Introduction to Commuting Local Hamiltonian Problem

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Based on [Bravyi-Vyalyi03] [Schuch11]
[Aharonov-Kenneth-Vigdorovich18] [Irani-Jiang23]
A philosophy question...

What makes quantum problem inherently different from classical?

- Uncertainty principle: non-Commuting ⇔ Quantum
- Today: Relationship between Commuting & Hardness of quantum problems
Outline

- Commuting Local Hamiltonian problems (CLHP) [Bravyi-Vyalyi03]
- More Motivation

- Overview of results
- Statement of the Structure Lemma
- 2-local CLHP

- Toric code & 4-local 2D CLHP
Local Hamiltonian problems (LHP)

- **LHP** is the quantum MAX-3SAT.
- **Input:** \((H,a,b)\)
  - \(k\)-Local Hamiltonian
  - \(b - a \geq 1/poly(n)\)
  - \(\lambda(H)\): ground energy of \(H\)
- **Output:**
  - “Yes” if \(\lambda(H) \leq a\)
  - “No” if \(\lambda(H) \geq b\)

- LHP is **QMA-Complete** (quantum NP) \([KKR06]\)
**Commuting LHP**

- **Input:** $(H,a,b)$
  - $k$-Local Hamiltonian
  - $b - a \geq 1/poly(n)$
  - $\lambda(H)$: ground energy of $H$

- **Output:**
  - “Yes” if $\lambda(H) \leq a$
  - “No” if $\lambda(H) \geq b$

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**Commuting LHP (CLHP)**

- $[h_i, h_j] = 0.$
- [Bravyi-Vyalyi03]

- **Examples:**
  - 3-SAT
  - Toric code, CSS code
  - Quantum double model
Is CLHP quantum (QMA) or classical (NP)?
Intuitive reasons

- NP-Complete
- Classical

“Simple” but quantum

- Uncertainty principle
Is CLHP quantum (QMA) or classical (NP)?

Intuitive reasons

- NP-Complete
- Classical

```
SAT
```

- “Simple” but quantum

```
CLHP
```

- QMA1-Complete
- Quantum.

```
LHP
```

- Uncertainty principle
- \{h_j\}_j diagonalize simultaneously
- But diagonalizing unitary may be complex.

- CLH can be very quantum!

- Toric code:
  - highly entangled
  - \(\Omega(\log n)\) depth quantum circuit for the ground state.
Main problem

CLHP: “Simple” but quantum

1) Complexity of CLHP:
   - NP? QMA1? QCMA?
2) Ground state of CLH:
   - Easy or hard to prepare?
   - Easy: Trivial ground state, constant-depth quantum circuit.
Reasons to be interested in CLH

CLH: “Simple” but quantum

Complexity

Test ground for hard problems:

- Quantum PCP [AE13] [ABN22]
- Mixing time for Gibbs preparation (Thermalization) [KB16] [BCG+23]
- Quantum Lovasz Local Lemma (Ground state preparation) [GS17]

Physics

Topological order

- Error correcting codes
  - Stabilizer codes
- Self-correcting Quantum memory [BT09][AHHH08]
Overview of known results

• “Some special case of CLHP is in NP”
• 2-local, **qudit** [Bravyi-Vyalyi03]
• 3-local, **qubits + qutrits** (“Nearly Euclidean”) [Aharonov-Eldar11]
• 4-local, 2D, **qubits** [Schuch11]
• [Aharonov-Kenneth-Vigdorovich18]
• 4-local, 2D, **qutrits** [Irani-Jiang23]
• Factorized, **qubits**, [Bravyi-Vyalyi03]
• Factorized-2D, **qudits**, [Irani-Jiang23]
Overview of known results

- “Some special case of CLHP is in NP”
- 2-local, qudit [Bravyi-Vyalyi03] (constructive)
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- 4-local, 2D, qutrits [Irani-Jiang23]
- Factorized, qubits, [Bravyi-Vyalyi03]
- Factorized-2D, qudits, [Irani-Jiang23] (constructive)

“Classical”
Trivial ground state

“Quantum”
No trivial ground state

**Next talk:** Isaac & Daniel classification of 2D gapped ground states
ground states of CLHs.
Overview of known results

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• 4-local, 2D, qutrits [Irani-Jiang23]
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Is CLHP in NP? QCMA? QMA?
Tool box & Technical challenges.

