What on earth does complexity theory have to do with gravity?

Quantum complexity in physics

Geometrically or k-local Hamiltonian H

$$|\psi(t)\rangle = e^{-iHt}|\psi\rangle$$

Hamiltonian simulation (e.g. trotterisation):

Complexity of $|\psi(t)\rangle$ is bounded as a (linear) function of t

Brown-Susskind conjecture:

For a sufficiently generic/chaotic H, complexity of $|\psi(t)\rangle$ does grow linearly for exponentially long times

Gravity:

Physics is "generally covariant" (independent of spacetime coordinates)

Determinism requires different coordinates describe same physical state

"Time evolution" is just a labelling change

Hamiltonian should equal zero for all physical states?

Wheeler-de Witt eqn

Cf Electromagnetism:

Gauss law: $\nabla \cdot E - \rho_q = 0$ ("charge is zero")

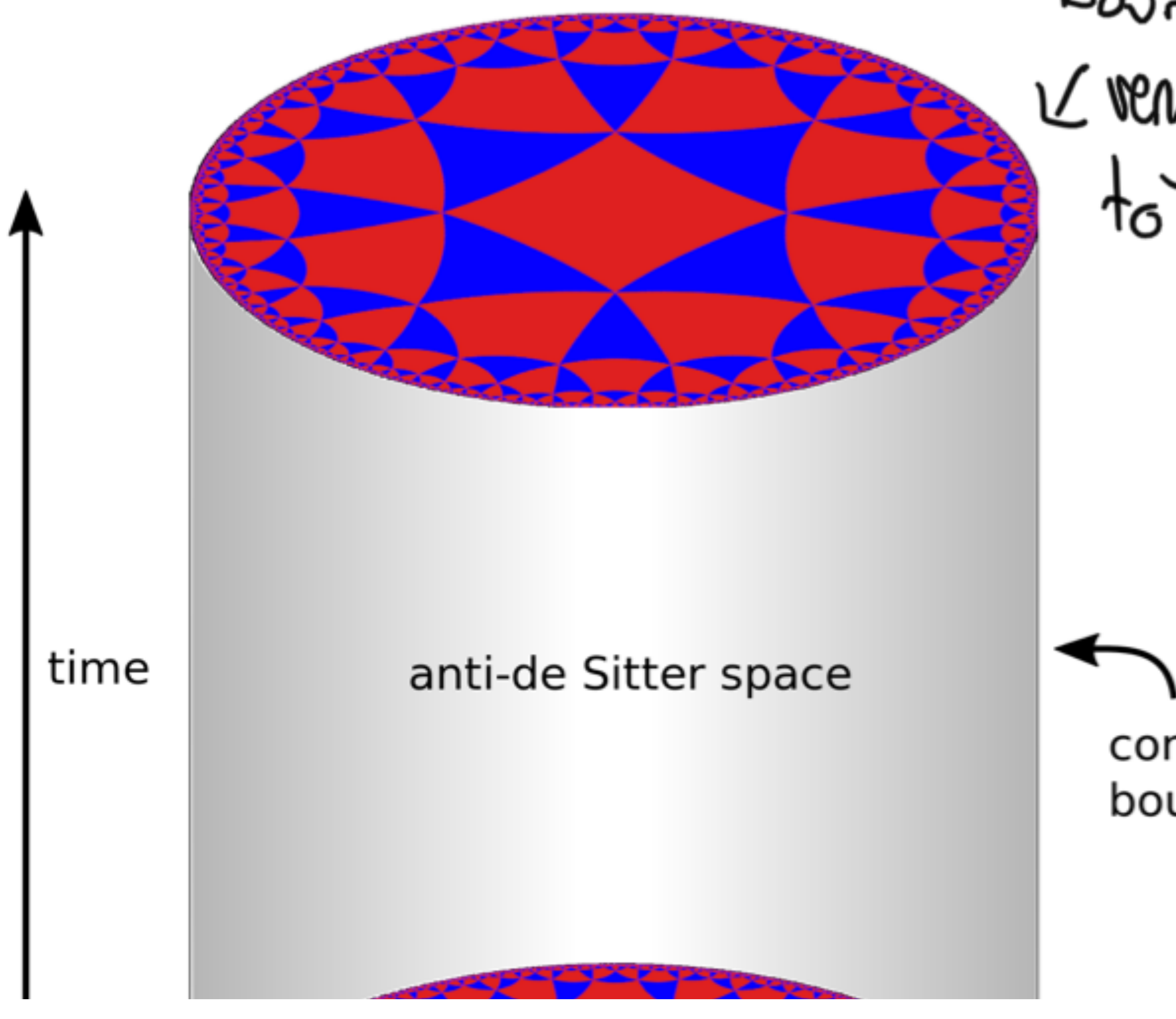
Compact manifolds must have zero net charge

