Quantum complexity of clique homology

With Tamara Kohler

https://arxiv.org/pdf/2311.17234

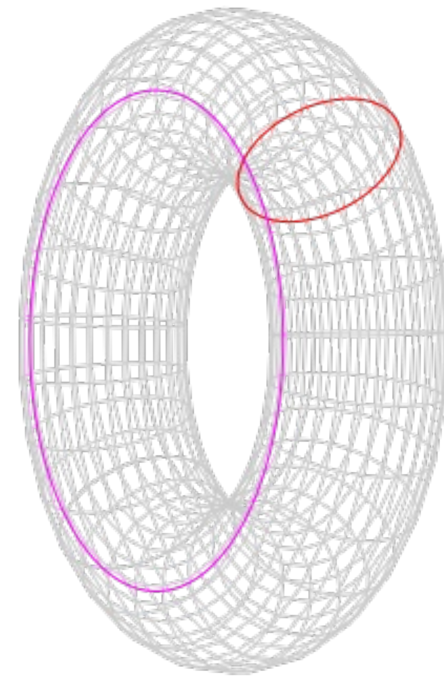
Quantum structure in homology

Homology = holes in spaces

Homology has **quantum mechanical** structure.

Given discrete space G and dimension k, decide if:

- (YES) G has a k-dimensional hole.
- (NO) G is far from having a k-dimensional hole.



...inside QMA and QMA1-hard.

Quantum algs for topological data analysis

ARTICLE

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Quantum algorithms for topological and geometric analysis of data

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Analyzing Prospects for Quantum Advantage in Topological Data Analysis

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A streamlined quantum algorithm for topological data analysis with exponentially fewer qubits

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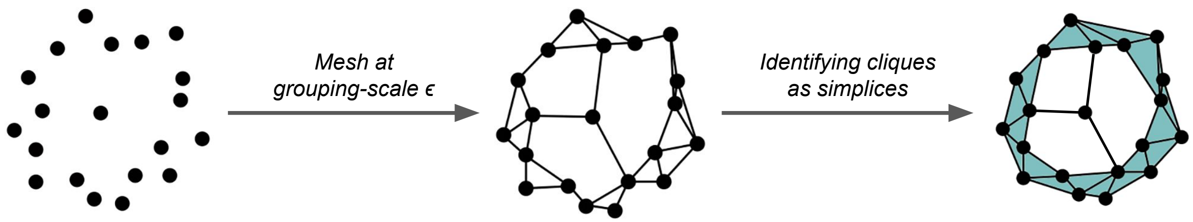
(Dated: September 27, 2022)

…quantum advantage?

Quantum structure in classical problems

- Hidden subgroup problem ⇔ quantum Fourier transform
- Jones polynomial ⇔ topological quantum field theory
- **Homology ⇔ supersymmetry?**

Topological data analysis

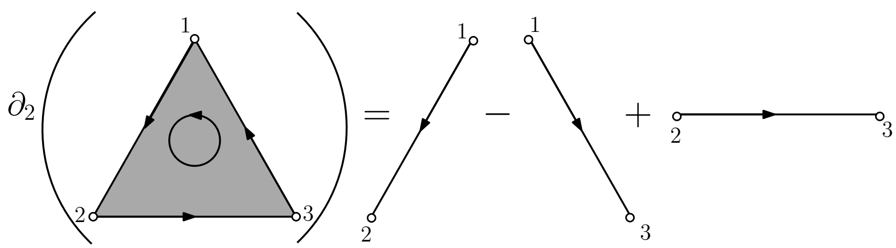


- k-simplices = $k + 1$-cliques

Homology

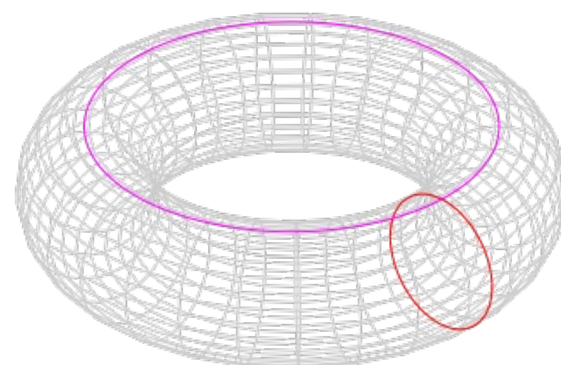
Chain-space: C_k = complex span of k-simplices

