Sublinear Subgraph Counting

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Sublinear graph algorithms

• How much of a graph needs to be seen for an (approximate) algorithmic task?

• How to sample a large graph?
Sublinear subgraph counting

- Approximate H-count in G
- G is simple, undirected
  - Think about G sparse, results hold in general
  - Not property testing!
- G stored as adjacency list
What’s the model?

• [Goldreich-Ron 02] “Standard sparse graph model”
• Vertex query: Get a uniform random vertex
• Degree query: For vertex \( v \), get degree \( d_v \)
• Neighbor query: For vertex \( v \), get a uar neighbor \( u \)
• Edge query: Given vertices \( u, v \), check if edge \( (u,v) \) present
• (Get uar edge)

n known initially
V is not known
Nothing else known

Algorithm can crawl/BFS from some random starting vertices
Tool #1: Heavy vertices/edges

Shedding weight
A simple question

- Estimate the average degree of a graph
- Consider “obvious procedure”: sample uniform random vertices, take the average
- \([\text{Feige 02}] \, O(\sqrt{n})\) samples give a 2-approximation
  - Average is in \([m/n, 2m/n]\)
  - For \(n = 10^8\), only 10,000 samples!

\[\text{Avg degree} = \frac{\sum_v d_v}{n}\]
\[\overline{d} = \frac{2m}{n}\]

\(m = \#\text{edges}\)
\(n = \#\text{vertices}\)
Why 2? And why $\sqrt{n}$?

Star graph

Average degree $\approx 2$

$o(n)$ samples only leaves, so empirical avg = 1

$n - c\sqrt{n}$ cycle

Average degree $\approx c/2$

$\ll \sqrt{n}/c$ samples will not hit clique. Empirical avg = 2
The variance problem

\[ \mathbb{E}[X_1] = \bar{d} \]

\[ \text{var}[X_1] \leq \frac{\sum_v d_v^2}{n} \]

Chebyshev

\[ \Pr \left[ |X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X] \right] \leq \frac{\text{var}[X]}{\varepsilon^2 \mathbb{E}[X]^2} = \frac{\text{var}[X_1]}{\varepsilon^2 k \mathbb{E}[X_1]^2} \]

\[ k \approx \frac{\text{var}[X_1]}{\mathbb{E}[X_1]^2} \]

Choose \( k \) iid samples, so \( \mathbb{E}[X] = k \cdot \bar{d} \)

\[ \text{var}[X] = k \cdot \text{var}[X_1] \]

\[ k \approx \frac{\sum_v d_v^2}{n \bar{d}^2} \]

We can have avg deg = \( O(1) \)

but avg sq deg = \( \Omega(n) \)

We can have numerator \( n^2 \)

but denominator \( \Theta(n) \)
The variance problem

- Need to reduce variance
- Can we simply drop “large” outcomes?
  - Word of the day: Winsorize

\[ k \approx \frac{\sum_v d_v^2}{nd^2} \]
But these are degrees!

- Avg degree of light vertices is \((1/2)\)-approx of avg degree
- But light degrees cannot be too large
  - So avg light degree has lower variance

\[d_v < \sqrt{m/\varepsilon}\]
\[d_v \geq \sqrt{m/\varepsilon}\]

- Sum of degrees of light vertices ≥ 
  - \(2 \times \text{(Light–light edges)} + \text{(Light-heavy edges)} \geq (1 - \varepsilon)m\)