Noncompact manifolds can have net charge but can be measured using electric flux at boundary/infinity

Gravity in AdS:

Nicest asymptotic boundary conditions are asymptotically anti-de Sitter space

region near boundary requires very large energy to excite

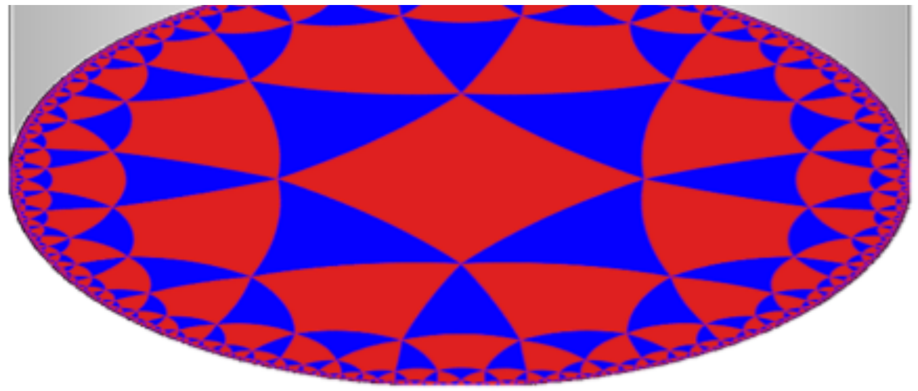


Hamiltonian is determined by metric at boundary

Boundary field $\phi(x,t)$

$$\phi(x,t) = e^{iHt}\phi(x,0)e^{-iHt}$$

Unitary evolution of boundary data



boundary coordinates (unlike bulk coordinates) are physical

Same degrees of freedom at boundary at all times ← Not true in classical gravity because Poisson evolution is nonanalytic

Holography: all degrees of freedom in nonperturbative quantum gravity are localised at boundary

Many explicit examples from string theory where gravity in AdS is described by an ordinary QFT on its boundary (AdS/CFT)

Semiclassical excitations deep in the bulk are described by complex boundary operators that effectively commute with simple boundary observables (quantum error correcting code)

$\boxed{\text{Lorentzian (QFT) causality is protected by (polynomial) complexity}}$

Black holes

"No hair theorem": Anything outside a black hole tends fall in. Black hole exterior equilibrates to a unique state with given mass, charge and angular momentum

Holographic dual of thermal equilibration

$$\text{Entropy} = \log(N_{\text{microstates}}) = A_{hor}/4G$$

The black hole interior however does not equilibrate

Instead,* it grows at the speed of light

*in a natural choice of time slicing

The interior describes history of how the thermal state was formed

Time ⟺ Space

$\boxed{\text{Size of interior} \sim \text{Complexity of state}}$ ⟵ Susskind++

Hawking radiation

Like any thermal system, black holes radiate

In QFT vacuum, nearby modes (incl modes near horizon) are entangled

Mode outside horizon can escape as thermal radiation, while mode inside remains trapped