$$\cdots \longrightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \longrightarrow \cdots$$



$$\partial_k \circ \partial_{k+1} = 0$$

$$H_k := \mathrm{Ker}\,\partial_k \big/ \mathrm{Im}\,\partial_{k+1}$$

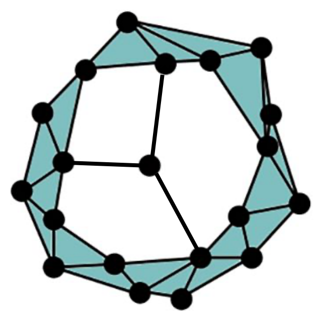


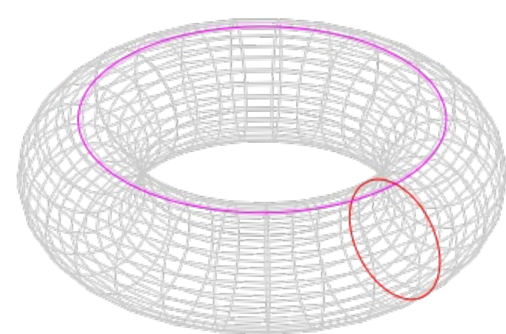
Complexity of homology?

Given graph, decide if:

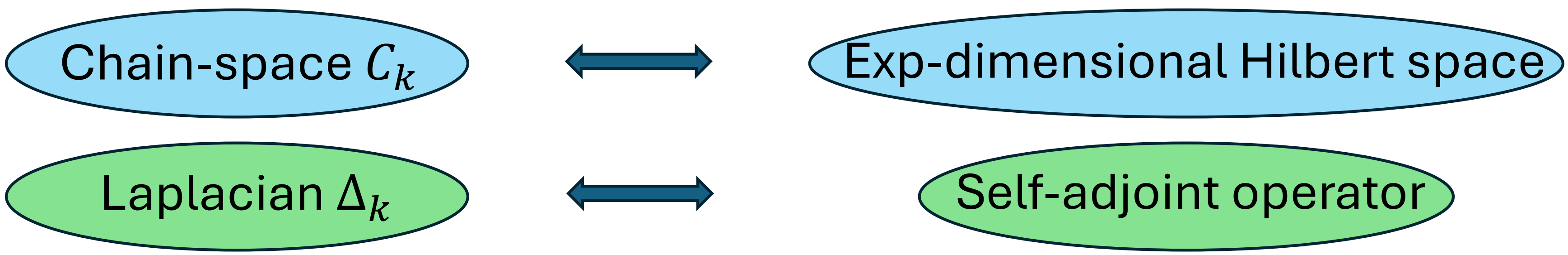
- (YES) G has a k-dimensional hole.
- (NO) G has no k-dimensional hole.

Computational complexity?





Homology ≅ Quantum Mechanics



- **Groundspace** of Δ_k = **homology**.
- Laplacian is **exponential**-dimensional **sparse** matrix encoded **succinctly** by graph.
- Homology becomes Hamiltonian problem on Δ_k.

Laplacian operator

Chain-space C_k ⟺ Exp-dimensional Hilbert space

Laplacian Δ_k ⟺ Self-adjoint operator

Definitions:

$$\Delta_k = \partial_k^\dagger \partial_k + \partial_{k+1} \partial_{k+1}^\dagger$$

$$H_k := \mathrm{Ker}\, \partial_k \big/ \mathrm{Im}\, \partial_{k+1}$$

Theorem: $\mathrm{Ker}\, \Delta_k \cong H_k$

Proof: $\langle \psi | \Delta_k | \psi \rangle = || \partial_k | \psi \rangle ||^2 + || \partial_{k+1}^\dagger | \psi \rangle ||^2$

Quantum complexity of homology

Gapped Clique Homology (GCH):

