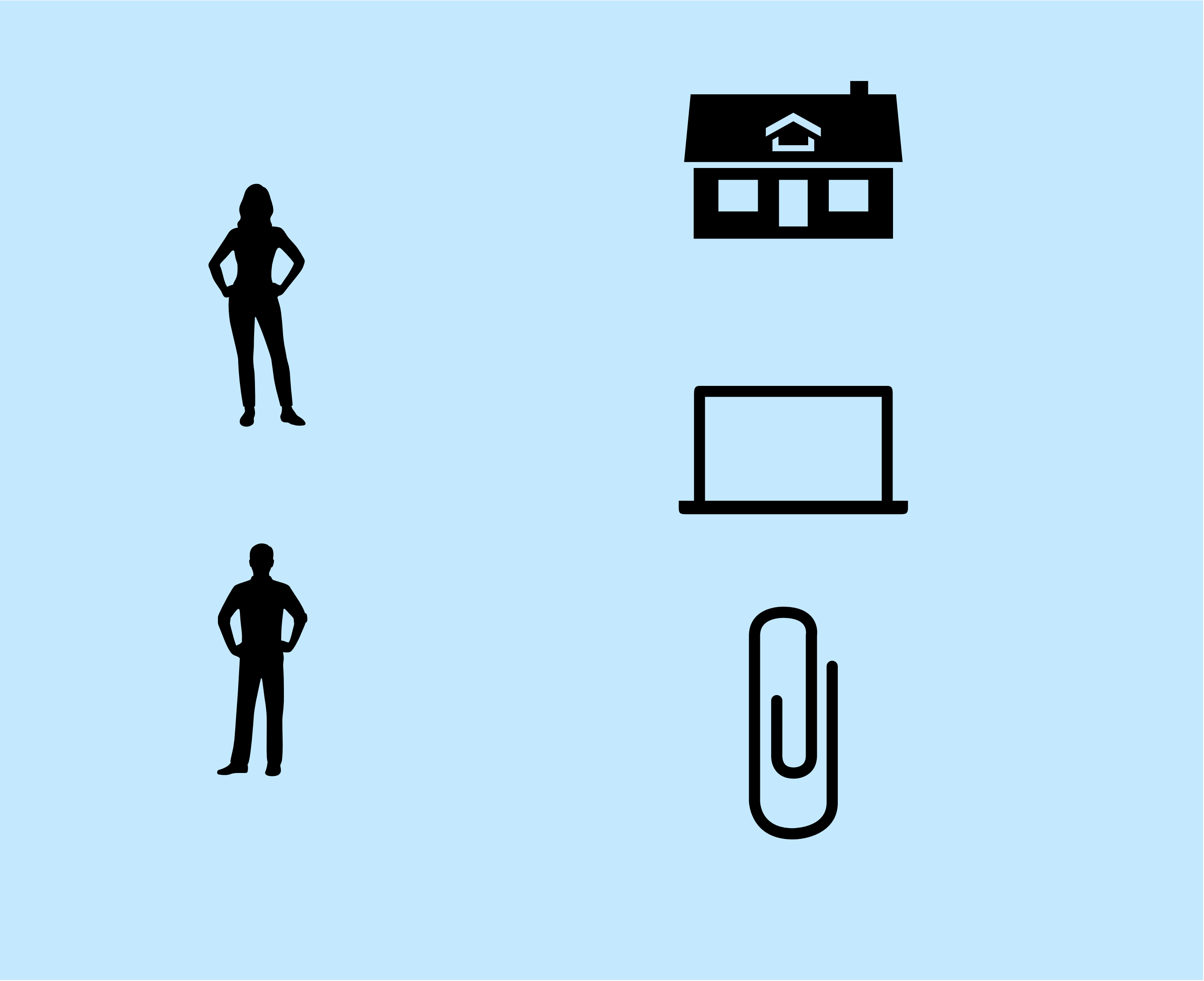
SAT modulo Symmetries

Fair allocation of goods

agents $\left\{ 1, \vdots, n \right.$ **allocation** $\left. \bullet, \vdots, \bullet \right\}$ set M of goods



- allocate to each agent i a set of goods $A_i \subseteq M$ such that

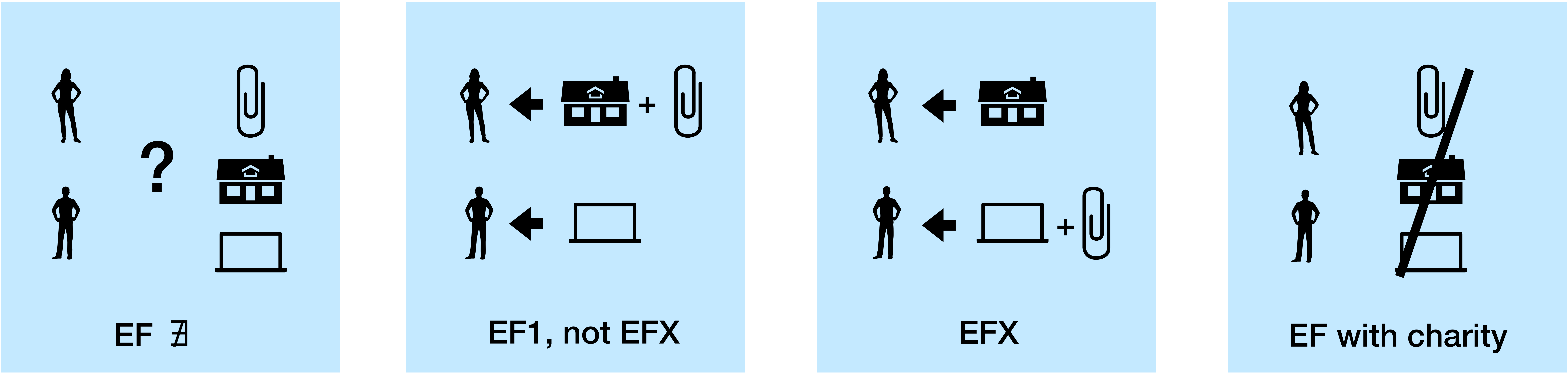
$$A_i \cap A_j = \varnothing \text{ and } \bigcup_i A_i = M$$

