Pseudo Entropy and Holography

Tadashi Takayanagi

Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics (YITP), Kyoto U.
This is a quantity which generalizes entanglement entropy in setups with post-selections. So it is different from pseudo entanglement in talks tomorrow [Aaronson-Bouland-Fefferman-Ghosh-Vazirani-Zhang-Zhou 2022,..]
Main Reference


with Yoshifumi Nakata (YITP, Kyoto)
Yusuke Taki (YITP, Kyoto)
Kotaro Tamaoka (Nihon U.)
Zixia Wei (Harvard U.)

Further progress to be mentioned in this talk


by Mollabashi-Shiba-Tamaoka-Wei-TT


Analogy between BH and Qubits

Blackhole Spacetime

Quantum Spin System

Area law

$S_{BH} = \frac{A_{BH}}{4G_N}$

Entangled!

Observer

Area law

Entanglement entropy $S_A$

Matter

$S_{BH}$ Spacetime

$A_{BH}$

BH entropy $S_{AH}$

Entanglement entropy $S_A$
Entanglement entropy (EE)

Divide a quantum system into two subsystems A and B:

\[ H_{\text{tot}} = H_A \otimes H_B. \]

Define the reduced density matrix by \( \rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| \).

The entanglement entropy \( S_A \) is defined by

\[
S_A = -\text{Tr}_A \rho_A \log \rho_A. \quad \text{(von-Neumann entropy)}
\]

Quantum Many-body Systems

Quantum Field Theories (QFTs)

Continuum Limit \( \varepsilon \to 0 \)

\[ \partial A = \partial B \]
Entanglement Entropy (EE) from operational viewpoint

\[ (|\Psi\rangle_{AB}\langle\Psi|)^\otimes M \Rightarrow (|\text{Bell}\rangle\langle\text{Bell}|)^\otimes N \]

Entangled in a very complicated way

Well-known fact in QI:

\[ S_A = \lim_{M \to \infty} \frac{N}{M} \]

\[ \rho_A \equiv \text{Tr}_B [|\Psi\rangle_{AB}\langle\Psi|] \]

[Bennett-Bernstein-Popescu-Schumacher 95, Nielsen 98]
Gravitational Holography

BH Entropy Formula

\[ S_{BH} = \frac{A_{BH}}{4G_N} \]

Degrees of freedom in Gravity $\propto$ Area

Holography

Gravity on $\mathcal{M}$ = Quantum Matter on $\partial \mathcal{M}$

Gravity = Matter

BH entropy ($\propto$ Area) = Thermal Entropy of Matter ($\propto$ Volume)

[ ’t Hooft 1993, Susskind 1994]
AdS/CFT Correspondence [Maldacena 97]

The most well-established holography:

AdS/CFT

(Quantum) Gravity on D+1 dim. AdS (anti de-Sitter space) = Conformal Field Theory (CFT) on D dim. Minkowski spacetime

Classical limit

Gapless quantum system

General relativity with $\Lambda < 0$

Large N + Strong coupling

Strongly interacting Quantum Many-body system
Holographic Entanglement Entropy

[Ver. 1] Holographic EE for Static Spacetimes

For static asymptotically AdS spacetimes:

\[ S_A = \min_{\partial \Gamma_A = \partial A} \left[ \frac{\text{Area}(\Gamma_A)}{4G_N} \right] \]

\( \Gamma_A \) is the minimal area surface (codim.=2) on the time slice such that \( \partial A = \partial \Gamma_A \) and \( A \sim \Gamma_A \).

(We omit the time direction.)

\[ z > \varepsilon \quad \text{(UV cut off)} \]
General Behavior of HEE(=d+1 dim. CFT EE)

[Ryu–TT 06,⋯]

\[ S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)} \Gamma(d/2)} \left[ p_1 \left( \frac{l}{\varepsilon} \right)^{d-1} + p_3 \left( \frac{l}{\varepsilon} \right)^{d-3} \right. + \cdots \]

\[ \cdots + \left\{ \begin{array}{l}
   p_{d-1} \left( \frac{l}{\varepsilon} \right)^2 + p_d \quad \text{(if } d+1 = \text{odd)} \\
   p_{d-2} \left( \frac{l}{\varepsilon} \right) + q \log \left( \frac{l}{\varepsilon} \right) \quad \text{(if } d+1 = \text{even)}
\end{array} \right. \]

where \( p_1 = (d-1)^{-1}, p_3 = -(d-2)/(2(d-3)), \ldots \)

\[ \ldots \quad q = (-1)^{(d-1)/2} (d-2)!!/(d-1)!! \quad . \]

A universal quantity (F) which characterizes odd dim. CFT.

