Introduction to Commuting Local Hamiltonian Problem

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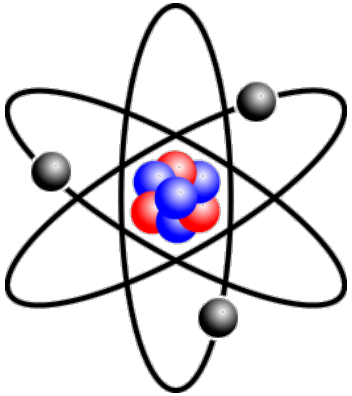
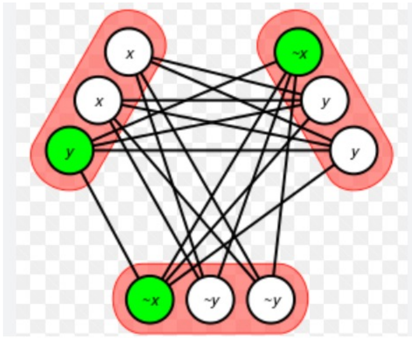
Based on [Bravyi-Vyalyi03] [Schuch11] [Aharonov-Kenneth-Vigdorovich18] [Irani-Jiang23]

A philosophy question...



(??) SAT-Solver



- Quantum world
- Quantum Chemistry
- (Challenging...)

- Classical world
- 3-SAT..
- (Many SAT-Solvers...)

What makes quantum problem inherently different from classical?

- ❖ Uncertainty principle: **non-Commuting** ⇔ Quantum
- ❖ Today: Relationship between Commuting & Hardness of quantum problems

Outline

- Commuting Local Hamiltonian problems (CLHP) [Bravyi-Vyalyi03]
- More Motivation

- Overview of results
- Statement of the Structure Lemma
- 2-local CLHP

- Toric code & 4-local 2D CLHP

Local Hamiltonian problems (LHP)

❖ LHP is the quantum MAX-3SAT.

❖ Input: (H,a,b)

- k-Local Hamiltonian

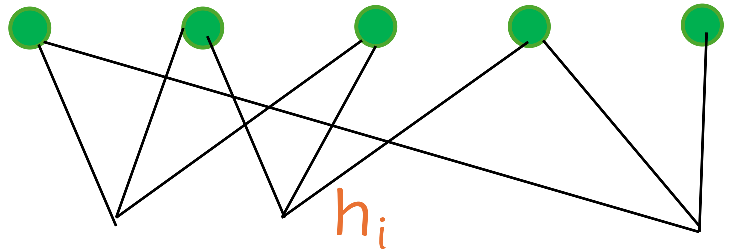
$$H = h_1 + \ldots + h_m$$

n-qudit Hermitian (H); k-qudit (h_m)

- $b - a \geq 1/poly(n)$
- $\lambda(H)$: ground energy of H

❖ Output:

- "Yes" if $\lambda(H) \leq a$
- "No" if $\lambda(H) \geq b$



❖ LHP is QMA-Complete (quantum NP) [KKR06]

Commuting LHP

❖ Input: (H,a,b)

- k-Local Hamiltonian

$$H = h_1 + \dots + h_m$$

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❖ **Commuting** LHP (CLHP)

- $[h_i, h_j] = 0.$
- [Bravyi-Vyalyi03]

▪ Examples:

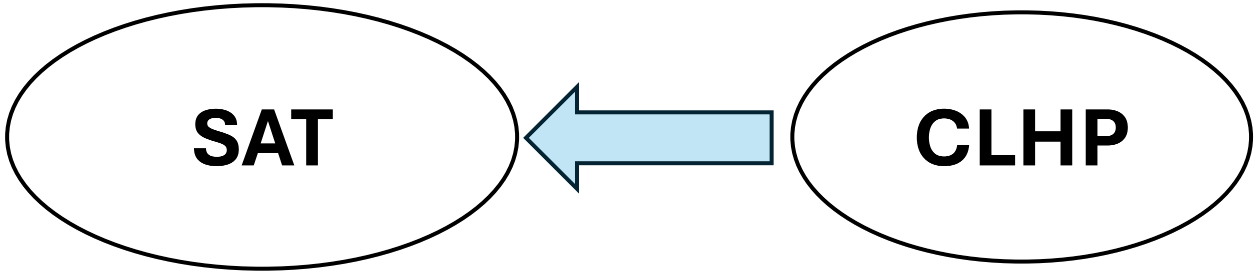
- 3-SAT
- Toric code, CSS code
- Quantum double model

Is CLHP quantum (QMA) or classical (NP)?

Intuitive reasons

- NP-Complete
- Classical

“Simple” but quantum



- Uncertainty principle

Is CLHP quantum (QMA) or classical (NP)?

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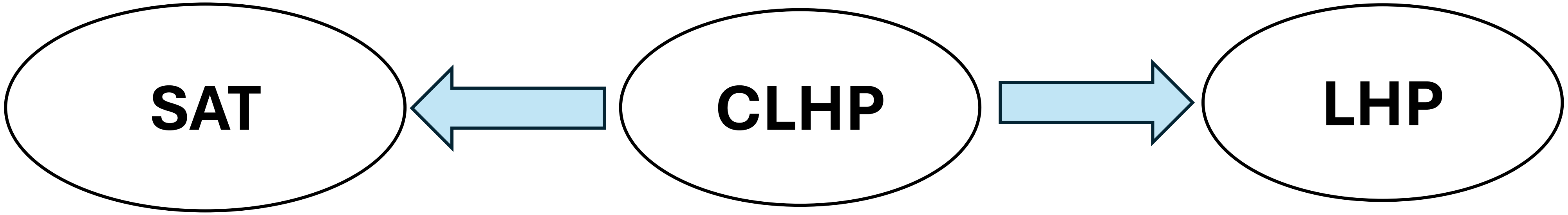
"Simple" but quantum

- QMA1-Complete
- Quantum.