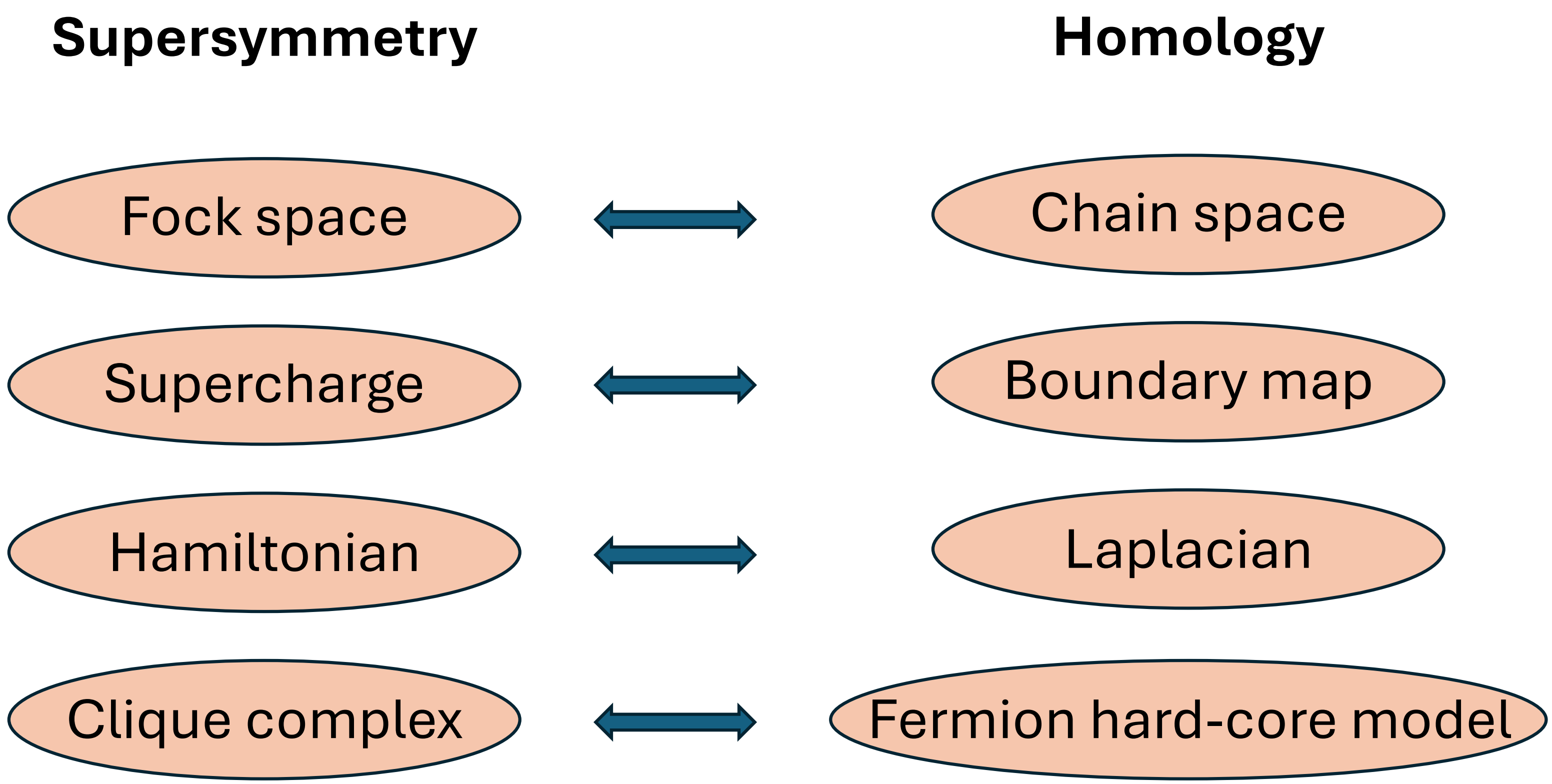
Given vertex-weighted graph G and dimension k, decide if:

- (YES) G has a k-dimensional hole.
- (NO) G has no k-dimensional hole, and min eigenvalue of Laplacian is at least $1/\mathrm{poly}(n)$.

GCH ∈ QMA

Our result: GCH is QMA1-hard

Supersymmetry



Dequantizing TDA?

- Quantum algorithms exploit Hamiltonian simulation and phase estimation of the Laplacian Δ_k

- Minimum eigenvalue of Δ_k is QMA1-hard

➔ Δ_k possesses no exploitable structure

➔ quantum TDA algorithms *cannot* be dequantized.