Agrees with conformal anomaly (central charge) in even dim. CFT.

Area law divergence
Holographic Proof of Strong Subadditivity

\[ S_{AB} + S_{BC} \geq S_{ABC} + S_B \]

\[ S_{AB} + S_{BC} \geq S_A + S_C \]

(Note: \( AB \equiv A \cup B \))

Algebraic properties in Quantum Information
\( \Leftrightarrow \) Geometric properties in Gravity
A generic Lorentzian asymptotic AdS spacetime is dual to a time dependent state $|\Psi(t)\rangle$ in the dual CFT.

The entanglement entropy gets time-dependent:

$$\rho_A(t) = \text{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \quad \rightarrow \quad S_A(t).$$

This is computed by the holographic formula:

$$S_A(t) = \text{Min}_{\Gamma_A} \text{Ext}_{\Gamma_A} \left[ \frac{A(\Gamma_A)}{4G_N} \right]$$
One qubit of entanglement for each Planck length area!

Spacetime may emerge from entangled Qubits!

Tensor Network (TN) realizes this idea!

\[ S_A = \frac{\text{Area}(\Gamma_A)}{4l_{pl}^{D-1}} \]

≈ \(10^{65}\) qubits per 1\(cm^2\)!
Tensor Network (TN) [DMRG: White 92, CTM: Nishino–Okunishi 96, PEPS: Verstraete–Cirac 04, …]

TN = Graphical description of quantum states
Quantum State = Network of quantum entanglement

[Ex. 1] MERA TN [Vidal 2005] — Describe CFT vacuum


Geometric Structure of Qubits = AdS [Swingle 2009, …]
**Ex.4 Path-integral Optimization**

- For AdS/CFT we need continuum theory as TN
- Time slice of AdS must be Euclidean (non-unitary)

**Basic Principle**

Minimize the computational cost of (discretized) path-integral.

\[
\lim_{T \to \infty} e^{-T \cdot H} |\psi\rangle = |\psi_0\rangle
\]

*ground state*

A time slice of AdS emerges.

Feynman Path-integral!
Upshot: Minimizing computational costs leads to gravity

Energetic source (=information source) distorts the spacetime
→ The essence of general relativity!

Comment
Emergent space in AdS/CFT seems to be related to non-unitary gates.
→ Relevant complexity class might be like PostBQP.

[Thanks to Tomoyuki Morimae and Ryu Hayakawa for suggestions]
Question: Ver 3. Holographic Entropy Formula?

Minimal areas in *Euclidean time dependent* asymptotically AdS spaces

= What kind of QI quantity (Entropy ?) in CFT?

The answer is Pseudo Entropy!
(3-1) Definition of Pseudo (Rényi) Entropy

Consider two quantum states $|\psi\rangle$ and $|\varphi\rangle$, and define the transition matrix:

$$\tau_{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$ 

We decompose the Hilbert space as $H_{tot} = H_A \otimes H_B$ and introduce the reduced transition matrix:

$$\tau_A^{\psi|\varphi} = \text{Tr}_B \left[ \tau_{\psi|\varphi} \right]$$

Pseudo Entropy

$$S \left( \tau_A^{\psi|\varphi} \right) = -\text{Tr} \left[ \tau_A^{\psi|\varphi} \log \tau_A^{\psi|\varphi} \right].$$

Rényi Pseudo Entropy

$$S^{(n)} \left( \tau_A^{\psi|\varphi} \right) = \frac{1}{1-n} \log \text{Tr} \left[ \left( \tau_A^{\psi|\varphi} \right)^n \right].$$
(3-2) Basic Properties of Pseudo Entropy (PE)

• In general, $\tau_A^{\psi|\varphi}$ is not Hermitian. Thus PE is complex valued.

