An Exponential Lower Bound for Linear 3-Query Locally Correctable Codes

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Based on joint work with:
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Locally **Correctable** Codes

LCC:

0	1	0	1	1	1

$b \in \{0,1\}^k$ $\rightarrow$

0	1	0	1	1	1	0	1	0	1	1	0	0	0	1	1

$x \in \{0,1\}^n$

$\downarrow$ δn errors

0	1	1	0	1	1	0	0	0	1	1	1	0	0	1	0

$y \in \{0,1\}^n$

q

$u \in [n]$ $\rightarrow$ Decoder $\leftarrow$ \$

$\hat{x}_u \in \{0,1\}$ $\leftarrow$ Decoder

(q, δ, ε)-**LCC**

Input: $y \in \{0,1\}^n$, $\Delta(x, y) \leq \delta n$
$u \in [n]$

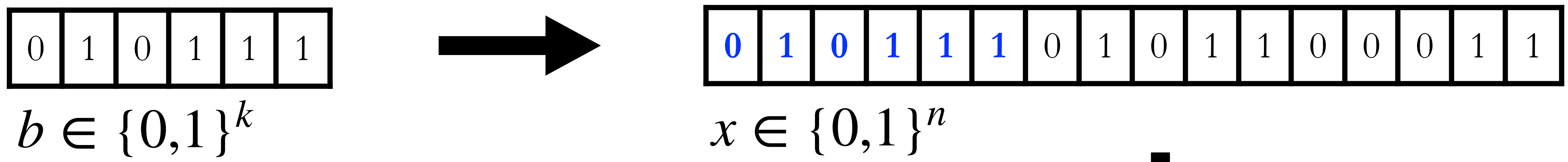
Read $\leq q$ bits of y

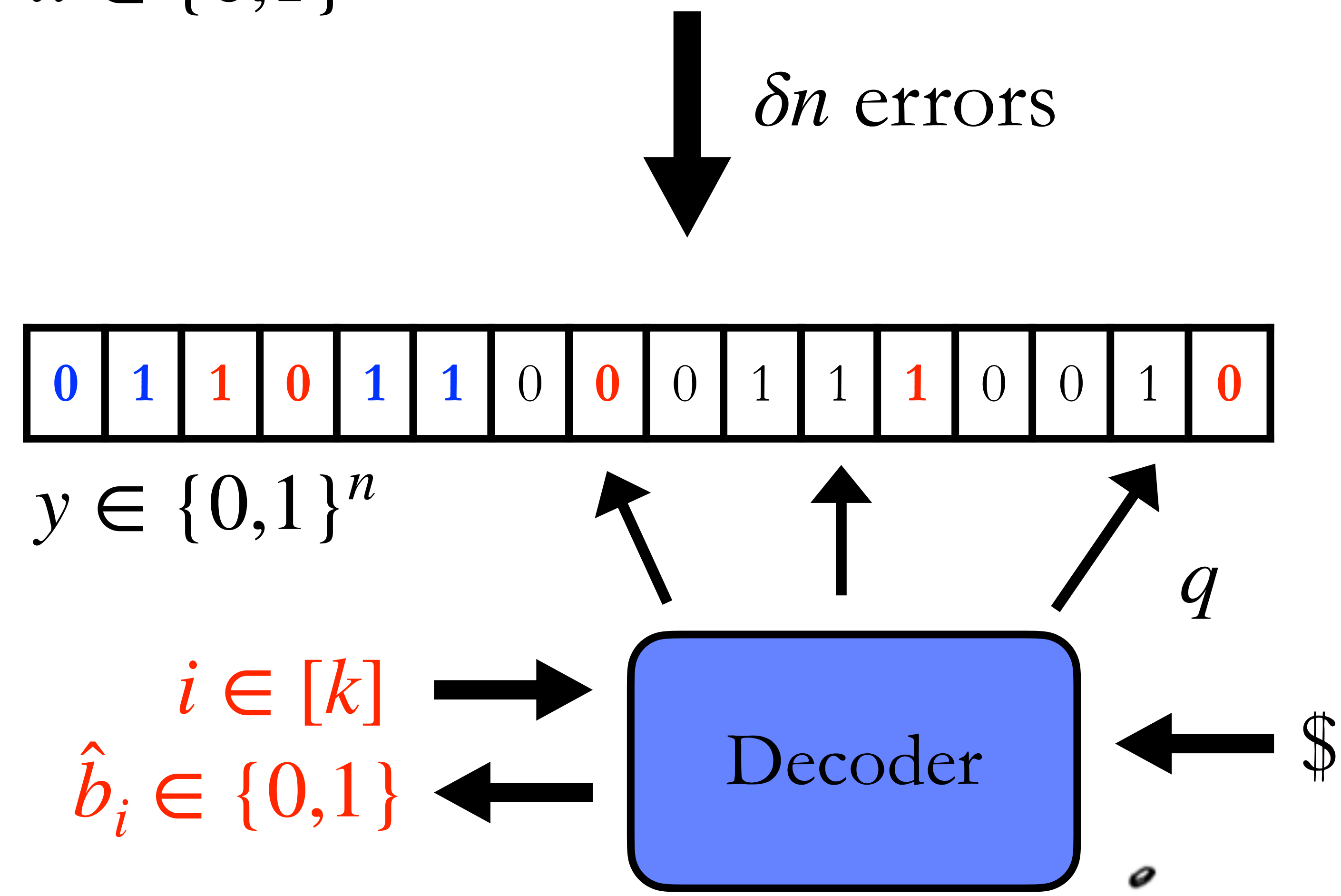
Output x_u w.p. $\geq 1 - \varepsilon$

Used in: program checking, PCPs, PIR, avg-case to worst-case, explicit rigid matrices, additive combinatorics, block designs

Locally **Decodable** Codes

LDC:





(q, δ, ε)-**LDC**

Input: $y \in \{0,1\}^n$, $\Delta(x, y) \leq \delta n$
$i \in [k]$

Read $\leq q$ bits of y

Output b_i w.p. $\geq 1 - \varepsilon$

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Locally **Correctable** Codes