Hardness construction overview

Reduce from **local Hamiltonian problem**

$$H = \sum_i |\phi_i\rangle\langle\phi_i|$$

- (YES) Ground energy $= 0$
- (NO) Ground energy $\geq 1/\mathrm{poly}(n)$

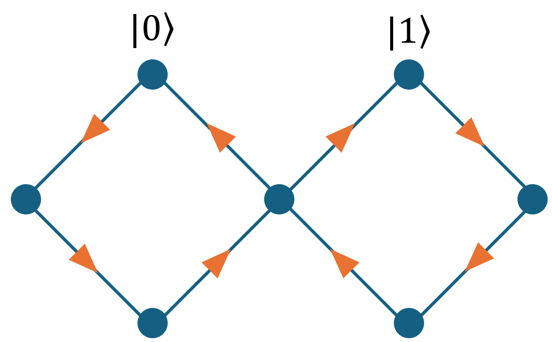
reduce →

Graph G, dimension k

- (YES) G has a k-dimensional hole.
- (NO) G has no k-dimensional hole, and $\lambda_{\min}(\Delta_k) \geq 1/\mathrm{poly}(n)$.

Hardness construction: 1 qubit

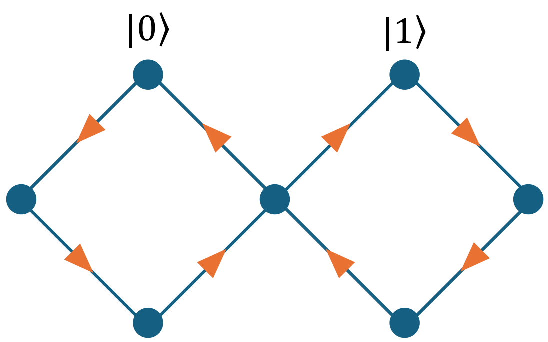
1-qubit graph G_1



$H_1(G_1) \cong \mathbb{C}^2$

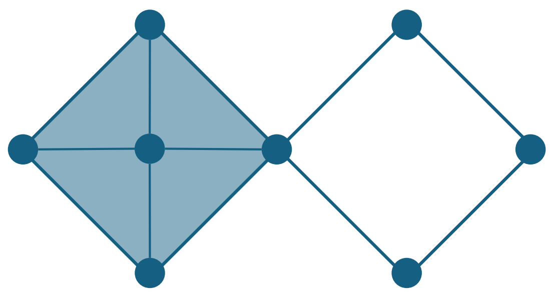
Hardness construction: 1 qubit

1-qubit graph G_1