• If either $|\psi\rangle$ or $|\varphi\rangle$ has no entanglement (i.e. direct product state), then
  
  $$S^{(n)} (\tau_A^{\psi|\varphi}) = 0.$$ 

• We can show
  
  $$S^{(n)} (\tau_A^{\psi|\varphi}) = [S^{(n)} (\tau_A^{\varphi|\psi})]^{\dagger}.$$ 

• We can show
  
  $$S^{(n)} (\tau_A^{\psi|\varphi}) = S^{(n)} (\tau_B^{\psi|\varphi}). \rightarrow \text{“} S_A = S_B \text{”}$$

• If $|\psi\rangle = |\varphi\rangle$, then
  
  $$S^{(n)} (\tau_A^{\psi|\varphi}) = \text{Renyi entropy}.$$
Comment: In quantum theory, transition matrices arise when we consider *post-selection*.

\[
\frac{\langle \varphi | O_A | \psi \rangle}{\langle \varphi | \psi \rangle} = \text{Tr} [O_A \tau_A^\psi | \varphi \rangle]
\]

This quantity is called **weak value** and is complex valued in general. [Aharanov-Albert-Vaidman 1988,...]

Thus, **pseudo entropy** = weak value of “modular operator”:

\[
S \left( \tau_A^\psi | \varphi \rangle \right) = \frac{\langle \varphi | H_A | \psi \rangle}{\langle \varphi | \psi \rangle} = \text{Area Operator}
\]

\[
H_A = -\log \tau_A
\]
**Holographic Pseudo Entropy (HPE) Formula**

\[
S \left( \tau_A^\psi|\varphi \right) = \text{Min}_{\Gamma_A} \left[ \frac{A(\Gamma_A)}{4G_N} \right]
\]

**Basic Properties**

(i) If \( \rho_A \) is pure, \( S \left( \tau_A^\psi|\varphi \right) = 0 \).

(ii) If \( \psi \) or \( \varphi \) is not entangled,
\[
S \left( \tau_A^\psi|\varphi \right) = 0.
\]

→This follows from AdS/BCFT [TT 2011]

(iii) \( S \left( \tau_A^\psi|\varphi \right) = S \left( \tau_B^\psi|\varphi \right) \). "SA=SB"
(3-4) Pseudo Entropy as Entanglement Distillation

In the special case where we can diagonalize $|\psi\rangle$ and $|\varphi\rangle$ at the same time, PE has a nice interpretation:

$$|\psi\rangle = \cos \theta_1 |00\rangle + \sin \theta_1 |11\rangle, \quad |\varphi\rangle = \cos \theta_2 |00\rangle + \sin \theta_2 |11\rangle$$

$$S \left( \tau_A^{\psi|\varphi} \right) = - \frac{\cos \theta_1 \cos \theta_2}{\cos(\theta_1 - \theta_2)} \cdot \log \frac{\cos \theta_1 \cos \theta_2}{\cos(\theta_1 - \theta_2)} - \frac{\sin \theta_1 \sin \theta_2}{\sin(\theta_1 - \theta_2)} \cdot \log \frac{\sin \theta_1 \sin \theta_2}{\sin(\theta_1 - \theta_2)}$$

In this case, we can show

$$S \left( \tau_A^{\psi|\varphi} \right) \approx \sum_k p_k \cdot \text{# of Bell Pairs in } |\text{Bell}_k\rangle$$

$$\approx \text{# of Bell Pairs in intermediate states}$$

$$\frac{\langle \varphi | \sum_k |\text{Bell}_k\rangle \langle \text{Bell}_k | \psi \rangle}{\langle \varphi | \psi \rangle} = p_k$$
Motivation: Improve PE so that (i) it become real and non-negative and (ii) it has a better LOCC interpretation.

\[
S_{SVD} \left( \tau_A^\psi|\varphi \right) = -\text{Tr} \left[ |\tau_A^\psi|\varphi \rangle \cdot \log |\tau_A^\psi|\varphi \rangle \right].
\]

Here, \(|\tau_A^\psi|\varphi \rangle \equiv \sqrt{\tau_A^\psi|\varphi \rangle \tau_A^\psi|\varphi \rangle}\)