LCC:

0	1	0	1	1	1

$b \in \{0,1\}^k$ →

0	1	0	1	1	1	0	1	0	1	1	0	0	0	1	1

$x \in \{0,1\}^n$

δn errors

0	1	1	0	1	1	0	0	0	1	1	1	0	0	1	0

$y \in \{0,1\}^n$

q

$u \in [n]$ → Decoder ← \$

$\hat{x}_u \in \{0,1\}$

(q, δ, ε)-LCC

Input: $y \in \{0,1\}^n$, $\Delta(x, y) \leq \delta n$
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Read $\leq q$ bits of y

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Used in: program checking, PCPs, PIR, avg-case to worst-case, explicit rigid matrices, additive combinatorics, block designs

Do LCCs/LDCs exist?

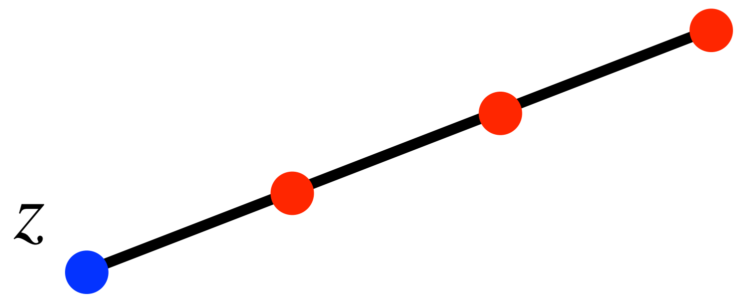
Purely combinatorial question
No natural random model for probabilistic method

Reed-Muller Codes

Simple q-LCC: Reed-Muller codes, $n = \exp(k^{1/(q-1)})$, and thus q-LDC

For 3-LCC: take deg 2 polys $f(z_1, \ldots, z_t)$ on $\mathbb{F}_4$

To decode $f(z)$, pick $z' \in \mathbb{F}_4^t$, look at $L_{z,z'} = \{z + \lambda z' : \lambda \in \mathbb{F}_4\}$



Binary code via $\mathrm{Tr} : \mathbb{F}_4 \to \mathbb{F}_2$

Parameters: $k \approx t^2/2$, $n = 4^t = 2^{O(\sqrt{k})} = 2^{2\sqrt{2k}}$

Can we do better?