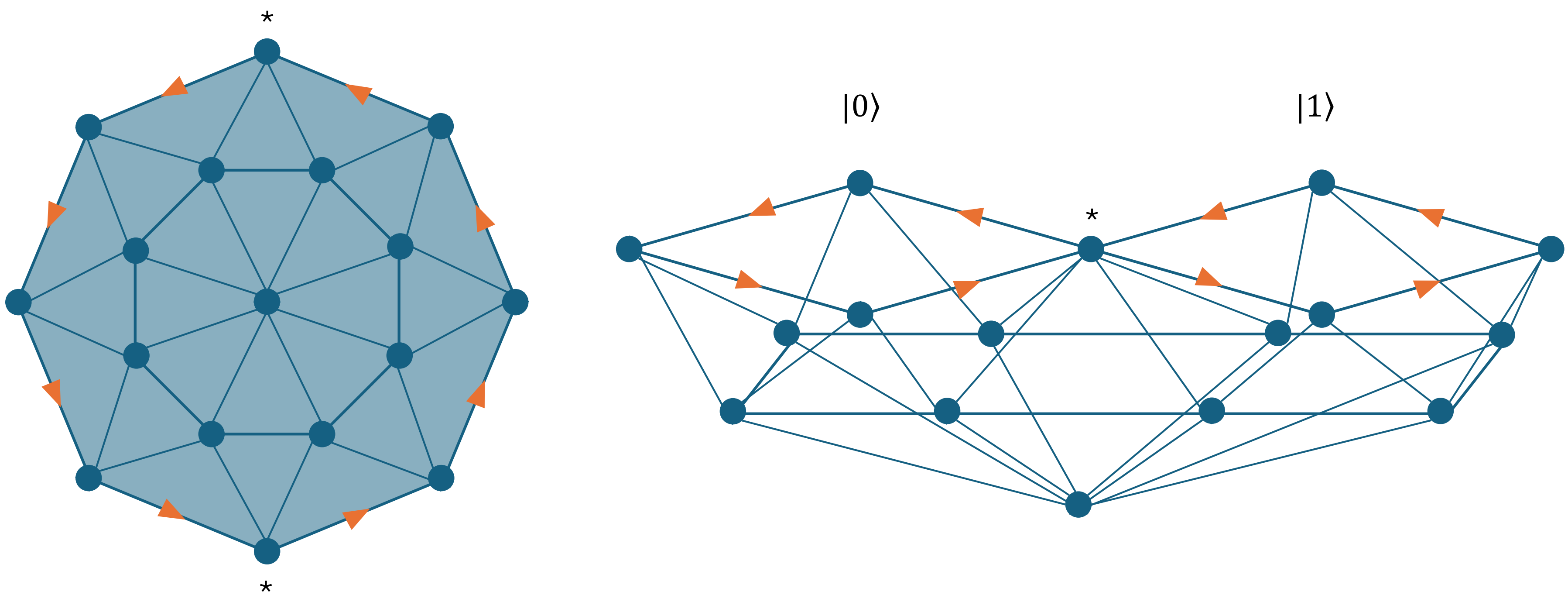


$H_1(G_1) \cong \mathbb{C}^2$

$|0\rangle\langle 0|$

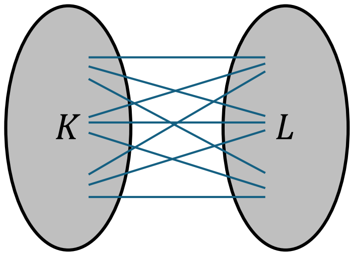


Hardness construction: $|-\rangle\langle-|$



Hardness construction: tensor products

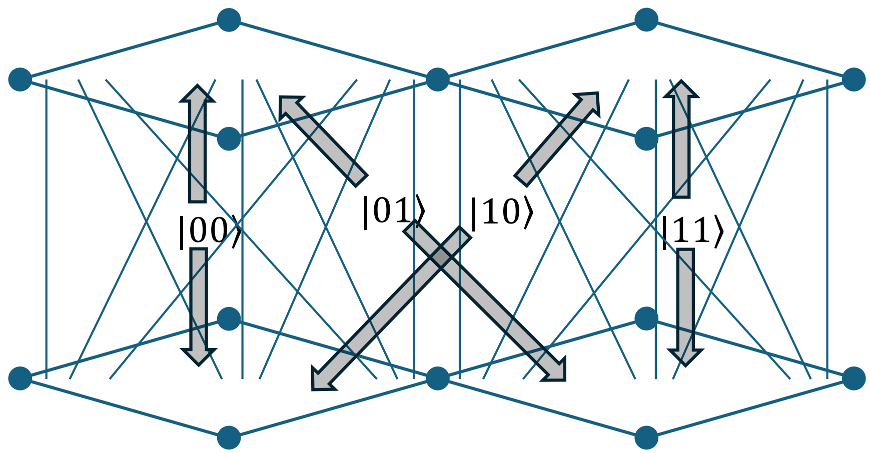
- **Join**: connect graphs all-to-all



$$\mathcal{K} * \mathcal{L}$$

- n-qubit graph $G_n = G_1 * \cdots * G_1$ (n times)

- G_2:



Weighting

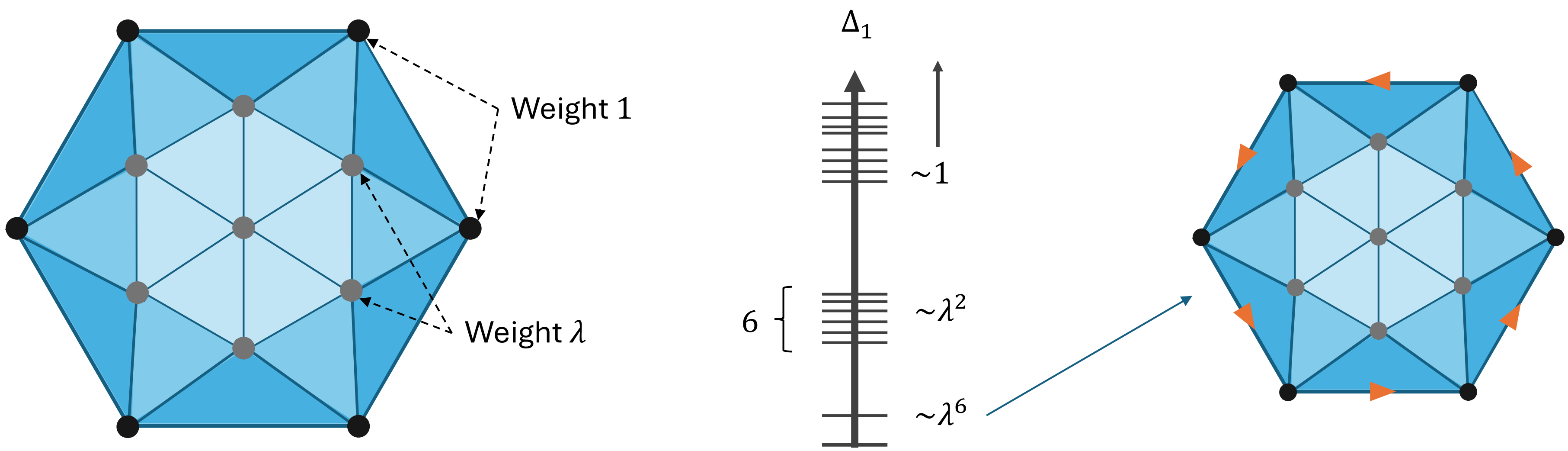
- Weight **vertices** of graph
- Topology unchanged
- Affects Laplacian Δ_k
- Qubit vertices weight $= 1$
- Gadget vertices weight $= \lambda$