“Classical”
Trivial ground state

“Quantum”
No trivial ground state
Toolbox covered in this talk

• [Bravyi-Vyalyi03]
• Structure Lemma: 2-local commuting $\Rightarrow$ Decoupling (5 mins)

• [Schuch11]
• Transform CLHP-2D to computing trace. (10-15 mins)

• [Irani-Jiang23]
• Non-constructive self-reduction for CLHP. (5 mins)
The structure Lemma [BV03]

• 2-local commuting $\Rightarrow$ Decoupling

$[h_1, h_2] = 0 \Rightarrow$ Decoupling

$\rho^g = \bigoplus_j \left( \frac{j}{3} \right)$

invariant under $h_1, h_2$

$
\begin{align*}
\text{GS of } h_1 + h_2 &\text{ can be prepared by constant depth circuit.} \\
\text{If no overlap.} \\
\text{GS: Tensor of 2-qudit state}
\end{align*}$
The structure Lemma [BV03]

• More general 2-local interaction

• 2-local CLH: Trivial ground state, prepared by constant depth circuit.
• NP witness: constant depth circuit for GS (constructive proof)
Limitation of structure lemma

- Can structure Lemma proves general CLHP ∈ NP?

- No it can’t...
  - Structure Lemma ⇒ Trivial GS
  - No trivial ground state for Toric code!
    - Toric code: 4-local CLH on 2D.
    - GS needs $\Omega(\log n)$ depth quantum circuits.
Outline: New ideas for qudit-CLHP-2D

- (?) qudit-CLHP-2D ∈ NP?
  - why structure lemma doesn’t work
  - Qubit-CLHP-2D [Sch11] (computing trace)
  - Qutrit-CLHP-2D, [IJ23] (non-constructive self-reduction)
  - Factorized-qudit-CLHP-2D [IJ23]
Qudit-CLHP-2D

- Assume \( p \) projections
- Qudit-CLHP-2D:
  \[
  H = \sum_{p} p
  \]
  Decide
  \[
  \lambda(H) = 0 \text{ or } \lambda(H) \geq 1
  \]

Goal: Prove \text{qubit-CLHP-2D} \in \text{NP}
Certify \( \lambda(H) = 0 \) classically.
Why structure Lemma doesn’t work?

2-local star-like interaction

4-local Incompatible decomposition
Non-constructive proof for $\text{qubit-CLHP-2D} \in \text{NP}$

- Transform the CLHP-2D to computing trace.

$\Pi_p :$ projection onto 0-eigenspace of $p$

$$\lambda(H) = 0 \iff \text{tr} \left( \prod_p \Pi_p \right) > 0.$$ 

$P_G = \text{product of gray } \Pi_p.$

$P_W = \ldots \ldots \text{ of white } \ldots$

$$\text{tr}(P_G P_W) > 0$$

- $H = \sum_p p$
- $p$ commuting projections
- Certify $\lambda(H) = 0$ classically.
Why proving trace is easier? \( tr(P_G P_W) > 0 \)

Take Toric code as an example

- Gray: \( \Pi_p = (I + X^{\otimes 4})/2 \)
- White: \( \Pi_p = (I + Z^{\otimes 4})/2 \)
- \( tr(P_G P_W) > 0 \)

Decompose w.r.t.
- \( X \) & \( Z \) basis

Gray: \( \Pi_p = (X+X^{\otimes 4})/2 \)
- \( tr(P_G P_W) > 0 \)

Easy: \( |<0| + 7|>^2n \)
Why **proving** trace is easier?
Take Toric code as an example

$$
\text{Proving } \sum \text{ non-negative terms} > 0 \text{ is equivalent to proving one term } > 0!
$$

**NP witness:** index of one non-negative term.
Qubit-CLHP-2D in NP

• General qubit-CLHP-2D: classify qubits

(a) Similar to Toric code
(b) two terms acts as Identity on q.