The Story in 2022

Simple q-LCC: Reed-Muller codes, $n = \exp(k^{1/(q-1)})$, and thus q-LDC

Can we do better?

$q = 2$: achieves $n \leq 2^k$, linear 2-LCC

tight $n \geq 2^{\Omega(k)}$ 2-LDC lower bound

[Katz Trevisan 00]
[Goldreich, Karloff, Schulman, Trevisan 02]
[Kerenidis, de Wolf 04]

$q = 3$: achieves $n \leq 2^{2\sqrt{2k}}$, linear 3-LCC

Matching vector code $n \leq 2^{2^{O(\sqrt{\log k \log\log k})}}$, linear 3-LDC of subexp length!

Is it a 3-LCC? Open Q **[Yekhanin 12]**

[Yekhanin 07]
[Efremenko 08]

Weak $n \geq k^2$ 3-LDC lower bound **[Kerenidis, de Wolf 04]**

"Techniques cannot beat k^2" **[Dvir, Gopi, Gu, Wigderson 19]**
[Alrabiah Guruswami 21]

The Story in 2022

Simple q-LCC: Reed-Muller codes, $n = \exp(k^{1/(q-1)})$, and thus q-LDC

Can we do better?

Yes for LDCs. What about LCCs? **We don't know!**

Conj: RM codes are optimal LCCs

True for $q = 2$. Hard to prove $q \geq 3$ **[Barkol, Ishai, Weinreb 10]**

[Hamada 74] Conj: RM codes are optimal design LCCs

Barrier: all lower bounds apply to Locally Decodable Codes

Recall: Matching Vector codes, 3-LDC, $n = \exp(k^{o(1)})$ **[Yekhanin 07]** **[Efremenko 08]**

The Story in 2024

(1) **[Alrabiah, Guruswami, Kothari, M 23]**: 3-LDC has $n \geq \tilde{\Omega}(k^3)$

(2) **[Kothari, M 23]**: **linear** 3-LCC has $n \geq 2^{\Omega(k^{1/8})} \rightarrow 2^{\Omega(k^{1/4})} \rightarrow 2^{\tilde{\Omega}(\sqrt{k})}$

Techniques do not work for LDCs!

MV codes are **not** LCCs

[Yankovitz 24]

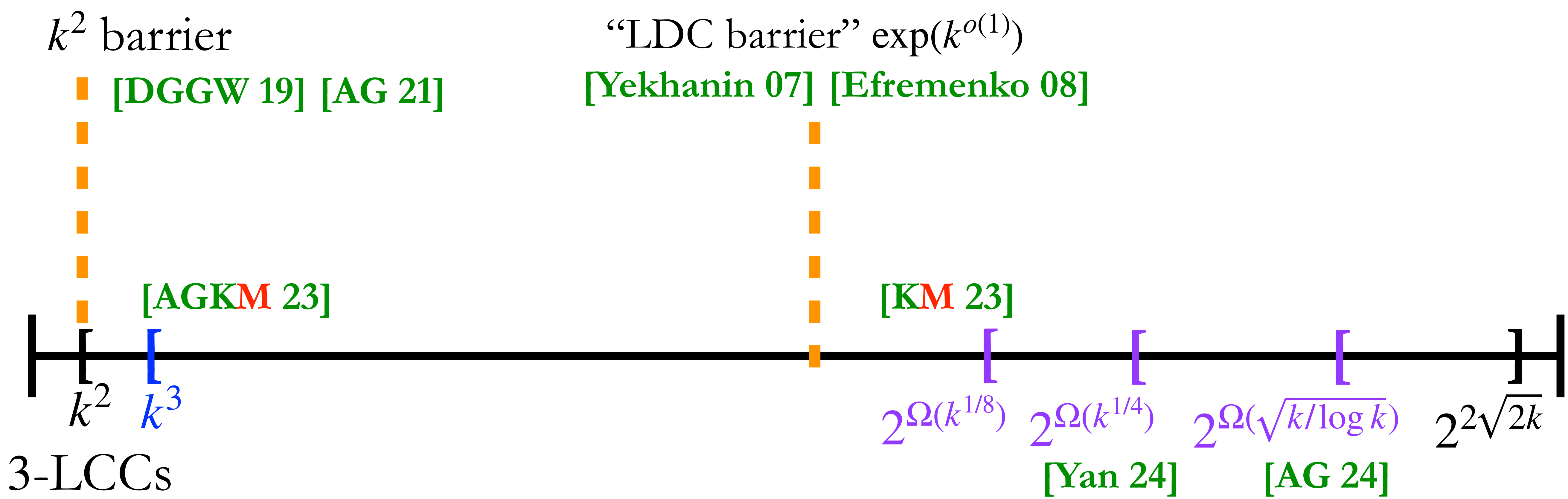
[Alrabiah Guruswami 24]