$$\lambda = 1/\text{poly}(n)$$

Definitions:

$$\Delta_k = \partial_k^\dagger \partial_k + \partial_{k+1} \partial_{k+1}^\dagger$$

Spectral sequences

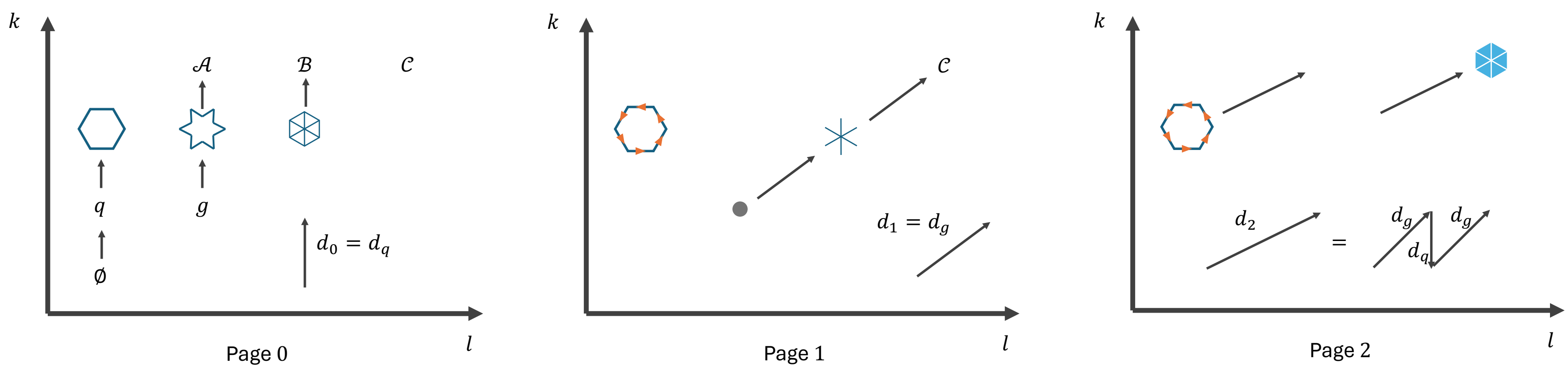


- Want **perturbative** expansion of **Laplacian** groundspace.

Spectral sequences

$$E_j^k = \{|\psi\rangle \in C_k \; : \; \exists\, |\psi_\lambda\rangle = |\psi\rangle + \lambda|\psi_1\rangle + \lambda^2|\psi_2\rangle + \ldots \; \text{s.t.}\; \langle\psi_\lambda|\Delta_k|\psi_\lambda\rangle = \mathcal{O}(\lambda^{2j})\}$$

- **Spectral sequences** give sequence of spaces e_j^k
- Theorem [Forman 1995]: $e_j^k \cong E_j^k$
- Difficult **topology/analysis** becomes easy **algebra**
- Replaces role of **perturbation theory** in perturbative gadgets



Conclusion

Laplacian Δ_k:

- Sparse matrix
- Encodes topology
- Efficient access from graph input
- Supersymmetric Hamiltonian
    - Locality = degree of complement graph, **not** constant for us...

- Towards PCPs: Discrete graph problem – gadgets for **gap-amplification**?