- a **partial** allocation allows some goods unassigned (or donated to *charity*)
- **valuation:** $v_i : 2^M \to \mathbb{R}_{\geq 0}$

			0	0
		✓	1	1
	✓		1000	1000
	✓	✓	1001	1001
✓			1000000	1000000
✓		1	1000001	1000001
✓	✓		1001000	1001000
✓	✓	✓	1001001	1001001

Envy-freeness

- EF - envy-free if $\forall i, j : v_i(A_i) \geq v_i(A_j)$
- EF1 - envy-free up to **one** good: $\forall i, j \exists g \in A_j : v_i(A_i) \geq v_i(A_j \setminus \{g\})$
- EFX - envy-free up to **any** good: $\forall i, j, \forall g \in A_j : v_i(A_i) \geq v_i(A_j \setminus \{g\})$

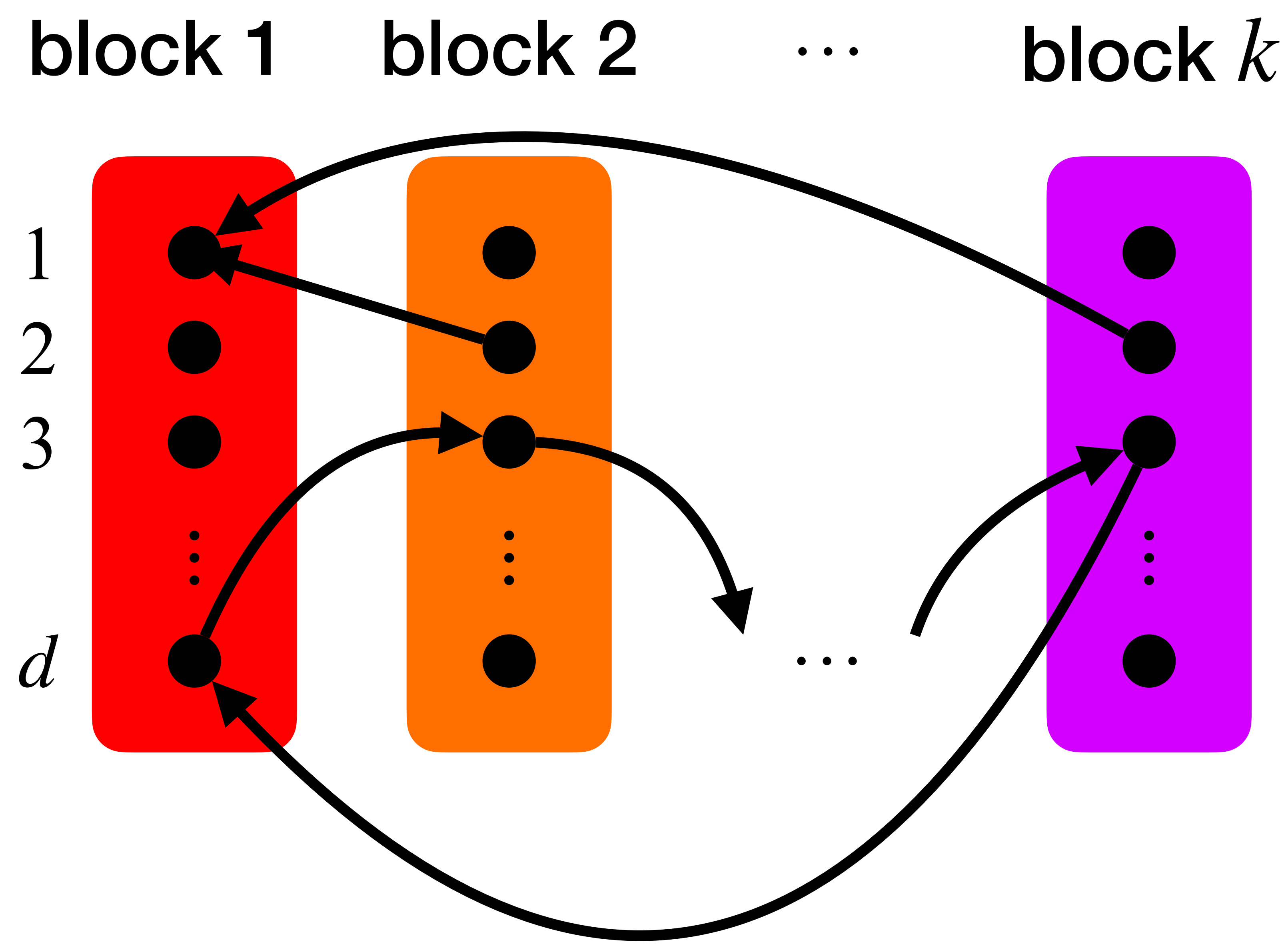


Does an EFX allocation always exist?

- **one of the most significant open questions in the field**
- Partial affirmative results include the cases
  - 2 agents [Plaut, Roughgarden 2018]
  - 3 agents, additive valuations [Chaudhury, Garg, Mehlhorn 2020]
- **Approximative EFX**
  - α-EFX for $\alpha \in (0,1]$: $\forall i, j, \forall g \in A_j : v_i(A_i) \geq \alpha \cdot v_i(A_j \setminus \{g\})$

The Rainbow Cycle Number

- The **rainbow cycle number** $R(d)$ is the largest integer k such that there exists a k-partite directed graph G with each block of size d such that
  - every vertex has an incoming edge from each other block (**in-property**)
  - there is **no rainbow cycle** (a cycle containing at most one vertex from each block)



α-EFX and the Rainbow Cycle Number

- Based on $R(d)$ one gets $(1-\varepsilon)$-EFX allocations with a sublinear number of unallocated items.

- Theorem [Chaudhury, Garg, Mehlhorn, Mehta, Misra 2021]

Let $\varepsilon \in (0,1/2]$ and let $g(y)$ be the smallest integer d such that $d \cdot R(d) \geq y$.