Via connection to "rainbow cycles"

[Hsieh Kothari Mohanty Munhá Sudakov 24]

"Rainbow cycle bound"

[Alon Bucić Sauermann Zakharov Zamir 23]



The Story in 2024

(1) **[Alrabiah, Guruswami, Kothari, M 23]**: 3-LDC has $n \geq \tilde{\Omega}(k^3)$

(2) **[Kothari, M 23]**: **linear** 3-LCC has $n \geq 2^{\Omega(k^{1/8})}$ $\rightarrow$ $2^{\Omega(k^{1/4})}$ $\rightarrow$ $2^{\tilde{\Omega}(\sqrt{k})}$

[Yankovitz 24]

Techniques do not work for LDCs!

MV codes are **not** LCCs

[Alrabiah Guruswami 24]

Via connection to "rainbow cycles"

[HKMMS 24]

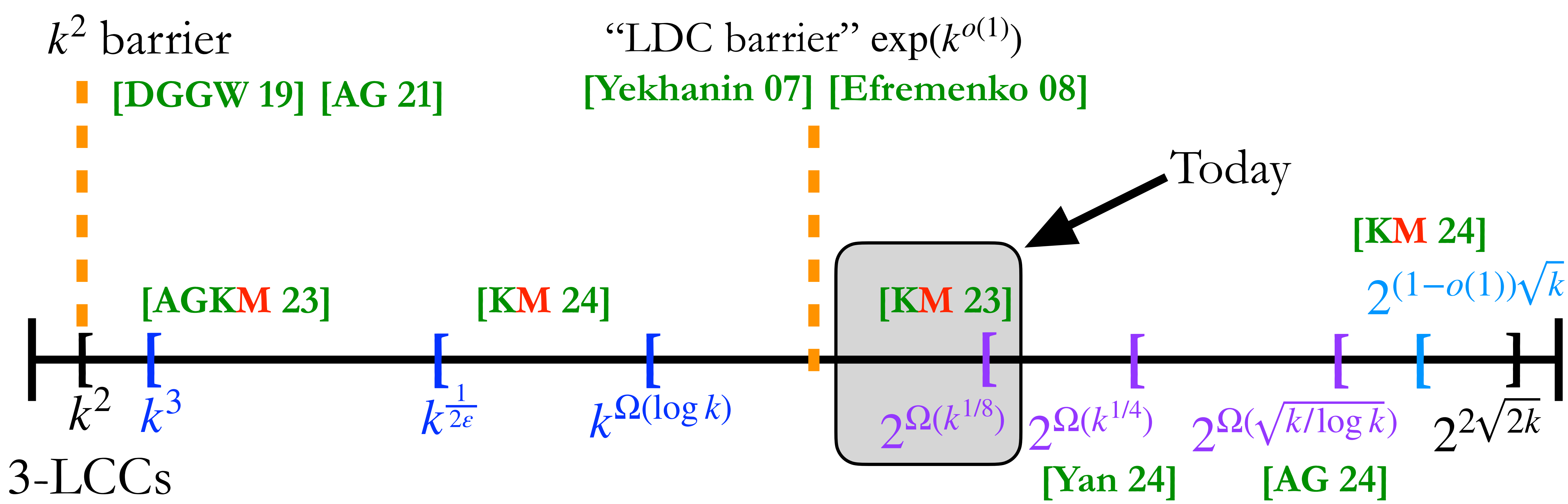
(3) **[Kothari, M 24]**: **design** 3-LCC has $n \geq 2^{(1-o(1))\sqrt{k}}$