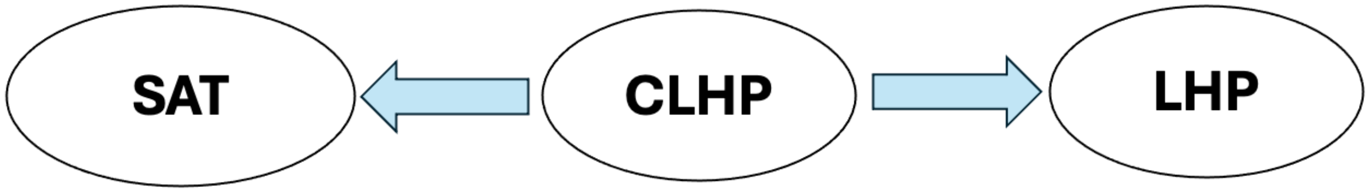


- Uncertainty principle
- $\{h_j\}_j$ diagonalize simultaneously
- But diagonalizing unitary may be complex.

- **CLH** can be very quantum!
- **Toric code:**
  - highly entangled
  - $\Omega(\log n)$ depth quantum circuit for the ground state.

Main problem

CLHP: “Simple” but quantum



1) Complexity of CLHP:
   - **NP**? QMA1? QCMA?
2) Ground state of CLH:
   - **Easy** or hard to prepare?
   - **Easy**: Trivial ground state, constant-depth quantum circuit.

Reasons to be interested in CLH

CLH: "Simple" but quantum

Complexity

Test ground for hard problems:

- **Quantum PCP** [AE13] [ABN22]
- Mixing time for **Gibbs preparation** (Thermalization) [KB16] [BCG+23]
- Quantum Lovasz Local Lemma (**Ground state preparation**) [GS17]

Physics

Topological order

- **Error correcting** codes
  - Stabilizer codes
- Self-correcting Quantum memory [BT09][AHHH08]

Overview of known results

- "Some special case of CLHP is in NP"
- 2-local, **qudit** [Bravyi-Vyalyi03]
- 3-local, **qubits** + **qutrits** ("Nearly Euclidean") [Aharonov-Eldar11]

"Classical"
Trivial ground state

- 4-local, 2D, **qubits** [Schuch11]
- [Aharonov-Kenneth-Vigdorovich18]
- 4-local, 2D, **qutrits** [Irani-Jiang23]

"Quantum"
No trivial ground state

- Factorized, **qubits**, [Bravyi-Vyalyi03]
- Factorized-2D, **qudits**, [Irani-Jiang23]

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- **Next talk:** Isaac & Daniel classification of 2D gapped ground states ground states of CLHs.

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No trivial ground state

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Is CLHP in NP? QCMA? QMA?

Tool box & Technical challenges.

Toolbox covered in this talk

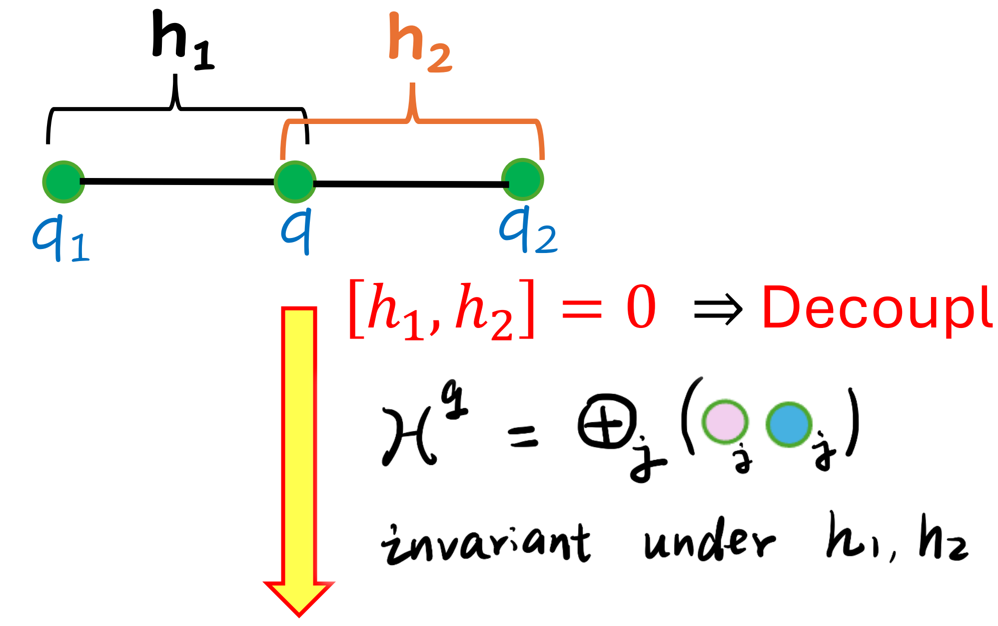
- [Bravyi-Vyalyi03]
- Structure Lemma : 2-local commuting ⇒ Decoupling (5 mins)

- [Schuch11]
- Transform CLHP-2D to computing trace. (10-15 mins)