Then, there is a partial $(1-\varepsilon)$-EFX allocation with at most

$$\frac{4n}{\varepsilon \cdot g(2n/\varepsilon)}$$

many unallocated items.

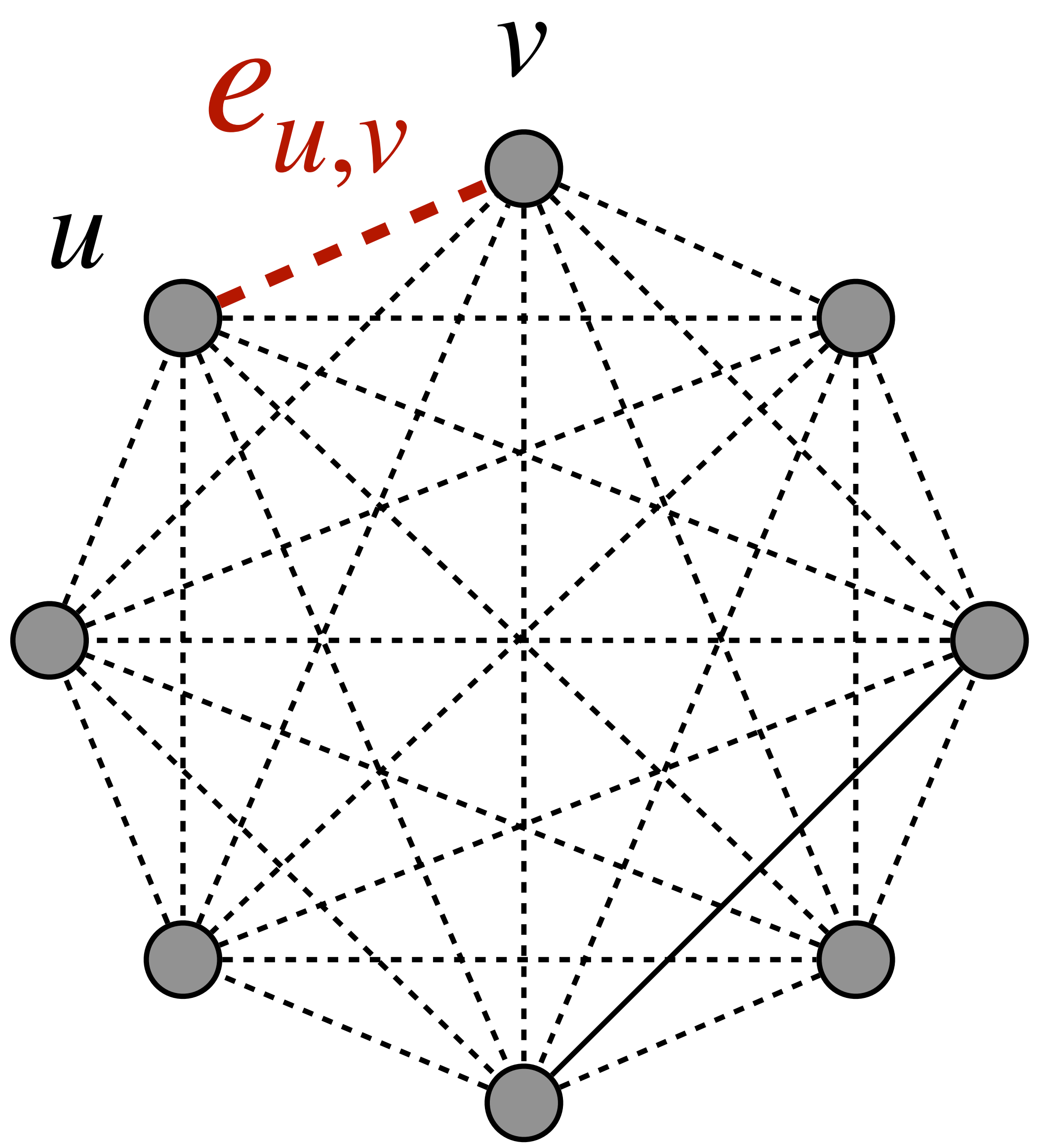
Bounds on the Rainbow Cycle Number

- [Chaudhury, Garg, Mehlhorn, Mehta, Misra 2021]
  - $d \leq R(d) \leq d^4 + d$
  - “We believe that finding better upper bounds on $R(d)$ is a natural combinatorial question”
  - “Better upper-bounds to $R(d)$ imply the existence of better relaxations of EFX allocations”
  - “Therefore investigating better upper bounds on the rainbow cycle number is of interest in its own right and we leave this as an interesting open problem”
- $R(2) = 2$
- $R(3) = 3$
- $R(4) \stackrel{?}{=} 4$ open