Proves Hamada's conj for 4-designs up to $2\sqrt{2}$-factor

"Rainbow cycle bound"

[ABSZZ 23]

(4) **[Kothari, M 24]**: **nonlin** $(3,\delta,\varepsilon)$-LCC has $n \geq \tilde{\Omega}(k^{\frac{1}{2\varepsilon}})$ $\rightarrow$ $k^{\Omega(\log k)}$



Proof Strategy

Theorem [KM 23]:
Let C be a linear $(3,\delta,\varepsilon)$-locally correctable code. Then, $n \geq \exp((\delta^2 k)^{1/8})$

Approach of **[AGKM 23]**:

(1) Reduce to proving unsatisfiability of XOR formulas with minimal randomness

(2) Use algorithm for CSPs:

[Abascal, Guruswami, Kothari 21]
[Guruswami, Kothari, M 22]

Semirandom CSP refutation
(CSPs with **minimal randomness**)

"Theory analogue" of using SAT solvers to prove theorems

Today: present ideas as reduction to 2-LDC

Linear 3-LCCs in "Combinatorial Form"

To decode x_u, the decoder (1) queries some entries $C \subseteq [n]$ of y

(2) computes a "predicate" on $y|_C$

For linear codes…

(1) "Predicate" is linear: output $\sum_{v \in C} y_v$

(2) Query sets H_u is $\Omega(n)$-sized 3-unif **matching** (C's are disjoint, $|C| = 3$)

$C_1, C_2, C_3, C_4 \in H_u$

0	1	0	1	1	1	0	0	0	1	1	0	0	0	0	1

"4-Sparse Parity check matrix":

$$\text{for each } u \in [n],\ C = \{v_1, v_2, v_3\} \in H_u$$

$$x_{v_1} + x_{v_2} + x_{v_3} = x_u \text{ for all codewords } x$$

From 3-LCCs to 2-LDC

Given: "parity check matrix" $H_1, \ldots, H_n$, 3-unif matchings of size $\Omega(n)$

Goal: construct 2-LDC of length $n^{O(\ell)}$

Then by 2-LDC lower bound, $k \leq O(\ell \log n)$

2-LDC encoding (high level):

(1) map $x \in \{0,1\}^n$ to $y \in \{0,1\}^N$, $N = \binom{n}{\ell}$

(2) set $y_S = \sum_{v \in S} x_v$

2-LDC if there exist **matchings** $M_1, \ldots, M_n$ on N of size $\Omega(N)$ s.t.

$(S,T) \in E(M_u)$ implies $y_S + y_T = \sum_{v \in S} x_v + \sum_{v \in T} x_v = x_u$ for all codewords x

From 3-LCCs to 2-LDC

Given: "parity check matrix" $H_1, \ldots, H_n$, 3-unif matchings of size $\Omega(n)$
Goal: construct 2-LDC of length $n^{O(\ell)}$
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2-LDC encoding (high level): $y_S = \sum_{v \in S} x_v$, $|S| = \ell$, $N = \binom{n}{\ell}$

2-LDC if there exist **matchings** $M_1, \ldots, M_n$ on N of size $\Omega(N)$ s.t.

$$(S, T) \in E(M_u) \text{ implies } y_S + y_T = \sum_{v \in S} x_v + \sum_{v \in T} x_v = x_u \text{ for all codewords } x$$

Define G_u : edge (S, T) if $y_S + y_T = x_u$ for all codewords x

(1) Clearly need G_u to have **avg** deg $d_u \gg 1$

If G_u has $\boxed{\textbf{max} \text{ deg } O(d_u)}$, then "greedy" matching $\geq d_u N / O(d_u) = \Omega(N)$

This is **false**!

