Attractor decompositions in automata and games

Marcin Jurdziński

Laure Daviaud Karoliina Lehtinen Thejaswini K.S.

Parity graphs

$$G = (V, E, \pi : V \to [d]) \qquad [d] = \{1, 2, \dots, d\}$$

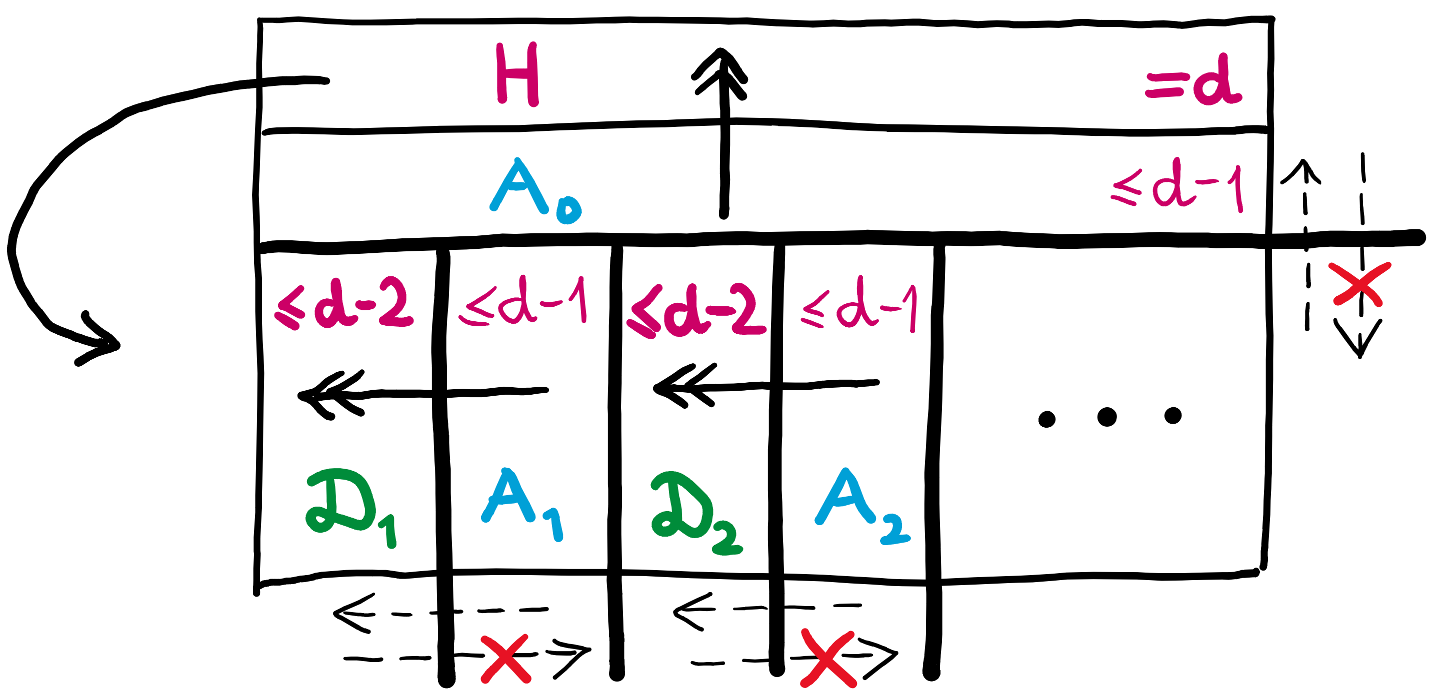
- max path $(v_1, v_2, v_3, \dots)$ in G is even