At most \(2\sqrt{\varepsilon m}\) heavy vertices
At most \(\varepsilon m\) heavy-heavy edges
The variance problem

\[
\begin{align*}
\text{\begin{array}{cccc}
\bullet & \bullet & & \\
\bullet & & \bullet & \\
& & \bullet & \\
\end{array}} \quad d_1 \leq d_2 \leq d_3 \leq d_4
\end{align*}
\begin{align*}
\text{\begin{array}{cccc}
& & \bullet & \\
& & \bullet & \\
\bullet & & & \\
\end{array}} \quad d_{n-1} \leq d_n
\end{align*}
\]

\[Y_1 = \begin{cases} 
  d_v & d_v < \sqrt{m/\varepsilon} \\
  0 & \text{else}
\end{cases} \]

\[k \approx \frac{\text{var}[Y_1]}{\mathbb{E}[Y_1]^2} \leq \frac{\max(Y_1)}{\mathbb{E}[Y_1]} \leq \frac{\sqrt{m}}{\bar{d}} \leq \frac{n}{\sqrt{m}}\]

\[
\text{var}[Y_1] \leq \mathbb{E}[Y_1^2] \leq \max(Y_1)\mathbb{E}[Y_1]
\]

• \(\sqrt{n}\) samples suffice to estimate average light degree
  • That gives (1/2)-approximation to true average degree

• Clean expression that deals with all (sparse to dense) cases

• But wait...how do we even sample \(Y\)?
A tale of two tails

\[ X = n*(\text{avg of k uar degrees}) \]
\[ E[X] = 2m \]

\[ Y = n*(\text{avg of k uar Winsorized degrees}) \]
\[ E[Y] = \text{sum of light degrees} \geq (1-\varepsilon)m \]

\[ X \geq Y \]

\[ \Pr[X < (1 - \varepsilon)m] \leq \Pr[Y < (1 - \varepsilon)m] \]

Chebyshev on Y, like previous slide

\[ \Pr[X \geq (1 + \varepsilon)(2m)] \leq 1 - \varepsilon \]

Markov on X ([Feige 02] is tighter)

Take min of \( O(1/\varepsilon) \) estimates for full proof

\[ \sqrt{n} \text{ samples give } (2+\varepsilon)-\text{approx to average degree} \]
Tool #2: Graph orientations

Get some direction
Back to a simple question

- Estimate the average degree of a graph
  - Beat the obvious procedure of sampling random degrees
  - Can we exploit graph structure?

- \([\text{Goldreich-Ron 08}] (1+\varepsilon)\)-approximation in \(O(\sqrt{n})\) time
Know thy neighbor

[Eden-Ron-S 17]

1. Pick u ar vertex u
2. Pick u ar neighbor v
3. If $d_u < d_v$, output $2d_u$
4. If $d_u > d_v$, output 0

(If equal, break ties consistently.)

Orient G into a DAG as follows:

$u < v$: if $d_u < d_v$ or $d_u = d_v$ and $id(u) < id(v)$
What do you expect?
What do you expect?
What do you expect?

We have an unbiased estimator for average degree

\[
E[Y_1] = \frac{1}{n} \sum_u d^+_u \cdot 2d_u = \frac{1}{n} \sum_u 2d^+_u = \frac{2m}{n} = \overline{d}
\]

- We have an unbiased estimator for average degree

\[
k \approx \frac{\text{var}[Y_1]}{E[Y_1]^2} \leq \frac{\max(Y_1)}{E[Y_1]}
\]
What’s the max?

\[ d_v \geq d_u \geq d_u^+ \]

\[ 2m \geq \sum_{\text{green}} v \ d_v \geq (d_u^+)^2 \]

\[ \max_u d_u^+ \leq \sqrt{2m} \]

\[ k \approx \frac{\text{var}[Y_1]}{E[Y_1]^2} \leq \frac{\max(Y_1)}{E[Y_1]} \leq \frac{\sqrt{2m}}{d} \leq \frac{n}{\sqrt{m}} \]

• So \( O(\sqrt{n}) \) queries suffice to get \((1+\varepsilon)\)-approx of average degree
Tool #3: Chiba-Nishizeki

A really really useful fact
Triangle counting

- Approximate triangle in G
- About as classic as it gets

- [Eden-Levi-Ron-S 15] \((1+\varepsilon)\)-estimate to \(t\) in time:

\[
O^*(\frac{n}{t^{1/3}} + \frac{m^{3/2}}{t})
\]