- [Irani-Jiang23]
- Non-constructive self-reduction for CLHP. (5 mins)

The structure Lemma [BV03]

- 2-local commuting ⇒ Decoupling



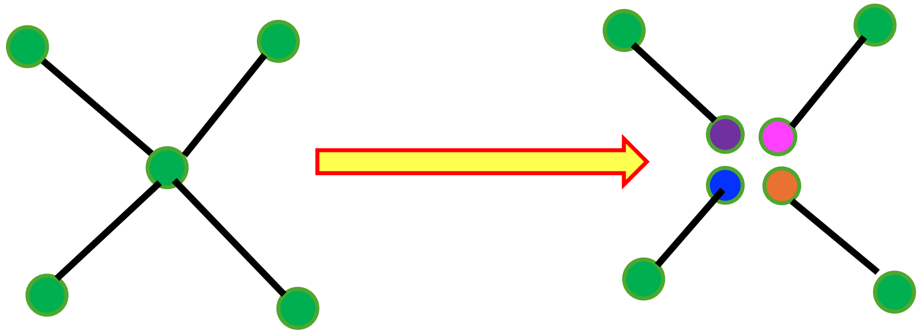
➢ (?) Ground state of $\boldsymbol{h_1} + \boldsymbol{h_2}$

➢ GS of $\boldsymbol{h_1} + \boldsymbol{h_2}$ can be prepared by constant depth circuit.

➢ If no overlap.
➢ GS: Tensor of 2-qudit state

The structure Lemma [BV03]

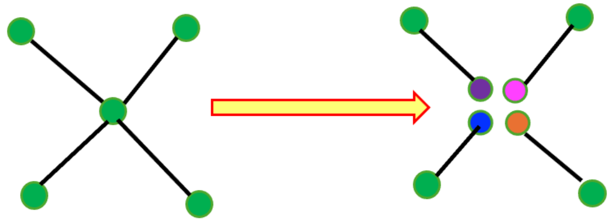
- More general 2-local interaction



- 2-local CLH: Trivial ground state, prepared by **constant depth** circuit.
- NP witness: **constant depth** circuit for GS (constructive proof)

Limitation of structure lemma

- Can structure Lemma proves general CLHP ∈ NP?



- No it can't…
  - Structure Lemma ⇒ Trivial GS
  - **No** trivial ground state for Toric code!
    - Toric code: **4-local CLH on 2D.**
    - GS needs $\Omega(\log n)$ depth quantum circuits.

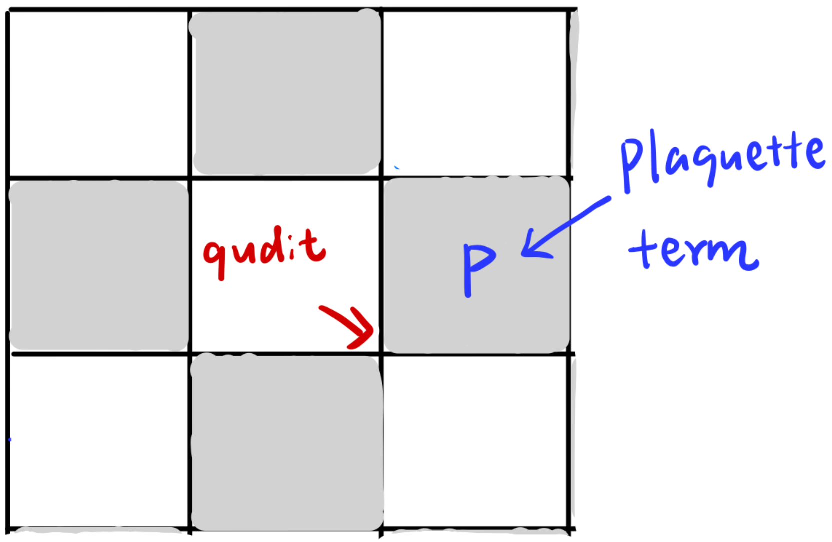
Outline: New ideas for qudit-CLHP-2D

- ❖ (?) qudit-**CLHP-2D** ∈ **NP**?

- ○ why structure lemma doesn't work
- ○ **Qubit**-CLHP-2D [Sch11] (computing trace)

- ○ **Qutrit**-CLHP-2D, [IJ23] (**non**-constructive self-reduction)
- ○ **Factorized**-qudit-CLHP-2D [IJ23]

Qudit-CLHP-2D



- Assume p projections
- Qudit-CLHP-2D:

$$H = \sum_p p$$

Decide

$$\lambda(H) = 0 \; or \; \lambda(H) \geq 1$$

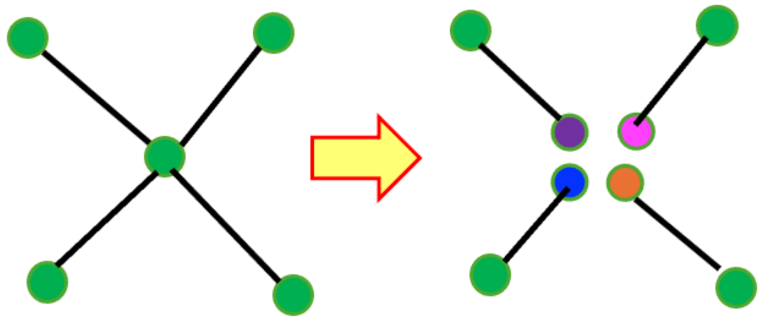
Goal: Prove qubit-CLHP-2D ∈ NP

Certify $\lambda(H) = 0$ classically.