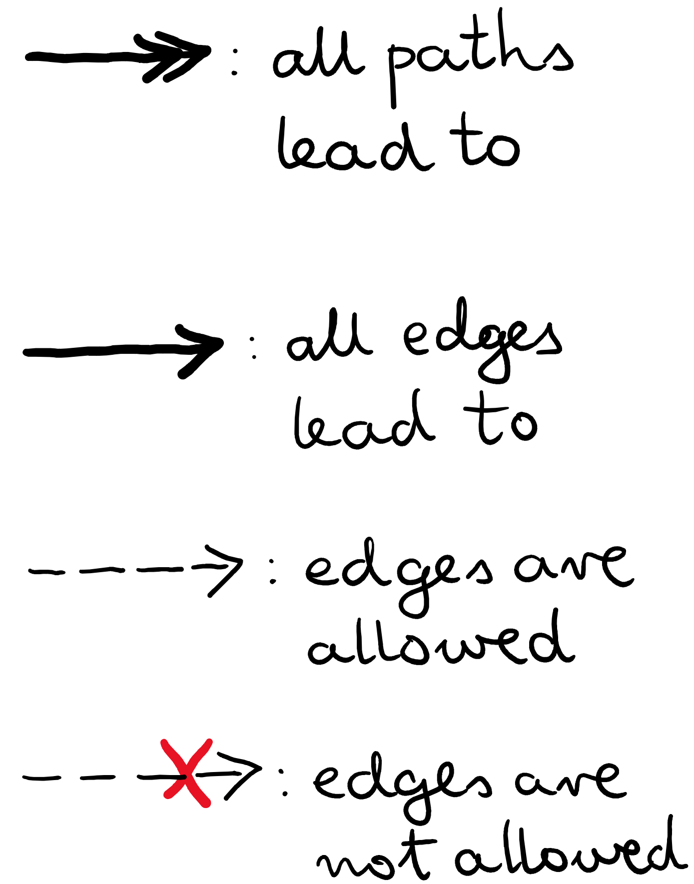
  <u>iff</u>

  $\limsup_{i \to \infty} \pi(v_i)$ is even

- G is even <u>iff</u> all max paths in G are even

Attractor decomposition $\mathcal{D}$ of G

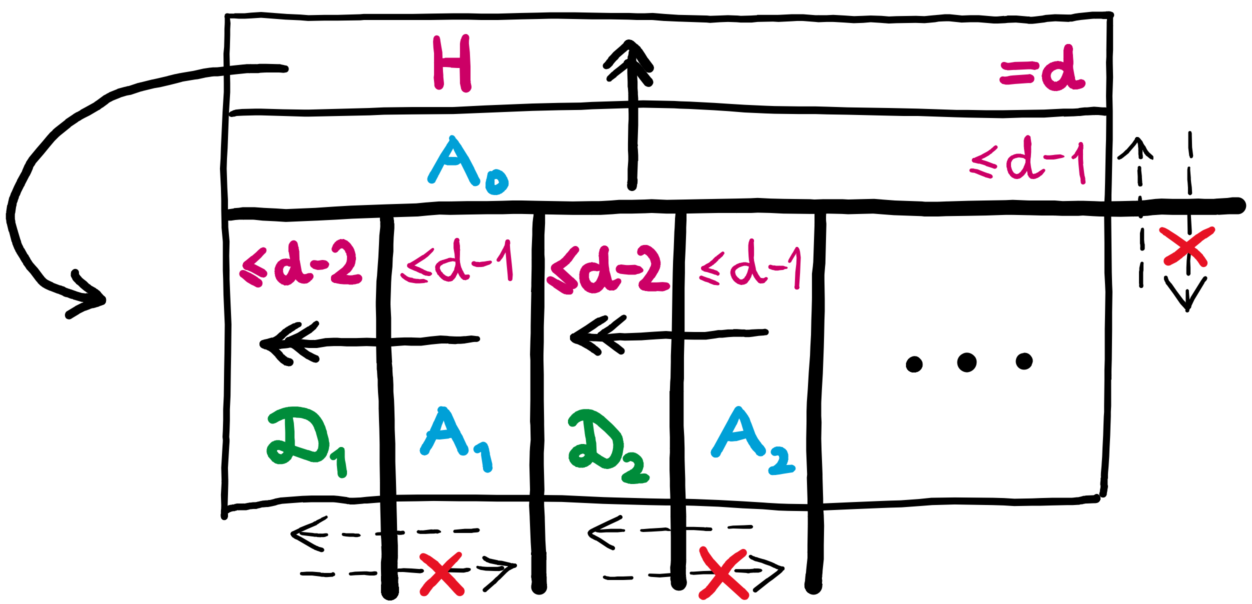




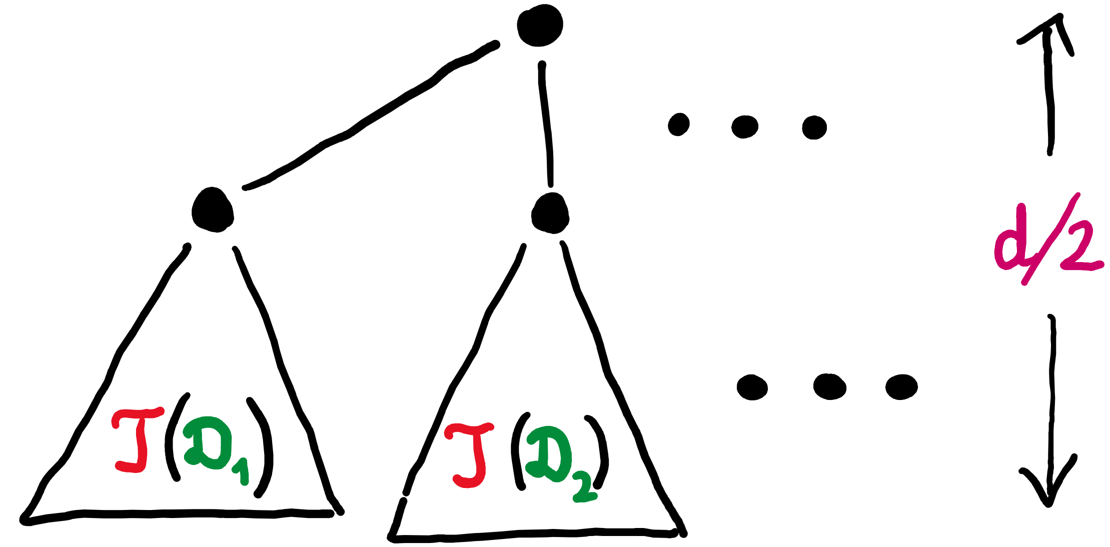