Optimal!
A simple estimator

1. Pick uar (u,v)
2. Pick uar neighbor w from lower degree endpoint
3. Check if (u,v,w) is a triangle

Success prob \[= \frac{1}{m}\sum_{e=(u,v)\in E} \frac{t_e}{\min(d_u,d_v)}\]

• Assume access to uar edges
  • [Assadi-Kapralov-Khanna 18]

• We want to estimate average \(t_e\), # triangles containing e
  • \(t = 3m(\sum e t_e/m)\)
An unbiased estimator

1. Pick uar \((u,v)\)
2. Pick uar neighbor \(w\) from lower degree endpoint
3. Check if \((u,v,w)\) is a triangle, output \(Y_1 = \min(d_w, d_v)\), else 0

Expectation

\[
\frac{1}{m} \sum_{e=(u,v) \in E} \frac{t_e}{\min(d_u, d_v)} \cdot \min(d_u, d_v) = \frac{1}{m} \sum_{e \in E} t_e
\]

\[
k \approx \frac{\text{var}[Y_1]}{\mathbb{E}[Y_1]^2} \leq \frac{\max(Y_1)}{\mathbb{E}[Y_1]} = \frac{\max_{(u,v) \in E} \min(d_u, d_v)}{\mathbb{E}[Y_1]}
\]
Chiba-Nishizeki to the rescue

\[ \sum_{(u,v) \in E} \min(d_u, d_v) \leq m \cdot \sqrt{2m} \]

• [Chiba-Nishizeki 85] In the context of clique counting and arboricity

• So average \( \min(d_u, d_v) \) is at most \( \sqrt{m} \)
An unbiased estimator

1. Pick uar (u,v)
2. Pick uar neighbor w from lower degree endpoint
3. Check if (u,v,w) is a triangle, output $Y_1 = \min(d_u, d_v)$, else 0

If $\min(d_u, d_v) \leq \sqrt{m}$

$$k \approx \frac{\text{var}[Y_1]}{E[Y_1]^2} \leq \frac{\max(Y_1)}{E[Y_1]} \leq \frac{\sqrt{m}}{E[Y_1]}$$

$$E[Y_1] = \frac{\sum_e t_e}{m} = \frac{m^{3/2}}{t}$$
Reducing variance

1. Pick uar (u,v)
2. Repeat \((1 + \min(d_u, d_v)/\sqrt{m})\) times
   a) Pick uar neighbor \(w\) from lower degree endpoint
   b) Check if \((u,v,w)\) is a triangle, set \(Z_i = \min(d_u, d_v)\), else 0
3. Output \(Y_1 = \text{average } Z_i\)

Variance of average of iid variables = Average of variance

\[
\text{var}[Y_1] = \frac{\text{var}[Z_1]}{\min(d_u, d_v)/\sqrt{m}} \leq \frac{\max(Z_1) \text{E}[Z_1]}{\min(d_u, d_v)/\sqrt{m}} = \sqrt{m} \text{E}[Y_1]
\]

\[
k \approx \frac{\text{var}[Y_1]}{\text{E}[Y_1]^2} \leq \frac{\sqrt{m}}{\text{E}[Y_1]}
\]
The punchline

1. Pick uar \((u,v)\)
2. Repeat \((1 + \min(d_u, d_v)/\sqrt{m})\) times
   a) Pick uar neighbor \(w\) from lower degree endpoint
   b) Check if \((u,v,w)\) is a triangle, set \(Z_i\) = \(\min(d_u, d_v)\), else 0
3. Output \(Y_1 = \) average \(Z_i\)

\[ k \approx \frac{\text{var}[Y_1]}{\mathbb{E}[Y_1]^2} \leq \frac{\sqrt{m}}{\mathbb{E}[Y_1]} = \frac{m^{3/2}}{t} \leq \sqrt{2m^{3/2}} \]

Runtime per sample = \[
\frac{1}{m} \sum_{(u,v) \in E} \left(1 + \frac{\min(d_u, d_v)}{\sqrt{m}}\right) = 1 + \frac{\sum_{(u,v) \in E} \min(d_u, d_v)}{m^{3/2}} \leq 3
\]
To finish...