(2) "Row pruning": find dense G'_u with **max** deg $O(d_u)$

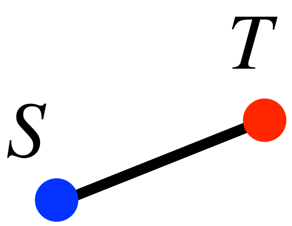
"Degree Heuristic"

Given: "parity check matrix" $H_1, \ldots, H_n$, 3-unif matchings of size $\Omega(n)$

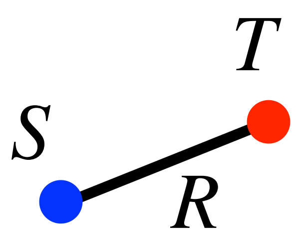
Goal: construct 2-LDC of length $n^{O(\ell)}$

Then by 2-LDC lower bound, $k \le O(\ell \log n)$

(1) Clearly need G_u to have **avg** deg $d_u \gg 1$



$$\Longrightarrow \sum_{v \in S \oplus T} x_v = \sum_{v \in S} x_v + \sum_{v \in T} x_v = x_u$$

"Labeled by R": 

$$\Longrightarrow \sum_{v \in R} x_v = x_u \, , \, S \oplus T = R$$

Each R contributes $\approx N \cdot (\ell/n)^{|R|/2}$ edges, or $(\ell/n)^{|R|/2}$ to the **density**

Recall: start with H_u s.t. $\sum_{v \in C} x_v = x_u$ for all $C \in H_u$, $|C| = 3$, $|H_u| \ge \Omega(n)$

$|E(G_u)|/N \approx (\ell/n)^{3/2} \cdot \Omega(n)$ ⟶ Need $d_u \approx (\ell/n)^{3/2} n \gg 1$ ⟶ $\ell = n^{1/3}$

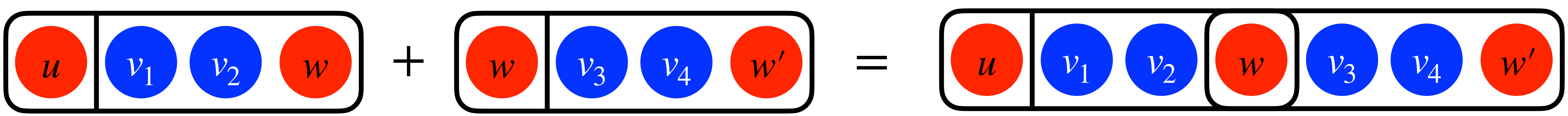
⟶ Get $k \le O(\ell \log n) = \tilde{O}(n^{1/3})$ **[AGKM 23]**

Boosting Density with 2-Chains

Given: "parity check matrix" $H_1, \ldots, H_n$, 3-unif matchings of size $\Omega(n)$
Goal: construct 2-LDC of length $n^{O(\ell)}$
Then by 2-LDC lower bound, $k \leq O(\ell \log n)$

(1) Clearly need G_u to have **avg** deg $d_u \gg 1$

Suppose we form a "chain":



$x_{v_1} + x_{v_2} + x_w = x_u$ $\qquad$ $x_{v_3} + x_{v_4} + x_{w'} = x_w$ $\qquad$ $x_{v_1} + x_{v_2} + x_{v_3} + x_{v_4} + x_{w'} = x_u$

$C = \{v_1, v_2, w\} \in H_u$ $\qquad$ $C \in H_w$ $\qquad$ $H_u^{(2)}$

$|H_u^{(2)}| = \Omega(n^2)$, arity 5

$H_u^{(2)}$ produces $N \cdot (\ell/n)^{2.5}\Omega(n^2)$ edges in G_u $\qquad$ **avg** deg is $(\ell/n)^{2.5}n^2 \sim \ell^{2.5}/n^{0.5}$

So, $\ell = n^{1/5} \implies k^5 \leq n$ $\qquad$ Small win! Beat k^3 !