Why structure Lemma doesn't work?

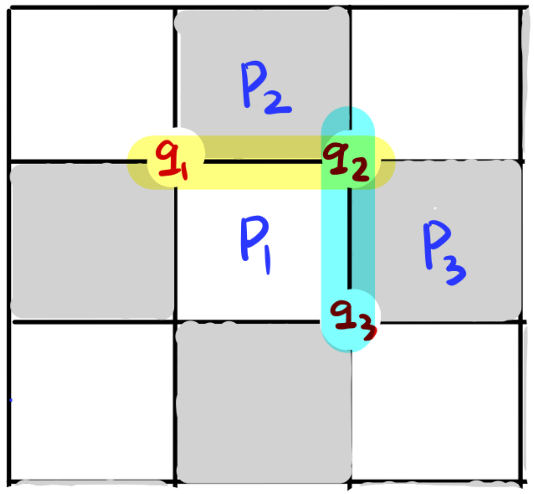
2-local

star-like interaction

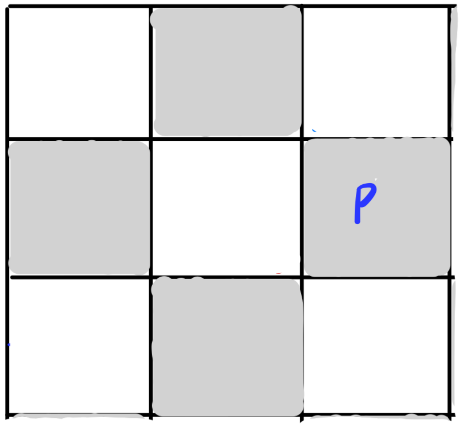


4-local

Incompatible decomposition



[Sch11]
Non-constructive proof for qubit-CLHP-2D ∈ NP



❖ [Sch11] Transform the CLHP-2D to computing trace.

Π_p : projection onto 0-eigenspace of p

$$\lambda(H) = 0 \Leftrightarrow tr\left(\prod_p \Pi_p\right) > 0,$$

$P_G =$ product of gray Π_p.

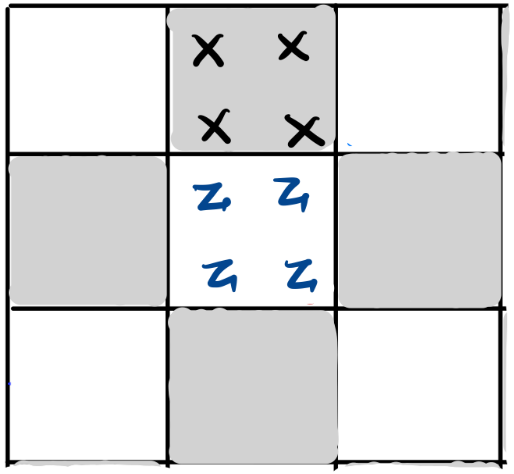
$P_W =$ … … of white …

$tr(P_G P_W) > 0$

- ❖ $H = \sum_p p$
- ❖ p commuting projections
- ❖ Certify $\lambda(H) = 0$ classically.