1. Pick uar (u,v)
2. Repeat \((1 + \min(d_u, d_v)/\sqrt{m})\) times
   a) Pick uar neighbor w from lower degree endpoint
   b) Check if \((u,v,w)\) is a triangle, set \(Z_i = \min(d_u, d_v)\), else 0
3. Output \(Y_1 = \text{average } Z_i\)

\[ k \approx \frac{\text{var}[Y_1]}{\text{E}[Y_1]^2} = \frac{m^{3/2}}{t} \]

Runtime per sample < 3

- \(m^{3/2}/t\) algorithm for estimating triangle count
  - Assuming uar edges
- Optimal!
Tool #4: Simulating edge samples

Fake it till you make it
Fake uar edge samples

• Query all degrees in R
• Set up data structure that:
  1. Samples u in R proportional to $d_u/d_R$
  2. Output uar edge incident to v (uar nbr of u)

• This gives uar edge incident to R, in $O(1)$ time
• Can we use these as generic “uar” edges?
What do we need?

\[ k \approx \frac{\text{var}[Y_1]}{\mathbb{E}[Y_1]^2} = \frac{\sum_{(u,v) \in E_R} \min(d_u, d_v)}{t_R} \]

1. Pick uar (u,v)
2. Repeat \((1 + \min(d_u, d_v)/\sqrt{m})\) times
   a) Pick uar neighbor w from lower degree endpoint
   b) Check if \((u,v,w)\) is a triangle, set \(Z_i = \min(d_u, d_v)\), else 0
3. Output \(Y_1 = \text{average } Z_i\)

• When is \(t_R\) good estimate for total triangle count?
  • Denominator \((t_R)\) should not too small

• Numerator is easy to deal with (Markov)
Tool #1: Heavy Vertices

- Can we simply drop “large” outcomes?
  - Word of the day: Winsorize
But these are degrees!

- At least \((1-\varepsilon)t\) triangles incident to light vertices
- Average \(t_v\) of light vertices gives \((1/3)\)-approx to average \(t_v\)

\[
k' \approx \frac{\text{var}[Y_1]}{E[Y_1]^2} \leq \frac{\max(Y_1)}{E[Y_1]} \approx \frac{t^{2/3}}{t/n} = \frac{n}{t^{1/3}}
\]
In total...

1. Pick uar (u,v)
2. Repeat \((1 + \min(d_u, d_v) / \sqrt{m}))\) times
   a) Pick uar neighbor w from lower degree endpoint
   b) Check if \((u,v,w)\) is a triangle, set \(Z_i = \min(d_w, d_v)\), else 0
3. Output \(Y_1 = \text{average } Z_i\)

\[
\frac{n}{t^{1/3}} + \frac{m^{3/2}}{t}
\]

• Direct analysis gives 3-approx for t
  • Optimal complexity for constant factor approx
• Getting \((1+\varepsilon)\)-approx needs little more work
  • Same tools, just need to determine whether vertex is heavy/light
Tool #1: Heavy Vertices

\[ d_1 \leq d_2 \leq d_3 \leq d_4 \]

Tool #2: Graph orientations

\[ d_u \text{ vs } d_v \]
Tool #3: Chiba-Nishizeki

\[ \sum_{(u,v) \in E} \min(d_u, d_v) \leq m \cdot \sqrt{2m} \]

Tool #4: Simulating edge samples
Some survey-ish slides

If you’re in the audience, I hope I cited you
Sublinear subgraph counting


\[
\frac{n}{C^{1/3}} + \frac{m^{k/2}}{C}
\]

• [Gonen-Ron-Shavitt 15, Eden-Ron-S 17] k-Star counting, standard model

\[
\frac{n}{C^{1/(k+1)}} + \frac{m}{C^{1/k}} \leq n^{1-1/(k+1)}
\]
The arboricity connection

• The degeneracy/arboricity $\alpha$ is: max min (or avg) degree of a subgraph
  • The $\sqrt{m}$ is really $\alpha$!