Boosting Density with Long Chains

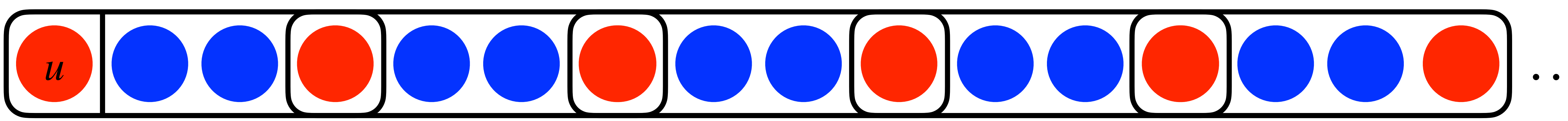
Given: "parity check matrix" $H_1, \ldots, H_n$, 3-unif matchings of size $\Omega(n)$

Goal: construct 2-LDC of length $n^{O(\ell)}$

Then by 2-LDC lower bound, $k \le O(\ell \log n)$

(1) Clearly need G_u to have **avg** deg $d_u \gg 1$

Suppose we form a "long chain" r steps:



$|H_u^{(r)}| = \Omega(n)^r$, arity $2r+1$ **always odd!**

$H_u^{(r)}$ produces $N \cdot (\ell/n)^{r+0.5}\Omega(n)^r$ edges in G_u

avg deg is $(\ell/n)^{r+0.5} \cdot n^r \sim \ell^r/n^{0.5}$

So, $\ell = n^{1/2r} \implies k \le n^{1/2r}$

Take $r = O(\log n) \implies k \le \log^8 n$

Big win: $n \ge \exp(k^{1/8})$!

Recall: 3-LCC to 2-LDC Reduction

Given: "parity check matrix" $H_1, \ldots, H_n$, 3-unif matchings of size $\Omega(n)$
Goal: construct 2-LDC of length $n^{O(\ell)}$
Then by 2-LDC lower bound, $k \leq O(\ell \log n)$

2-LDC encoding (high level): $y_S = \sum_{v \in S} x_v$, $|S| = \ell$, $N = \binom{n}{\ell}$

2-LDC if there exist **matchings** $G_1, \ldots, G_n$ on N of size $\Omega(N)$ s.t.

$(S, T) \in E(G_u)$ implies $\sum_{v \in S} x_v + \sum_{v \in T} x_v = x_u$ for all codewords x

Form chains!

Main technical part!

(1) Clearly need G_u to have **avg** deg $d_u \gg 1$

If G_u has $\boxed{\textbf{max} \text{ deg } O(d_u)}$, then "greedy" matching $\geq d_u N / O(d_u) = \Omega(N)$

This is **false**!

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"Row Pruning"

Given: "parity check matrix" $H_1, \dots, H_n$, 3-unif matchings of size $\Omega(n)$

Goal: construct 2-LDC of length $n^{O(\ell)}$

(2) "Row pruning": find dense G'_u with **max** deg $O(d_u)$

[KM 23]: random S (vertex) has degree $O(d_u)$ w.h.p.

Proof uses Kim-Vu style conc inequality, "high moments", needs $\ell \geq \log^5 n$

$+ \log^2 n$ loss from "heavy pairs" $= \log^8 n$

[Yan 24] (rephrased): second moment $\mathbb{E}_S[\deg(S)^2]$ suffices!

Requires $\ell \geq \log^2 n + \log n$ loss from "heavy pairs" $= \log^4 n$

[KM 24]: if H is a block design, can take $\ell = \log n$

Necessarily requires some changes to G_u in **[KM 23, Yan 24]**

3-LCC to 2-LDC Reduction

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Open Problems

What happens for larger q ?

Degree heuristic: $\ell = n^{1-2/(q-1)} \ll n^{1-2/q}$

"Plug in $q - 1$ in LDC lower bound"

Non-linear codes?

Form "adaptive chains", loses success prob: $1 - \varepsilon$ to $1 - r\varepsilon$

Cannot use "standard reductions" from **[Katz Trevisan 00]**

Need full power of spectral refutation

Conj: RM codes are optimal linear 3-LCCs

[KM 24]: design case up to $2\sqrt{2}$

[AG 24]: almost $(2^{\Omega(\sqrt{k/\log k})}$ vs $2^{O(\sqrt{k})})$

(1) Exp lower bounds for non-linear 3-LCCs?

(2) Beat k^3 for (linear) 3-LDCs or 4-LCCs?

Thanks!

(arXiv 2311.00558 and 2404.06513)