Why proving trace is easier? $tr(P_G P_W)>0$

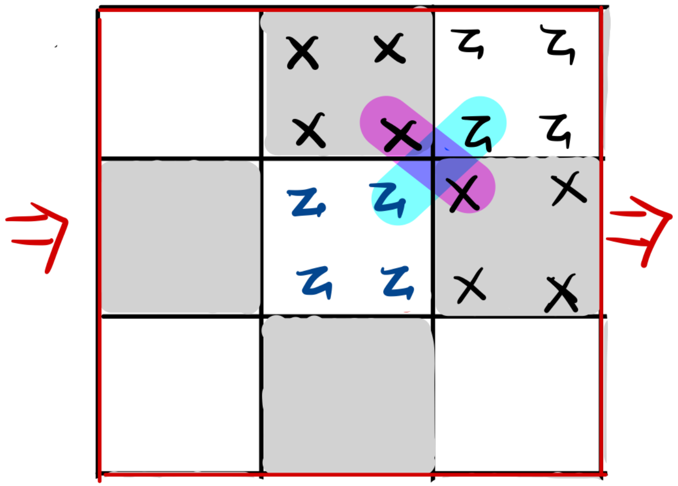
Take Toric code as an example



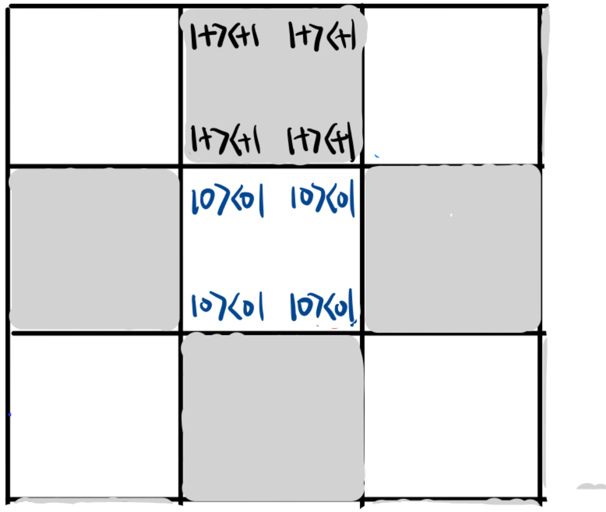
Gray: $\Pi_p = (I + X^{\otimes 4})/2$

white: $\Pi_p = (I + Z^{\otimes 4})/2$

- $tr(P_G P_W) > 0$



Decompose w.r.t X & Z basis



Gray: $\Pi_p = |+\rangle\langle+|^{\otimes 4}$

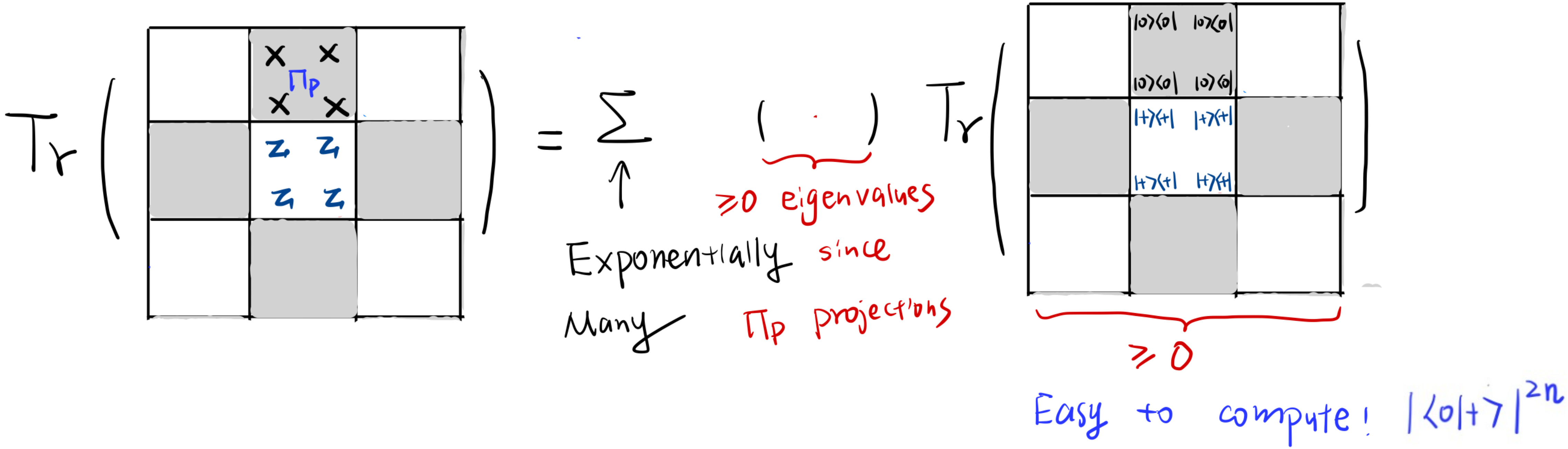
white: $\Pi_p = |0\rangle\langle 0|^{\otimes 4}$

- $tr(P_G P_W) > 0$

Easy: $|\langle 0|+\rangle|^{2n}$

Why **proving** trace is easier?

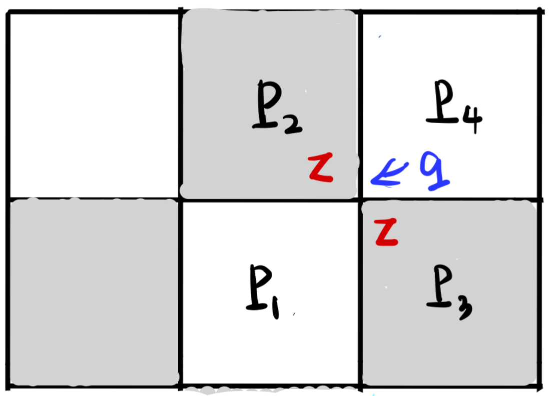
Take Toric code as an example



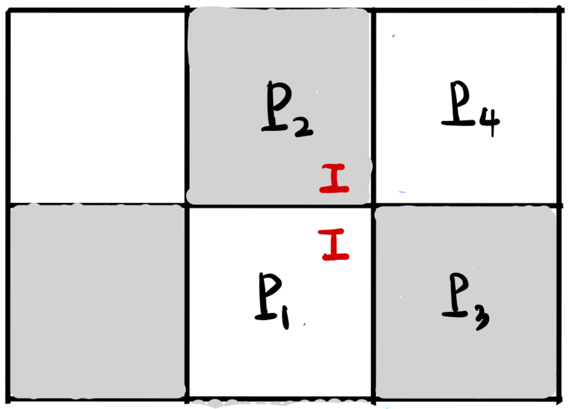
- ❖ Proving ∑ non-negative terms >0 is equivalent to proving one term >0!
- ❖ NP witness: index of one non-negative term.

Qubit-CLHP-2D in NP

- General qubit-CLHP-2D: classify qubits



(a) Similar to Toric code



(b) two terms acts as Identity on q.

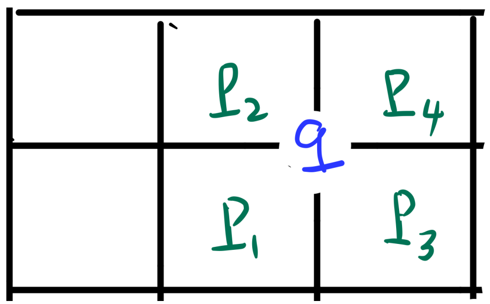
$tr(P_G P_W) > 0$

Decompose the trace w.r.t to all qubits in (a)
Each sub-trace will be of 1D structure because of the identity.