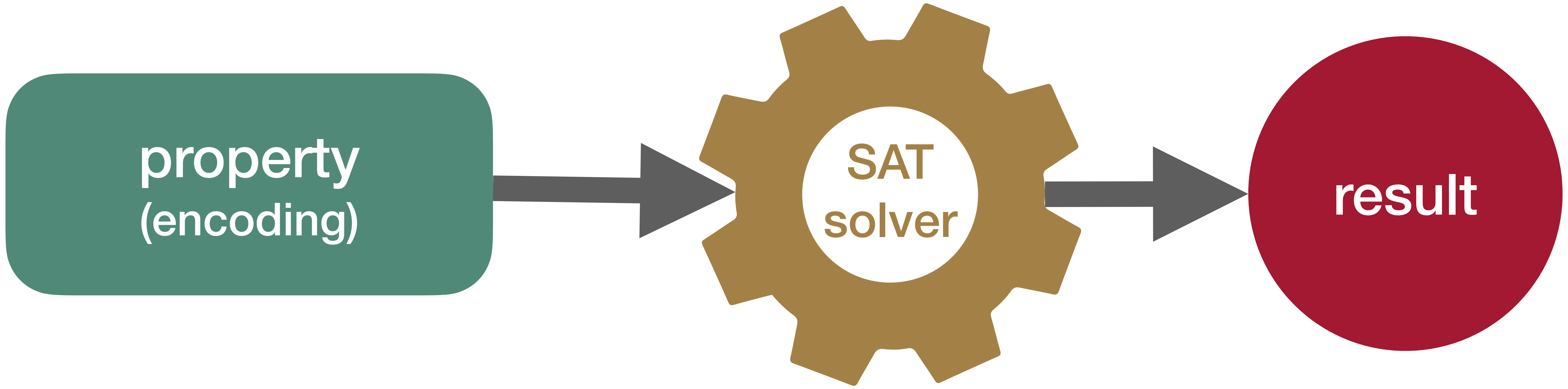
Showing $R(d) = d$ for small d

- checking that every $d+1$-partite graph with d vertices per block that satisfies the in-property contains a rainbow cycle.
- enumerate all such graphs modulo isomorphism, say with Nauty?
- $d = 4$ implies $n = 20$
- there are more than 2.4×10^{34} undirected graphs with 19 vertices, modulo isomorphism
- generate-and-test not feasible!
- generate only graphs without rainbow cycles?

Graph search as a synthesis problem

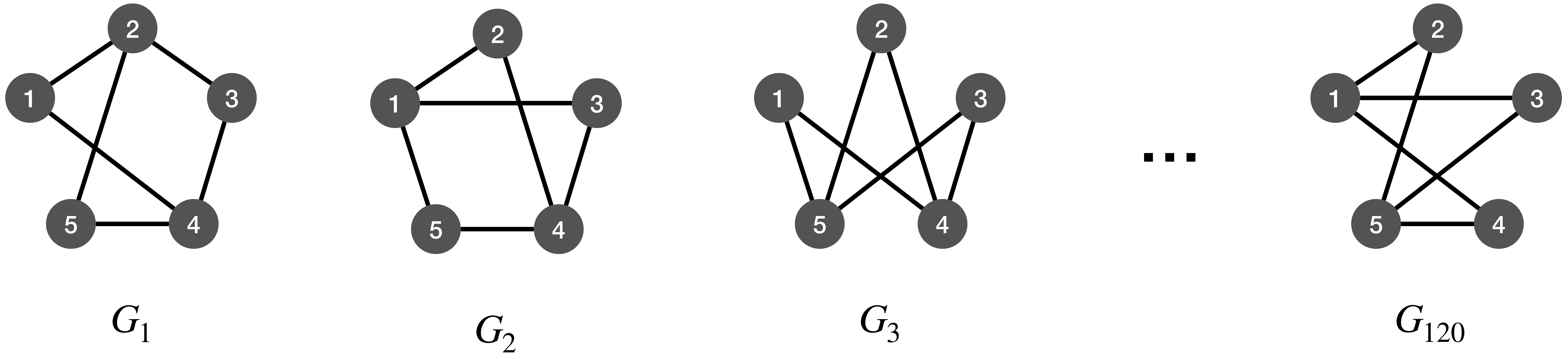


- We fix the number n of vertices, this gives $\binom{n}{2}$ many possible edges
- Each edge $\{u, v\}$ is represented by a propositional variable $e_{u,v}$ which is true iff the edge exists



Isomorph-Free Generation

- **Isomorph-free generation:** Number of objects explode quickly
- **Canonization:** map each object G to a unique representative $\alpha(G)$ of its isomorphism class
- **Canonical Objects:** Only generate objects G with $\alpha(G) = G$



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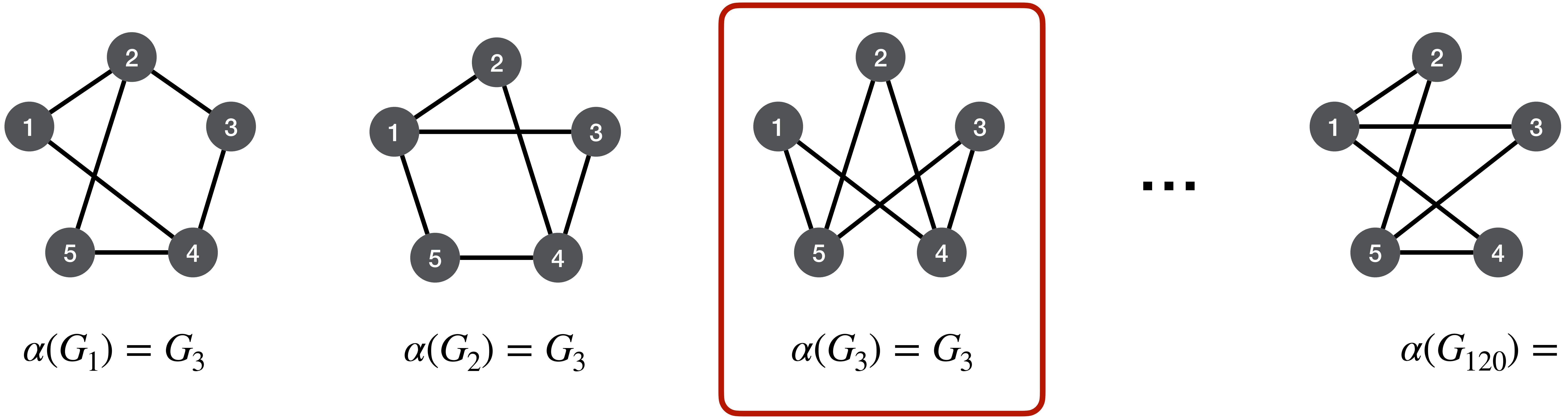
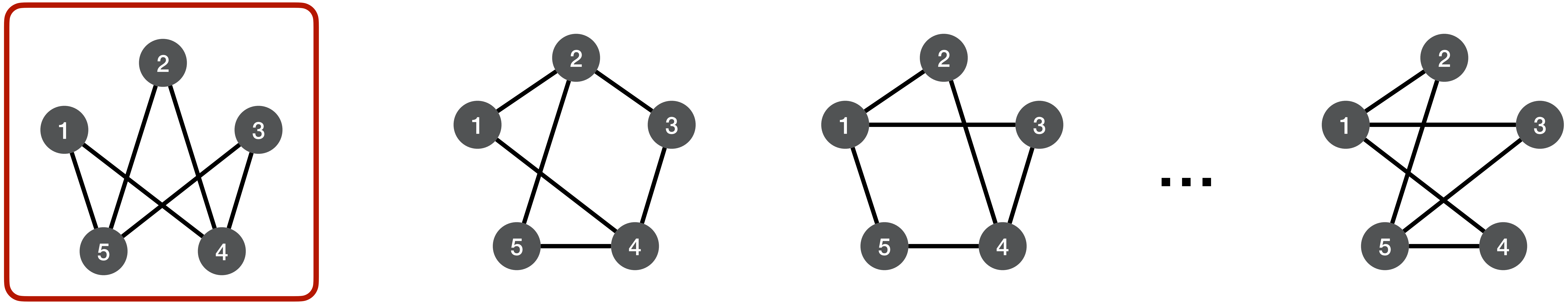