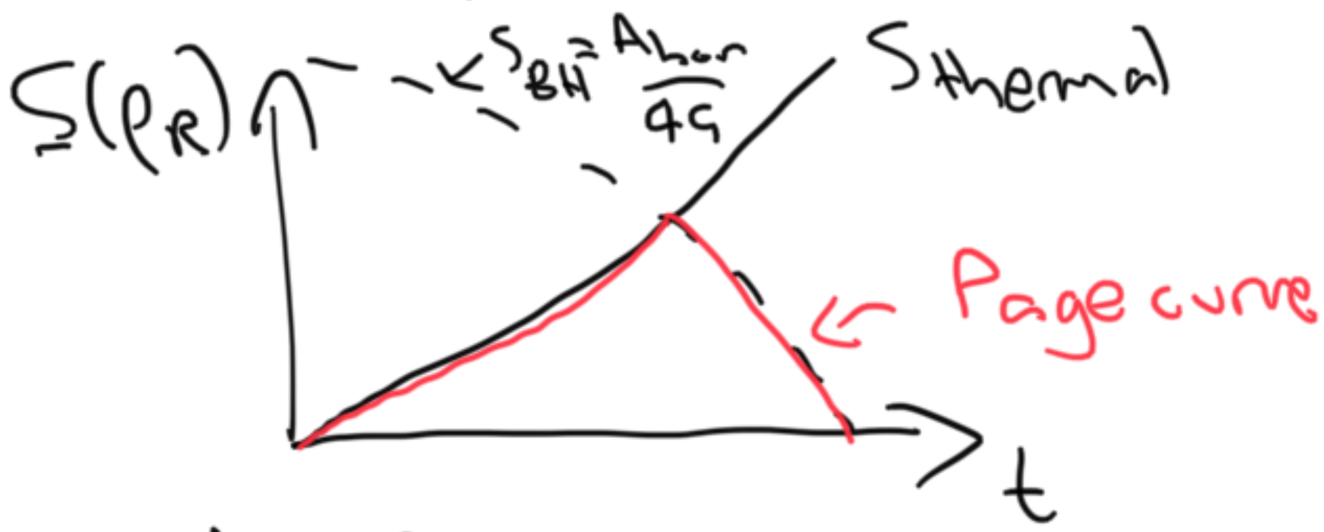
Over time ever more radiation is emitted. Entanglement entropy grows linearly

Information problem

Eventually radiation has more degrees of freedom than black hole

Unitary evolution $\Rightarrow$ radiation cannot be perfectly thermal. Needs to encode information that fell in

Simple proof:



Entanglement entropy cannot become larger than $\log(N_{micro}) = \frac{A_{hor}}{4G}$ for the black hole

Paradox?

$|\psi\rangle$ needs to be encoded in R for unitarity

But $|\psi\rangle$ is spacelike-separated from R

should commute?

Resolution:

operators that recover $|\psi\rangle$ from R (e.g Petz map) create topology changing "spacetime wormholes" that break semiclassical causality

Unlike Hamiltonian evolution this cannot be seen in (perturbative) semiclassical gravity

Why isn't semiclassical causality breaking all around us?