NP witness:

Qubit-CLHP-2D: Equivalence to Toric code permitting boundary
Constructive proof [AKV18]

- **Ground state** can be prepared similarly as **Toric code** by poly-size quantum circuit.
- **"Measure & Correct."**



- "Interior qubit" q: p_1, p_2, p_3, p_4 act non-trivially on q.

$$p_1, p_4 = I_q \otimes () + X_q \otimes ();$$
$$p_2, p_3 = I_q \otimes () + Z_q \otimes ();$$

4-local: qubit to qudit.

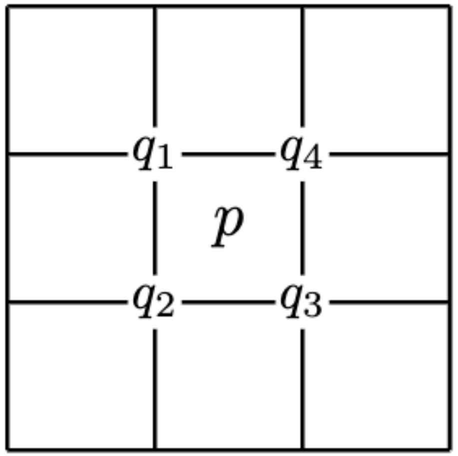
- Qudit/Qutrit is very different from qubit.
  - +1 dimension introduces degeneracy.
  - Sub-trace: ~~1D structure.~~

New for 4-local qudit [IJ23]

- ❖ Non-constructive proof.
  - ○ "rounding" for self-reduction

Theorem 1 *Qutrit-CLHP-2D is in* **NP**.

Theorem 2 *Factorized qudit-CLHP-2D is in* **NP**. *Furthermore, Hamiltonian in factorized qudit-CLHP-2D is equivalent to direct sum of qubit stabilizer Hamiltonian.*



- $p = p^{q_1} \otimes p^{q_2} \otimes p^{q_3} \otimes p^{q_4}$

4-local qudit [IJ23]

Theorem 1 *Qutrit-CLHP-2D is in* **NP**.

- Non-constructive proof.
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Theorem 2 *Factorized qudit-CLHP-2D is in* **NP**. *Furthermore, Hamiltonian in factorized qudit-CLHP-2D is equivalent to direct sum of qubit stabilizer Hamiltonian.*

- Factorized Hamiltonian can look very different from stabilizer Hamiltonian.
  - Stabilizer H: Regular way of commuting

$$X^{\otimes 4} Z^{\otimes 4} = Z^{\otimes 4} X^{\otimes 4}, XZ = -ZX.$$

  - Factorized H: Singular way of commuting

$$h_1 := \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \otimes A \qquad h_1 h_2 = 0 = h_2 h_1$$

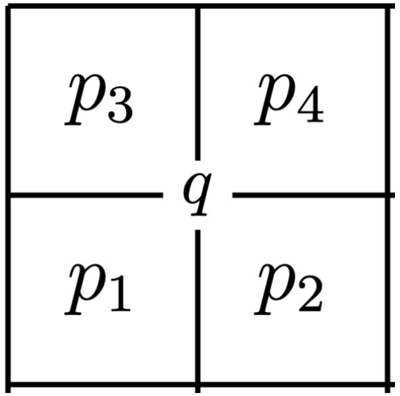
$$h_2 := \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \otimes B \qquad \text{for any } A, B.$$

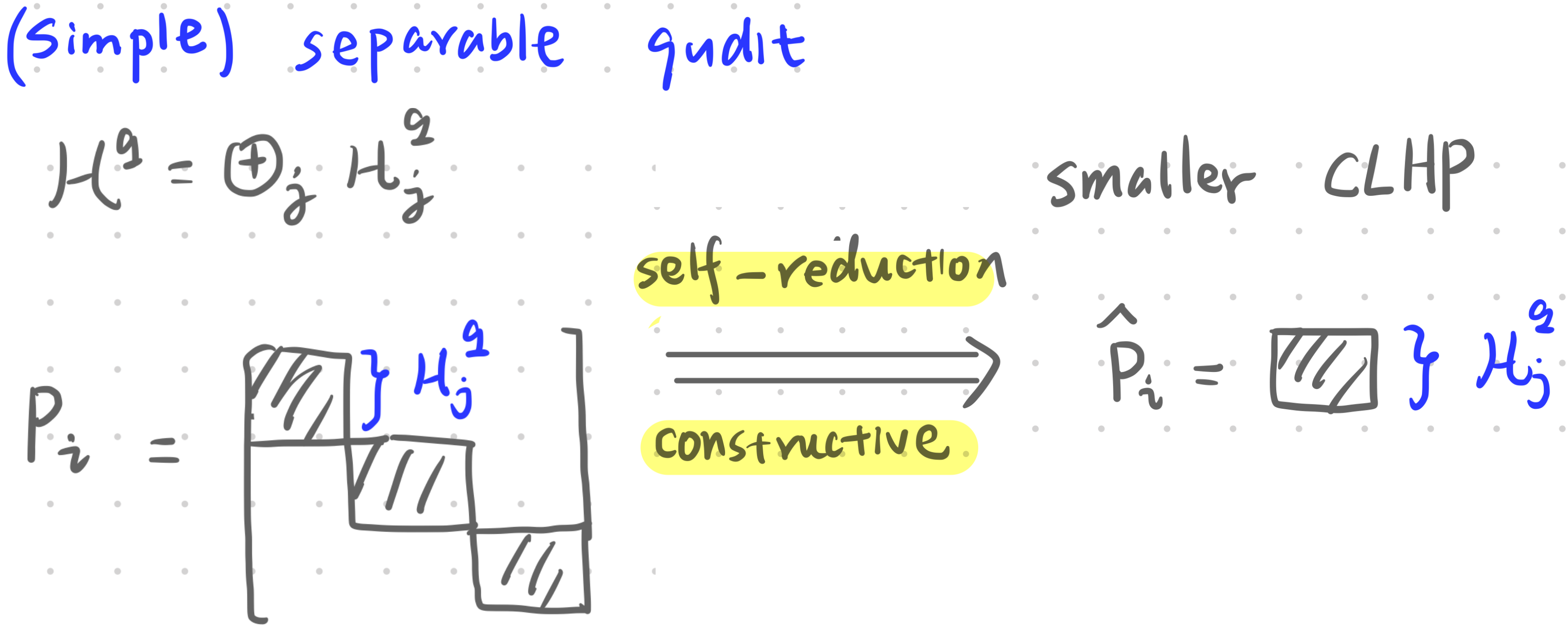
- We prove the ground space of factorized-CLHP looks like stabilizer Hamiltonian.