$\alpha(G_1) = G_3$ $\alpha(G_2) = G_3$ $\alpha(G_3) = G_3$ $\alpha(G_{120}) = G_3$

Canonical if it has the lex-smallest adjacency matrix



0	0	0	1	1
0	0	0	1	1
0	0	0	1	1
1	1	1	0	0
1	1	1	0	0

<lex

0	1	0	1	0
1	0	1	0	1
0	1	0	1	0
1	0	1	0	1
0	1	0	1	0

<lex

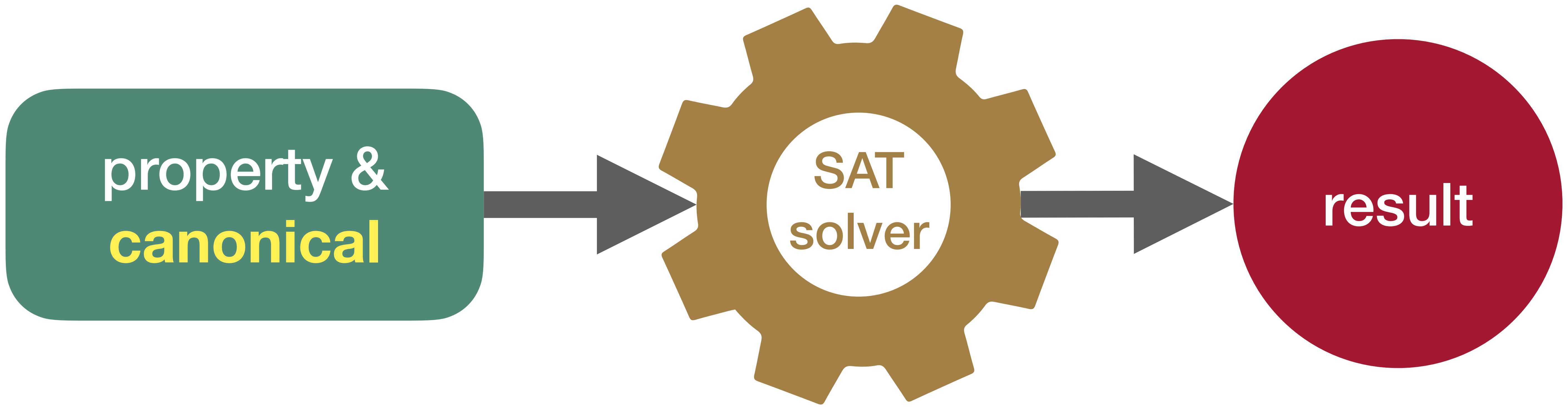
0	1	1	0	1
1	0	0	1	0
1	0	0	1	0
0	1	1	0	1
1	0	0	1	0

<lex

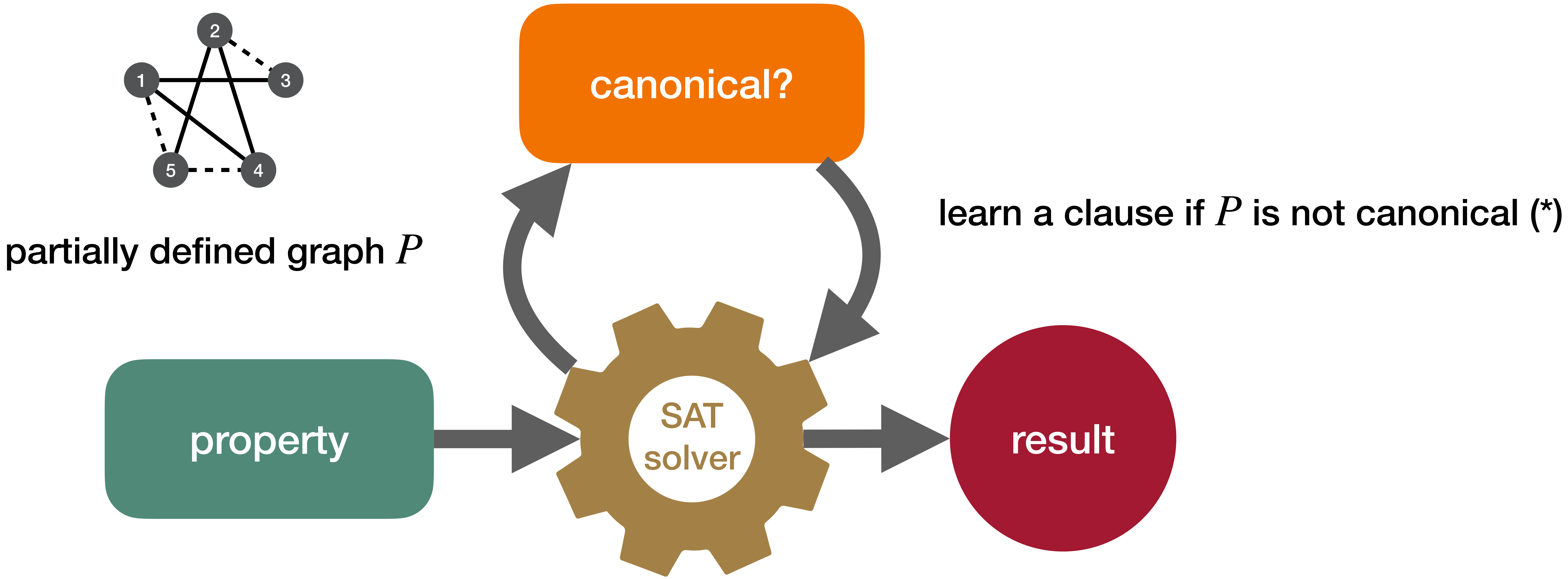
0	1	1	1	0
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
0	1	1	1	0

Static SAT approach

- Problem: no polynomial size encoding for canonicity is known!