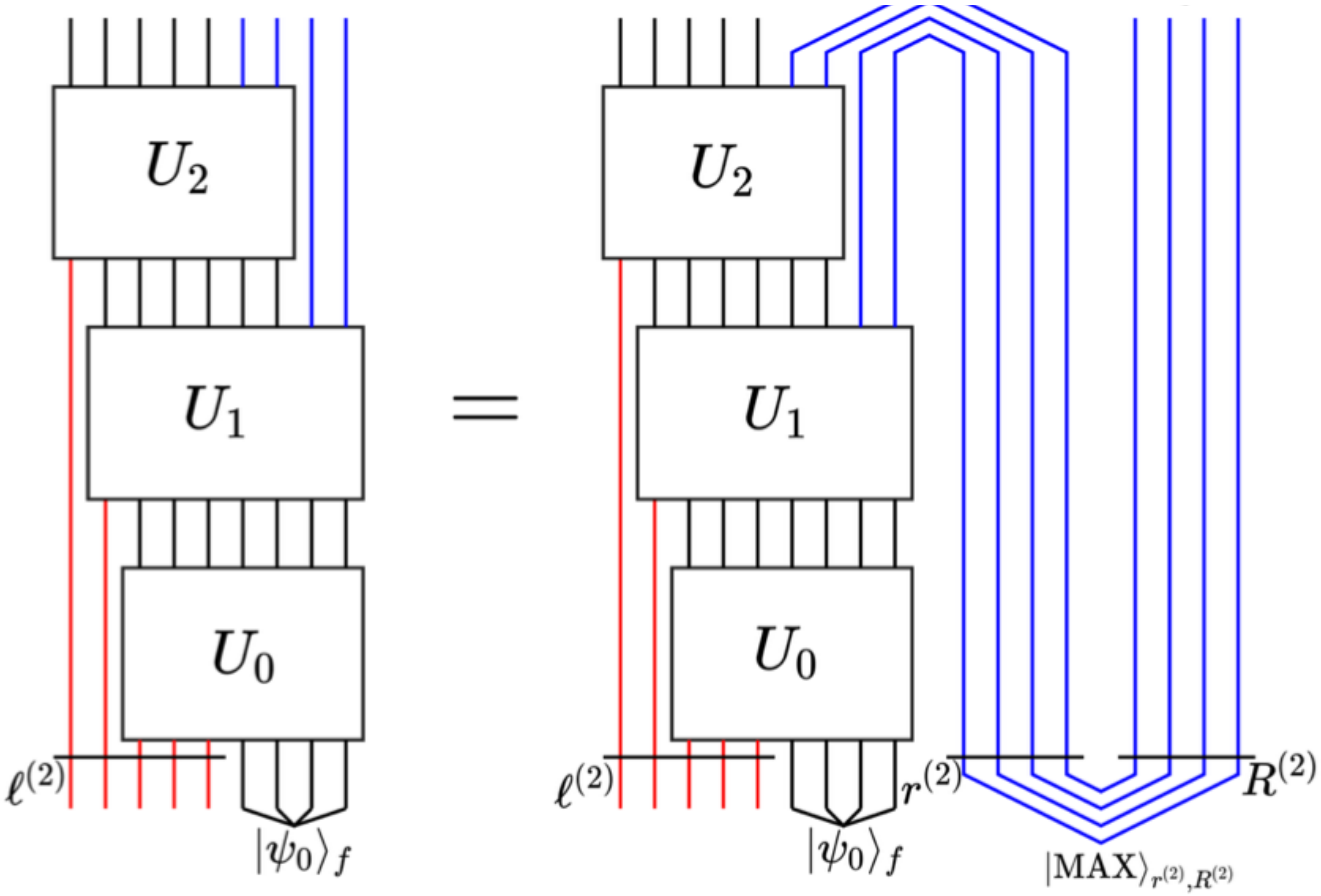
Perturbative gravitational causality is protected by exponential complexity

Harlow, Hayden

Two pictures of thermal evaporation:

$B^{(2)}$ $R^{(2)}$

$B^{(2)}$ $\sqrt{|P^{(2)}|}\langle\mathrm{MAX}|_{R^{(2)},r^{(2)}}$ $R^{(2)}$



Unitary evolution with some qubits falling in and some being radiated

Bending some legs around makes things look more like a semiclassical BH interior postselected into an $e^{S_{BH}}$-dimensional Hilbert space

At late times ℓ (stuff that fell in) can be recovered from R (radiation) once $|R| \gg |B|$

But fastest known algorithm for doing so is

(minimum)

Grover search ... number of postselected qubits

Complexity $\sim 2^{n_{post}/2}$

Python's lunch conjecture: (Brown, Gharibyan, GP, Susskind)

Complexity of recovering information is exponential in "gravitational postselection" (difference of areas/generalised entropies between two extremal surfaces)

Also "explains" results such as Bouland, Fefferman, Vazirani

Summary:

Three important roles for complexity in gravity

1. Polynomial complexity required for a boundary observer to reverse time ("protects semiclassical causality")
2. Black hole records its history on a spatial slice ("size ~ complexity")
3. Exponential complexity required to recover info

from radiation after Page time ("protects perturbative gravity")

Complexity theory questions with interesting implications for gravity:

- Does complexity really grow linearly with time?
- Are there generically more efficient reconstruction algorithms than Grover search? Sub-exponential?
- Can we find provable lower bounds on reconstruction complexity?