- This is always non-negative and is bounded by \( \log \dim HA \).
- From quantum information theoretic viewpoint, this is the number of Bell pairs distilled from the intermediate state:

\[
\tau_A^\psi|\varphi = U \cdot \Lambda \cdot V,
\]

\[
\frac{\langle \phi|V^\dagger \sum_k |EPR_k \rangle \langle EPR_k| U^\dagger |\psi \rangle}{\langle \phi|V^\dagger U^\dagger |\psi \rangle} = p_k, \quad \sum_k p_k = 1
\]

\[
S_{SVD} \approx \sum_k p_k \cdot \# \text{ of Bell Pairs in } |EPR_k \rangle
\]
Basic Properties of Pseudo entropy in QFTs

[1] Area law

\[ S_A \sim \frac{\text{Area}(\partial A)}{\mathcal{E}^{d-1}} + \text{(subleading terms)}, \]

[2] The difference

\[ \Delta S = \text{Re} \left[ S \left( \tau_A^\dagger |^2 \right) \right] - \left( S(\rho_A^1) + S(\rho_A^2) \right)/2 \]

is negative if \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are in a same phase.

What happen if they belong to different phases?
Can \( \Delta S \) be positive?
(4–2) Quantum Ising Chain with a transverse magnetic field

\[ H = -J \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z - \hbar \sum_{i=0}^{N-1} \sigma_i^x, \]

\[ \Psi_1 \rightarrow \text{vacuum of } H(J_1) \]
\[ \Psi_2 \rightarrow \text{vacuum of } H(J_2) \]
(We always set \( \hbar = 1 \))

J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

N=16, NA=8

\[ J_1 = 1/2 \]
\[ J_1 = 1 \]
\[ J_1 = 2 \]

We find
\[ \Delta S = S(\tau_A^{1|2}) + S(\tau_A^{1|2}) - S(\rho_A^1) - S(\rho_A^2) > 0 \]
when \( \Psi_1 \) and \( \Psi_2 \) are in different phases!
Two gapped phases are separated by a gapless phase.

\[ \Psi_1 \quad \text{Ψ2} \quad A \]

The gapless interface (edge state) also occurs in topological orders.

\[ \Gamma_A \quad \text{AdS Dual of Gapless Interface} \]

Topological pseudo entropy \[ [\text{Nishioka–Taki–TT 2021}] \].

End-of-the-world brane

Time $t$

AdS geometry

Non-unitary dynamics

Unitary Projection

Holographic Model?

(i) $p < p_*$: $S_A \propto t$,
(ii) $p = p_*$: $S_A \propto \log t$,
(iii) $p > p_*$: $S_A = \text{finite}$,
Double Wick Rotation and Time Evolution

Imaginary valued scalar field on EOW brane → Dissipation

\[ \rho(t) = e^{-\left( \frac{\beta + it}{4} \right) H} B(\varphi_0 + \Delta \varphi) \langle B(\varphi_0) | e^{-\left( \frac{\beta - it}{4} \right) H} \]

\[ |B\rangle = \text{Boundary state} \sim \text{Direct product state} \]

\( \Delta \varphi < \Delta \varphi^* \)

\( \Delta \varphi = \Delta \varphi^* \)

\( \Delta \varphi > \Delta \varphi^* \)

(a) BTZ phase

\[ S_A \approx \frac{c}{6} \cdot \frac{\pi}{2\alpha} \cdot t \]

(b) Poincare AdS phase

\[ S_A \approx \frac{c}{3} \cdot \log t \]

(c) Thermal AdS phase

\[ S_A \approx \text{const.} \]

A = a half space

\( \tau = \beta / A \)

\( \tau = -\beta / A \)
Conclusions

Pseudo entropy (PE) is a generalization of entanglement entropy.

- PE depends on both the initial and final state.
- PE is in general complex valued.
- PE for diagonalizable states measures the amount of quantum entanglement in the intermediate states.
- In AdS/CFT, PE is equal to the minimal surface area.
  ➤ Emergence of space from PE
- PE can detect the difference of quantum phases.
  ➤ New quantum order parameter
Future directions

• Quantum information meaning of the complex values of PE?
  
  Gravity formula suggests that if EE is useful in QI, so does PE...
  
  We might think “pseudo pseudo entanglement”.

• Implications to quantum gravity including de Sitter spaces?
  
  ▶ Imaginary part of PE= Emergent time in holography?

  [Doi-Harper-Mollabashi-Taki-TT. PRL 130(2023)031601]
Thank you very much!