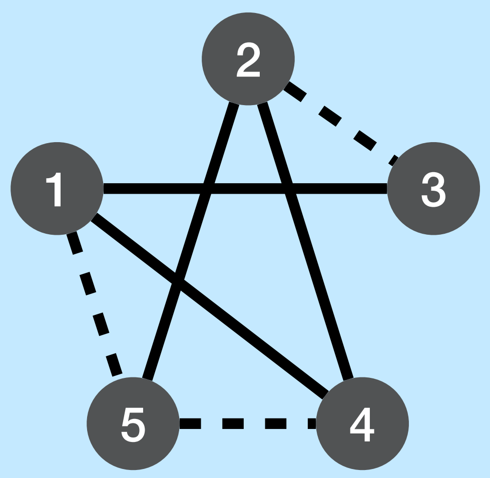
Dynamic SAT approach: SMS



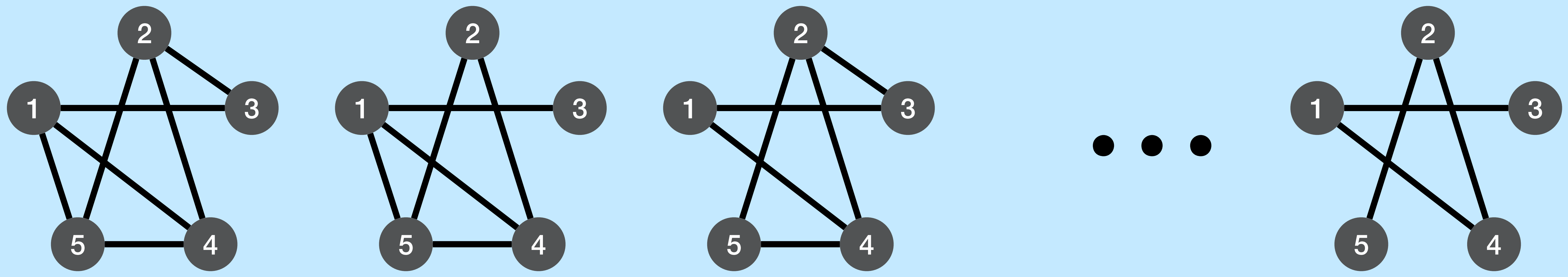
- SAT modulo Symmetries [Kirchweger, Sz. 2021]
- IPASIR-UP interface [Fazekas et al. 2023]