Key technique for qutrit-CLHP-2D in NP

Non-constructive self-reduction for CLHP: qutrit to qubit?

Goal: $H = \sum_{p_i} p_i$ (projection) prove $\lambda(H) = 0$; Decrease dimension of qudit





Key technique for qutrit-CLHP-2D in NP

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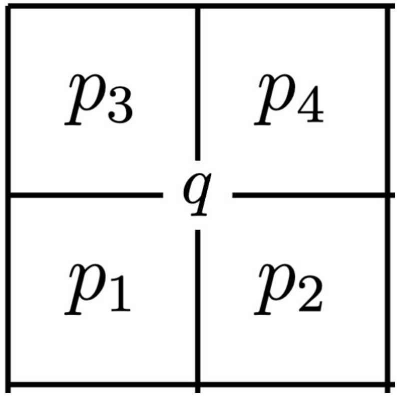
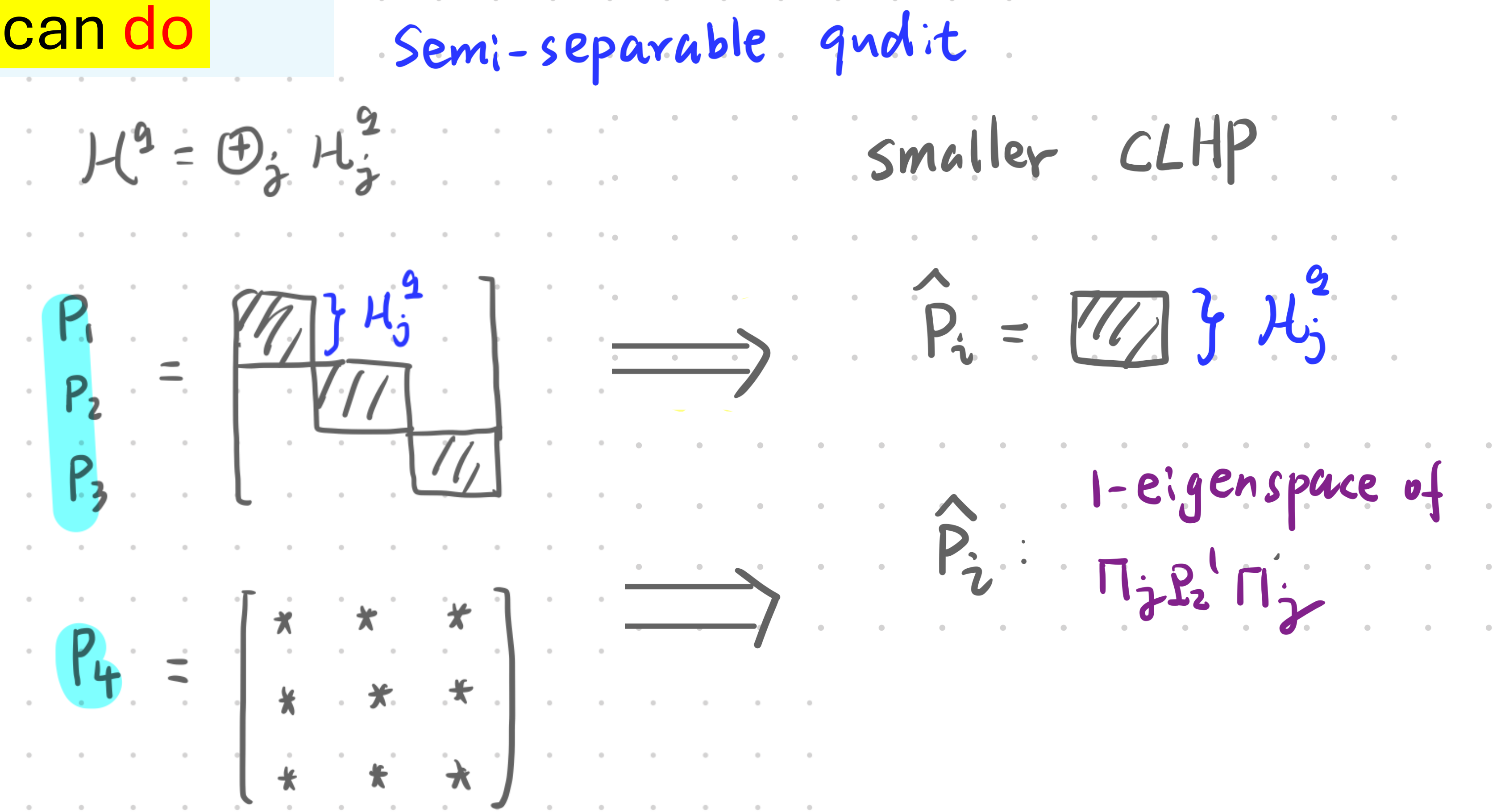
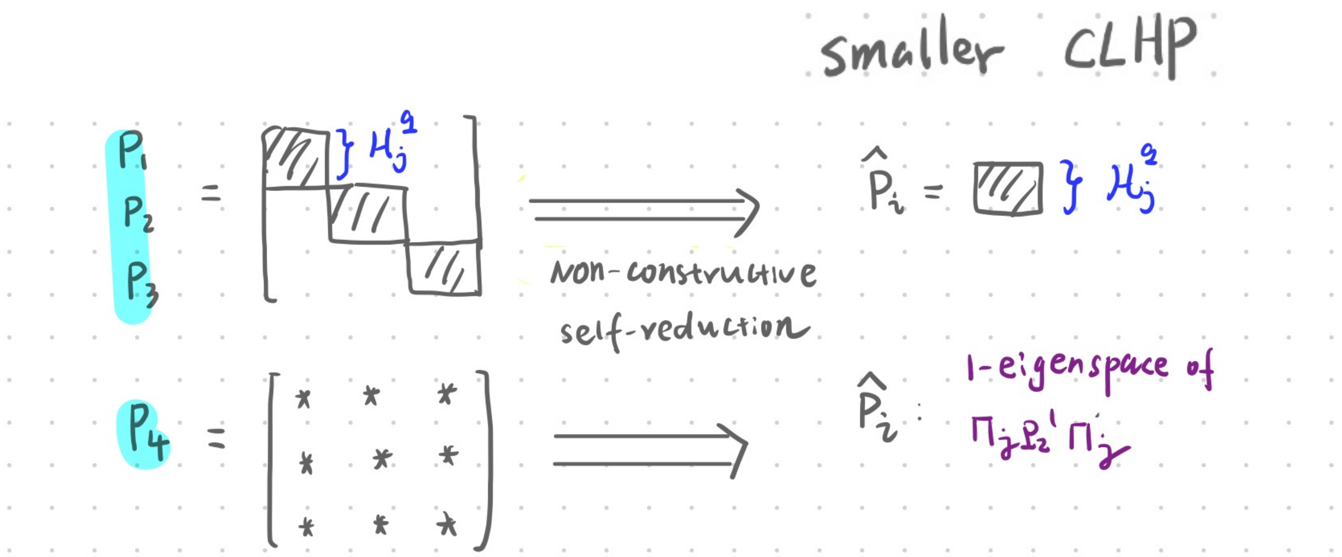
For CLHP we can do more!