$\mathcal{D}_1, \mathcal{D}_2, \ldots$ are attractor decompositions

<u>Fact</u> G is even iff G has an attr. decomp.

$\mathcal{D}$



$\mathcal{T}(\mathcal{D})$



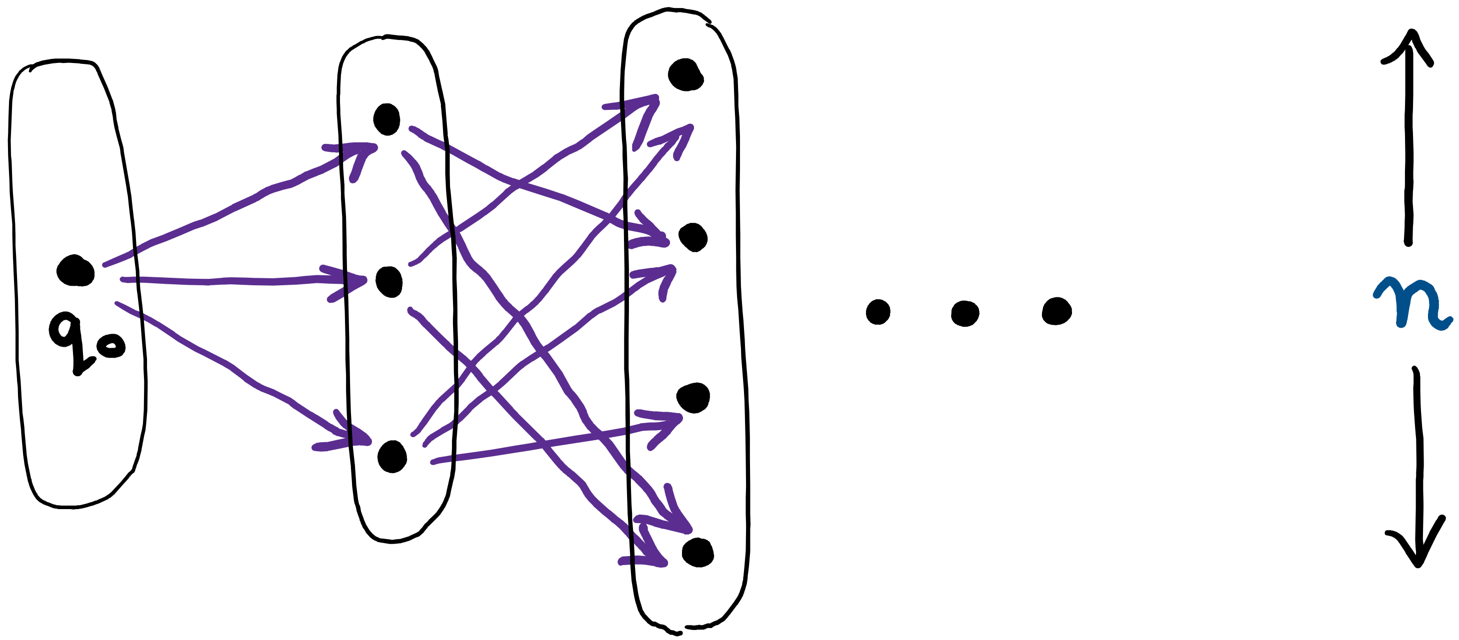
Attractor decomposition: $\delta : V \longrightarrow \mathcal{T}$