Canonicity of partially defined graphs

partially defined graph P

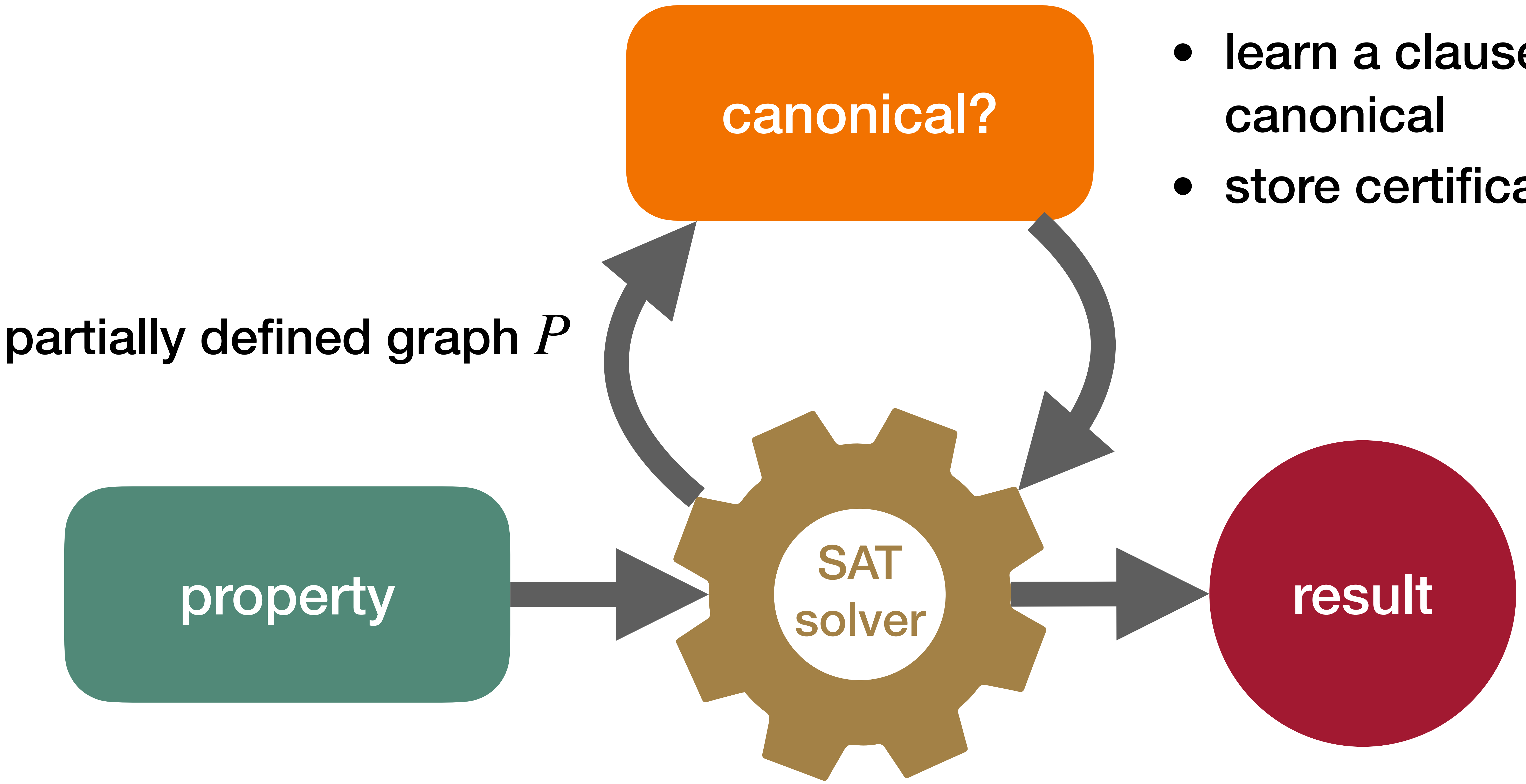


$X(P)$: set of all fully defined graphs P can be extended to



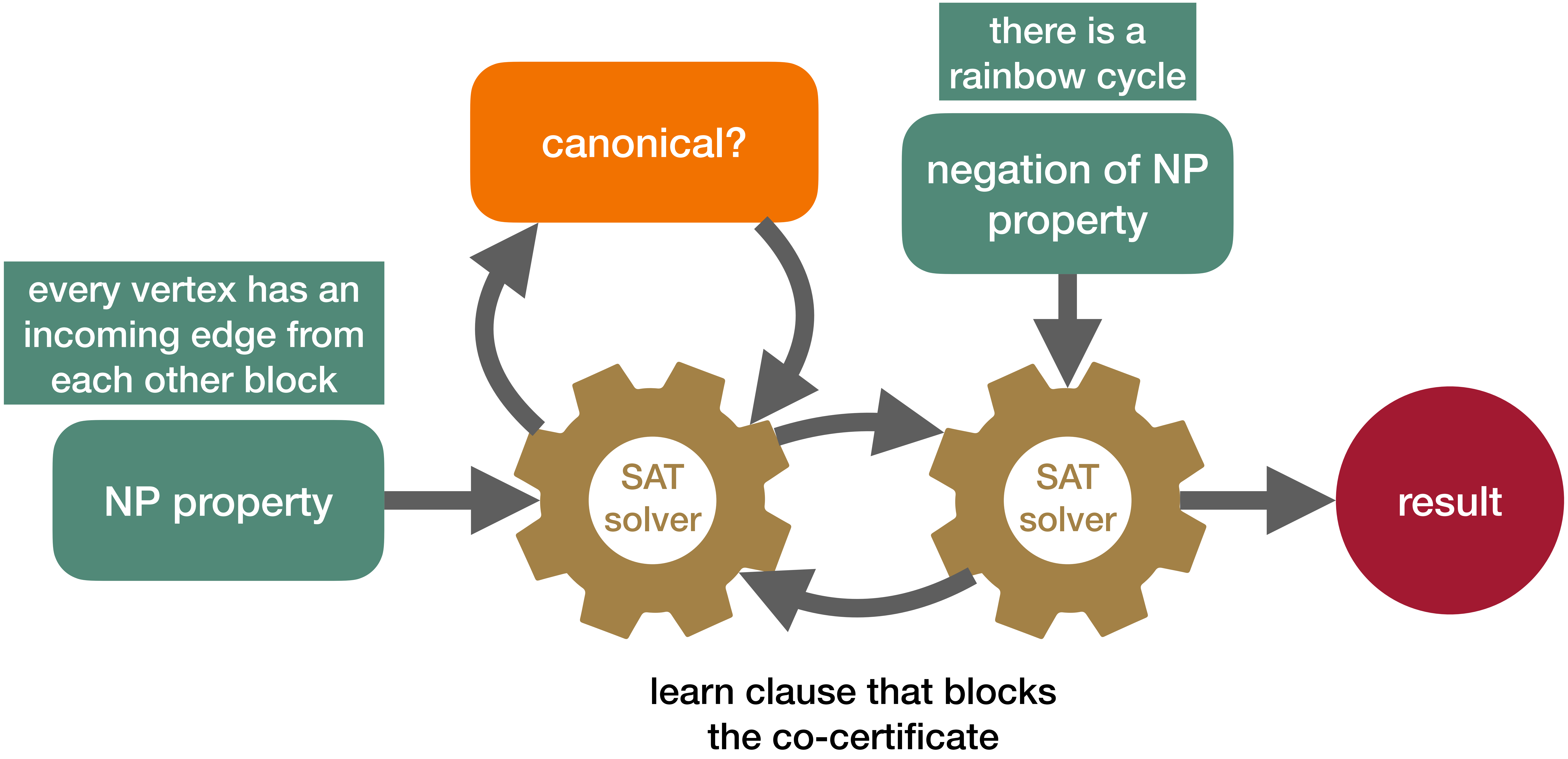
- P is **non-canonical** if $\forall G \in X(P)\, \exists \pi : \pi(G) <_{\text{lex}} G$
- G is **certified non-canonical** if $\exists \pi \forall G \in X(P) : \pi(G) <_{\text{lex}} G$
  - we have an efficient constraint-propagation algorithm for computing π
  - with π we keep a certificate for non-canonicity of P

Dynamic SAT approach: SMS



- learn a clause if P is not certified canonical
- store certificate π

SMS with co-certificate learning



- [Kirchweger, Peitl, Sz. 2023]

Results for showing $R(d) = d$

- "$R(3) \geq 4$" is unsatisfiable, within 1 second
- "$R(4) \geq 5$" is unsatisfiable, within 23 minutes
- "$R(5) \geq 6$" didn't terminate within 300h

Invariant pruning

- assuming max indegree is $\Delta := d(k-1)$
- w.l.o.g., vertex 1 has indegree Δ
- if UNSAT, add constraints that limit indegrees to $\Delta - 1$