Decompose the trace w.r.t to all qubits in (a)
Each sub-trace will be of 1D structure because of the identity.

\[ tr(P_G P_W) > 0 \]

NP witness:
Qubit-CLHP-2D: **Equivalence to Toric code** permitting boundary
Constructive proof [AKV18]

- **Ground state** can be prepared similarly as **Toric code**
  by poly-size quantum circuit.
- “**Measure & Correct.**”

```
  l_2  l_4
   q
  l_1  l_3
```

- “**Interior qubit**” q: $p_1, p_2, p_3, p_4$ act non-trivially
  on q.

  \[
  p_1, p_4 = I_q \otimes () + X_q \otimes ();
  \]

  \[
  p_2, p_3 = I_q \otimes () + Z_q \otimes ();
  \]
4-local: qubit to **qudit**.

- **Qudit/Qutrit** is very different from qubit.
  - +1 dimension introduces **degeneracy**.
  - Sub-trace: 1D structure.
New for 4-local qudit [IJ23]

Theorem 1 \( \text{Qutrit-CLHP-2D is in NP.} \)  
\( \star \) Non-constructive proof.  
\( \circ \) “Rounding” for self-reduction

Theorem 2 \( \text{Factorized qudit-CLHP-2D is in NP. Furthermore,} \)  
Hamiltonian in factorized qudit-CLHP-2D is equivalent to  
\( \text{direct sum of qubit stabilizer Hamiltonian.} \)

\[ p = p^{q_1} \otimes p^{q_2} \otimes p^{q_3} \otimes p^{q_4} \]
4-local qudit [IJ23]

Theorem 1  \textit{Qutrit-CLHP-2D is in NP.}

- Non-constructive proof.
  - “rounding” for self-reduction

Theorem 2  \textit{Factorized qudit-CLHP-2D is in NP. Furthermore, Hamiltonian in factorized qudit-CLHP-2D is equivalent to direct sum of qubit stabilizer Hamiltonian.}

- Factorized Hamiltonian can look very different from stabilizer Hamiltonian.
  - Stabilizer H: Regular way of commuting
    \[ X^4 Z^4 = Z^4 X^4, \quad XZ = -ZX. \]
  - Factorized H: Singular way of commuting

\[
\begin{align*}
h_1 &:= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes A & \quad h_1 h_2 = 0 = h_2 h_1 \\
h_2 &:= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes B \\
\text{for any } A, B.
\end{align*}
\]
Key technique for qutrit-CLHP-2D in NP
Non-constructive self-reduction for CLHP: qutrit to qubit?

Goal: $H = \sum_{p_i} p_i \text{ (projection)} \text{ prove } \lambda(H) = 0$; Decrease dimension of qudit
Key technique for qutrit-CLHP-2D in NP

Non-constructive self-reduction for CLHP: qutrit to qubit?

Goal: \( H = \sum p_i p_i \) (projection) prove \( \lambda(H) = 0 \); Decrease dimension of qudit

For CLHP we can do more!

\[
H^q = \bigoplus_j H_j^q
\]

Semi-separable qudit

Smaller CLHP

\[
P_i = \prod_j H_j^q
\]

1-eigenspace of

\[
P_i \cap P_1 \cap P_2 \cap P_3
\]
Non-constructive Self-reduction:
  - We get a CLHP with smaller dimension in qudit q.
    - \( \lambda(H) = 0 \) iff exists \( i \), \( \lambda(\hat{H}_i) = 0 \)

This lemma works for any qudit, any geometry.
Open problems

“Some special case of CLHP is in NP”

- Qudit-CLHP-2D:
  - Complexity & Ground state preparation
    - In NP? Non-trivial class in QCMA?
    - Can GS be prepared by poly-size quantum circuit?

- (?) 2D Area Law ⇒ poly-size quantum circuit?
  - 2D Area law: for GS, entanglement entropy across any cut is proportional to the boundary.
    - 2D Area law (?) poly-size ground state (?)
    - Qudit-CLHP-2D: Area law (yes) & poly-size ground state (?)

Thanks for listening. Question?

“Some special case of CLHP is in NP”
Reference


• [AE13] Dorit Aharonov and Lior Eldar. The commuting local hamiltonian on locally-expanding graphs is in NP.


• [BCG+23] Ivan Bardet, Angela Capel, Li Gao, Angelo Lucia, David P´erez-Garc´ıa, and Camb yogse Rouz´e. Rapid thermalization of spin chain commuting hamiltonians.


• [BV03] Sergey Bravyi and Mikhail Vyalyi. Commutative version of the k-local hamiltonian problem and common eigenspace problem.