$$\begin{bmatrix}\text{structural}\\ \text{complexity}\\ \text{of } G\end{bmatrix} \sim \min_{\substack{\text{attr. decomp.}\\ \delta : V \to T}} \begin{bmatrix}\text{structural}\\ \text{complexity}\\ \text{of } T\end{bmatrix}$$

Alternating parity automata

- $[d]$-APW: $A = (Q, q_0, \Sigma, \tau, \pi : Q \to [d])$ $n = |Q|$

- Accepting run G of A:



even DAG of width $\leq n$

Alternating parity automata translations

Thm [KV'98]

[d]-APW $\rightsquigarrow$ [2]-APW

n states $\qquad$ n^{d} states

Thm [DJL'19]

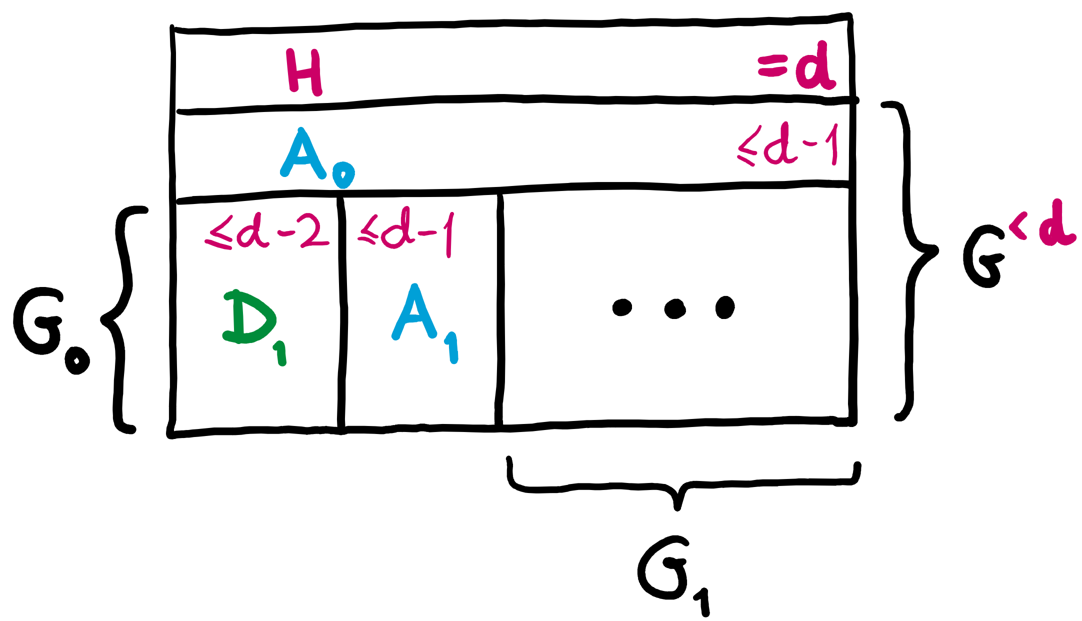
[d]-APW $\rightsquigarrow$ [2]-APW

n states $\qquad$ $n^{\lg\left(\frac{d}{\lg n}\right)+O(1)}$ states

Small attractor decompositions

Fact [DJL'19] An acc. run G of A
has an attr. decomp. $\delta : V \to T$
s.t. $|\text{leaves}(T)| \leq n$