- assuming max indegree is $\Delta - 1$
- w.l.o.g., vertex 1 has indegree $\Delta - 1$
- if UNSAT, add constraints that limit in-degrees to $\Delta - 2$

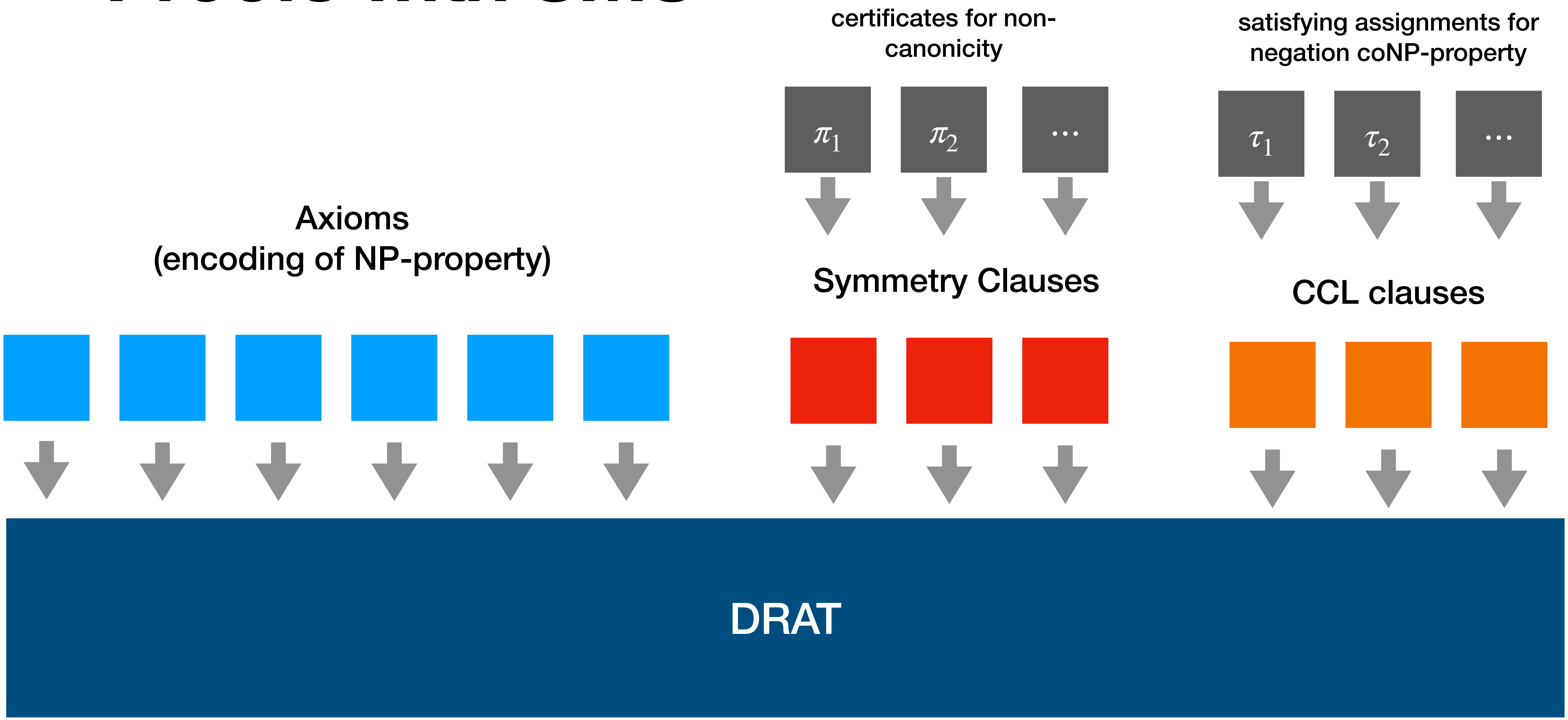
etc.

$R(d)$ with invariant pruning

- "$R(4) \geq 5$"
  - showing in-degree ≤ 4 within 3 seconds
  - showing unsatisfiability with in-degree ≤ 4 then takes half a minute
  - almost 50-fold speedup

- "$R(5) \geq 6$"
  - showing in-degree ≤ 6 within 105h
  - showing unsatisfiability with in-degree ≤ 6 didn't terminate within 300h

Proofs with SMS



[Wetzler, Heule, Hunt 2014]

Summary

- Fair division of goods, EFX
- Connection between rainbow cycle numbers and α-EFX
- Computing rainbow cycle numbers with SAT modulo Symmetries
- Determined $R(4) = 4$, with DRAT proof
- Invariant pruning provides speedup, but $R(5) = 5$ remains open

Applications of SMS

[Kirchweger-Szeider CP'21]	**Simon-Murty Conjecture** on diameter-2 critical graphs, verified up to 18 vertices
[Kirchweger-Scheucher-Szeider SAT'22]	**Rota's Basis Conjecture** for matroids, DRAT proofs for SMS
[Kirchweger-Peitl-Szeider SAT'23]	**Erdős-Faber-Lovász Conjecture** for hypergraphs
[Kirchweger-Scheucher-Szeider SAT'23]	**Planar graphs and digraphs, enumeration**—planar Turán numbers, Earth-Moon problem, integer sequences for OEIS
[Kirchweger-Peitl-Szeider IJCAI'23]	**Co-certificate learning**—3D Kochen-Specker vector systems have at least 24 vectors
[Zhang-Szeider CP 2023]	**Universal graphs** and **universal tournaments**—templates, induced 7-universal graphs have at least 17 vertices
[Zhang-Peitl-Szeider SAT 2024]	**smallest unsatisfiable CNF formulas** with bounded occurrence of variables
[Kirchweger-Szeider CP 2024]	**Rainbow cycle numbers**

Resources

Tool https://github.com/markirch/sat-modulo-symmetries/

Documentation https://sat-modulo-symmetries.readthedocs.io/

Presentation https://simons.berkeley.edu/talks/stefan-szeider-tu-wien-2023-04-18