• [Eden-Ron-S 18, Eden-Ron-S 20] One can get $\alpha$ in all the complexities

• For any minor-free family graphs:
  • Clique estimation in $O(n/C)$
  • $k$-Star estimation in $n^{1-1/k}$ (instead of $n^{1-1/(k+1)}$)
Sampling uar edge/clique


• [Fichtenberger-Gao-Peng 20, Eden-Ron-Rosenbaum 22] Sampling cliques
  • [FGP20] does arbitrary subgraphs but needs uar edges
Model with uar edges

• Access to uniform random edges

• [Aliakbarpour-Biswas-Goulekis-Peebles-Rubinfeld-Yodpinyanee 18] k-star counting
  \[
  \frac{n}{C^{1/(k+1)}} + \frac{m}{C^{1/k}}
  \]

  \[
  \frac{n}{C^{1/3}} + \frac{m^{k/2}}{C}
  \]

• [Chierichetti-Dasgupta-Kumar-Lattanzi-Sarlos 16, Tetek-Thorup 22] Full neighbor list in one query
Independent set queries

• A different model, but again, you don’t see the whole graph

• [Beame-HarPeled-Ramamoorthy-Rashtchian-Sinha 18] Edge estimation
  • [Addanki-McGregor-Musco 22]

• [Bhattacharya-Bishnu-Ghosh-Mishra 21] Triangles with tripartite queries
But is it practical?

Constants matter, only when they don’t
What do you mean?

• Algorithms can be implemented “off the shelf”?  
  • Small constant factors

• Ideas can be used for faster algorithms?  
  • Coding sublinear sampling algorithms

• Write papers in other conferences?  
  • Design algorithms that non-TCS people care about

• Solve algorithmic problems others care about?  
  • Either write SciPy package everyone uses, or make some money
Estimating the degree distribution

- [Eden-Jain-Pinar-Ron-S 18] Total of 0.01n degree queries in all cases
Sublinear triangle counting (for real)

• [Bera S 20] Sublinear triangle counting
• In the real-world, one cannot sample uniform random vertices
  • Need to use random walks from “seed vertices”
  • Assume mixing time bounds
• Need to couple random walk with Tools #1 - #4
• [Bera-Choudhari-Haddadan-Ahmadian 24] General clique counting

Accuracy over 100 independent runs
3% of edges seen, graphs have 300M – 30B edges
Estimating $m$ (or $n$)

• [Dasgupta-Kumar-Sarlos 14, Chierichetti-Dasgupta-Kumar-Lattanzi-Sarlos 16, BenEliezer-Eden-Oren-Fotakis 22]

• What is the right model?
• [Goldreich-Ron 02] G is bounded degree, stored as adjacency list
  • n vertices, d degree bound
  • You can select (random) vertices/seeds
  • You can crawl from these seeds
    • BFS, Random walks
  • You can look up edges

But that's how all your experiments are run!
The query models...?

- You start with one/few random vertices
- You can crawl from these seeds
  - BFS, Random walks
- You can look up edges
- Mixing time of graph is small
Words of wisdom

“All of us (applied researchers) are really running sublinear graphs algorithms, because our data collection is incomplete.

Our data is a random snapshot of the ground truth”
A deep question

“If I run my favorite graph algorithm on the sample, what does that say about the whole?
How should I collect my graph data?”
Concrete sublinear questions

• Triangle statistics and clustering coefficients
• Distribution of PageRank values
• Cluster/community structure of the graph
Less concrete sublinear questions

• Output of Graph Neural Net

• Output of downstream ML task
Time for coffee?

Dana, Talya, and I are working on a survey