Proof sketch



H: vertices of highest (even) priority d

A_0: vertices with fin. many descendants in $G^{<d}$

D_1: $(d-1)$-safe vertices in G_0

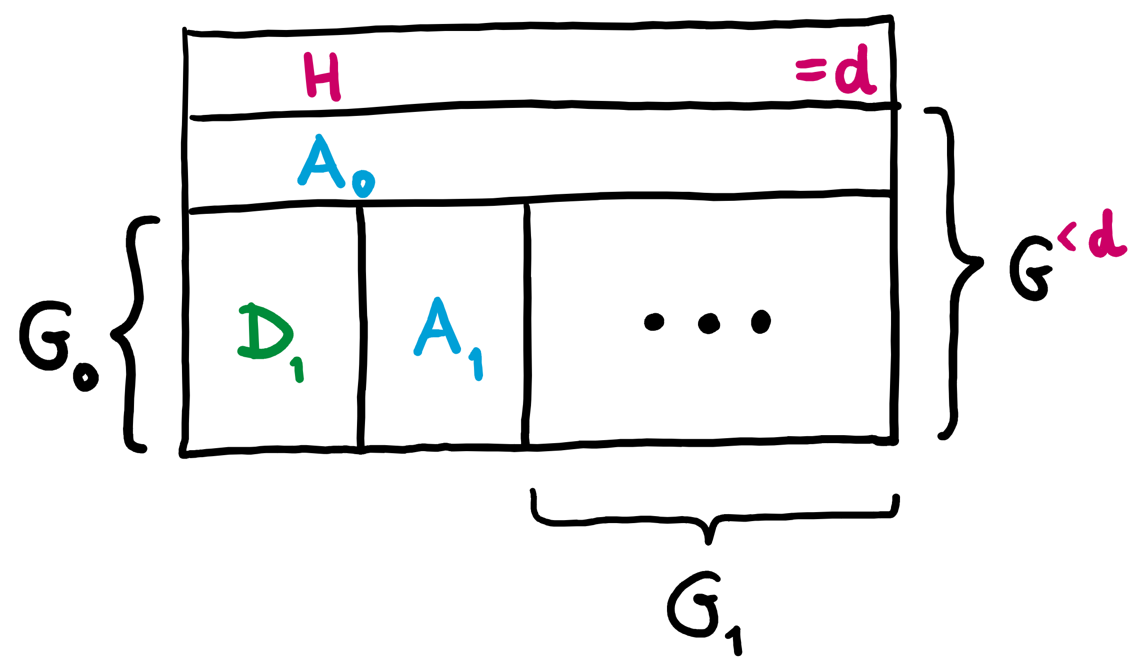
A_1: vertices with fin. many descendants in $G_0 \setminus D_1$

...

Small attractor decompositions

<u>Fact</u> [DJL'19] An acc. run G of A has an attr. decomp. $\delta : V \to \mathcal{T}$ s.t. $|\text{leaves}(\mathcal{T})| \leq n$

<u>Proof sketch</u>



$$\begin{aligned} |\text{leaves}(\mathcal{T})| &= \sum_i |\text{leaves}(\mathcal{T}_i)| \\ &\leq \sum_i \text{width}(D_i) \\ &\leq \text{width}(G) \\ &\leq n \end{aligned}$$

Thm [DJL'19] [d]-APW → [2]-APW

n states → $n^{\lg\left(\frac{d}{\lg n}\right)+O(1)}$ states

- guess acc. run G
- guess attr. decomp. $\delta : V \to \mathcal{T}$
- guess embedding $\varepsilon : \mathcal{T} \to \mathcal{U}$
  s.t. $\mathcal{U}$ is $\left(n, \frac{d}{2}\right)$-universal tree
- use 2 priorities to ensure leaving attractors

Thm [JL'17] There is (n,h)-univ. tree of size $n^{\lg\left(\frac{h}{\lg n}\right)+O(1)}$