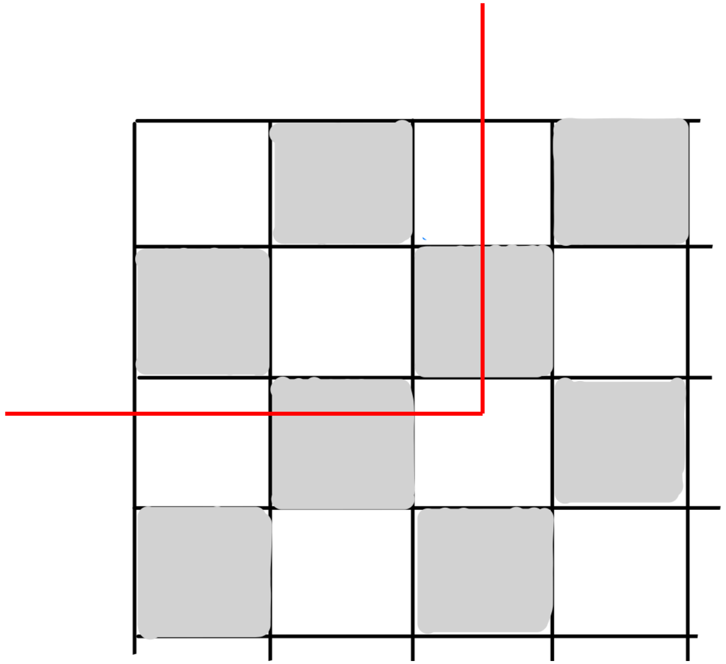
- **Non-constructive** Self-reduction**:**
  - We get a **CLHP** with **smaller** dimension in qudit q.
  - $\lambda(H) = 0$ iff exists i, $\lambda(\widehat{H}_i) = 0$

- This lemma works for any **qudit**, any **geometry**.

Open problems

"Some special case of CLHP is in NP"

Thanks for listening. Question?



- Qudit-CLHP-2D:
  - Complexity & Ground state preparation
    - In **NP**? Non-trivial class in **QCMA**?
    - Can GS be prepared by poly-size quantum circuit?
- (?) 2D Area Law ⇒ poly-size quantum circuit?
  - 2D Area law: for GS, **entanglement entropy** across **any cut** is proportional to the boundary.
    - 2D Area law (?) poly-size ground state (?)
    - Qudit-CLHP-2D: Area law (yes) & poly-size ground state (?)

Reference

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