Thm [DJL'19] [d]-APW $\rightsquigarrow$ [2]-APW

n states $\rightsquigarrow$ $n^{\lg\left(\frac{d}{\lg n}\right)+O(1)}$ states

Open Improve $\Omega(n \log n)$ lower bound

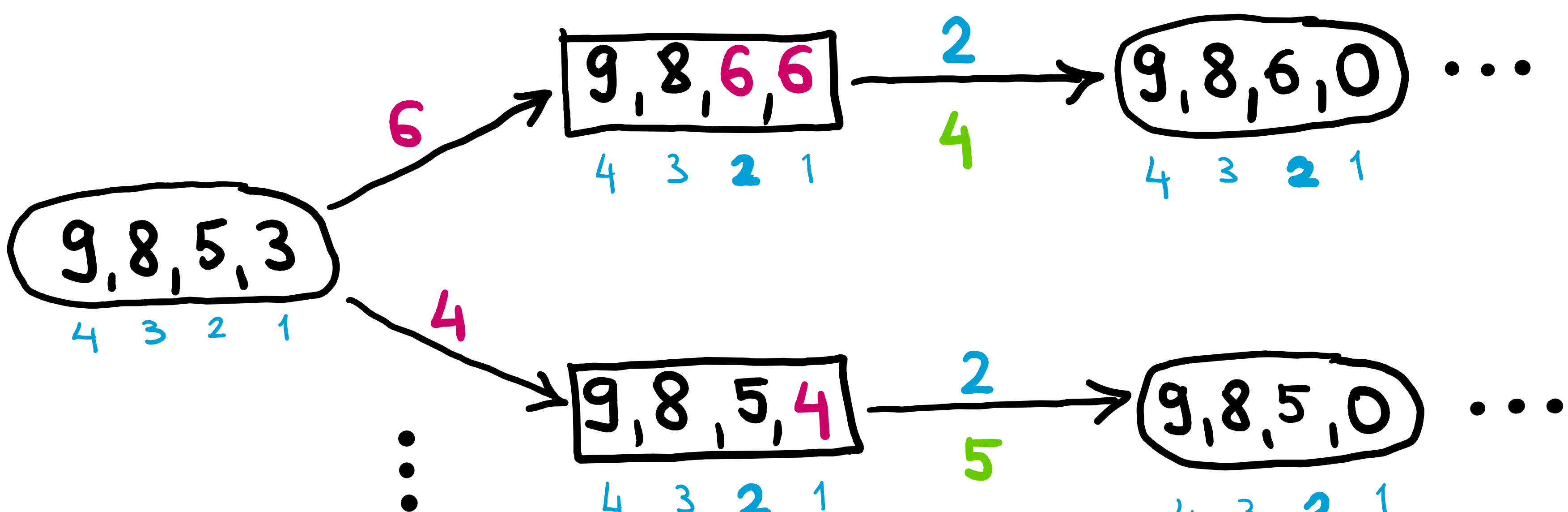
Note [CDFJLP'19] $n^{\lg\left(\frac{d}{\lg n}\right)+\Omega(1)}$ lower bounds for:

- (n, d)-universal trees
- (non-) deterministic (n, d)-separating automata

Def [L'18] k-register parity transducer $\mathcal{R}_d^k$

- states : $[d]^k$
- inputs : $[d]$
- control : $[k]$
- outputs : $[2k+1]$

$$[d]^\omega \longrightarrow [2k+1]^\omega$$



Register number of a parity game

Def [L'18]

For $[d]$-parity game G,

Register(G) is the smallest k, s.t.

games G and $G \rhd R^k_d$ have same winners

Thm [L'18]

If a $[d]$-parity game G has n vertices

then Register$(G) \leq \lg n$

Register number and algorithms

<u>Fact</u> [d]-PG of register number k can be solved in time $O\left(\left(n \cdot d^k\right)^k\right)$

<u>Corollary</u> [d]-PG can be solved in time

$$n^{O(\log n \cdot \log d)}$$

Strahler number

<u>Thm</u> [DJT'20]

$$\text{Register}(G) = \text{Strahler}(G)$$

$$\|df$$

$$\min_{\substack{\text{attr. decomp.} \\ \delta : V \to \mathcal{T}}} \text{Strahler}(\mathcal{T})$$

Strahler universal trees

<u>Thm</u> [DJT'20]

There is an (n, h, k)- Strahler universal tree of size $n^{O(1)} \cdot \left(\frac{h}{k}\right)^k$

<u>Corollary</u> Complexity of solving parity games

mid-2010's: $\boxed{\left(\frac{n}{d}\right)^d}$

mid-2020's: $\boxed{n^{O(1)} \cdot \left(\frac{d}{k}\right)^k}$

When are modern algorithms poly-time?

$$\boxed{k \cdot \log\left(\frac{d}{k}\right) = O(\log n)}$$

- [CJKLS'17, JL'17]: $d = O(\log n)$
- [L'18]: $k = O(1)$
- $k = O(\sqrt{\log n})$ $\quad d = \underbrace{2^{O(\sqrt{\log n})}}_{\text{super-polylog}}$

What next?

- Hypothesis 1: Strahler$(G) \leq 4$ in the wild
- Hypothesis 2: sublinear Strahler universal trees suffice in the wild
- Strahler number and register parity transducer deserve more study