Quantized-Constraint Concatenation and The Covering Radius of Constrained Systems

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Joint work with:
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Combining error correction and constraints is needed

Motivation

- **Constrained codes** are often employed in communication and storage systems in order to mitigate the occurrence of data-dependent errors.
Combining error correction and constraints is needed

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Combining error correction and constraints is needed

Motivation

- **Constrained codes** are often employed in communication and storage systems in order to mitigate the occurrence of data-dependent errors.
- In many channels some patterns are more prone to error than others, and we avoid them by using constrained codes.
- This reduces the number of errors, however the transmitted data may still be corrupted by data-independent errors, requiring additional error-correcting codes.
This is relevant for DNA storage

<table>
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<tbody>
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<td>• Homopolymer runs</td>
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<td>• GC content</td>
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  - Only a handful of ad-hoc examples are known.
  - A non-constructive (hard to compute) lower bound on the rate (Marcus and Roth, T-IT '92).

- Separate the error-correcting code and the constrained code, and combine them using a concatenation scheme (e.g., concatenation, or reverse concatenation).

- Many issues need to be resolved (see book draft by Marcus, Roth, and Siegel).

- In the known schemes, the error-correction capabilities are quite limited: the state-of-the-art method (Gabrys, Siegel and Yaakobi ISIT '18) allows for a correction of $O(\sqrt{n})$ errors (where $n$ is the block length).
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Quantized-Constraint Concatenation (QCC)
QCC is different

Conventionally: (concatenation, reverse concatenation)

- A constrained word represents the data to be transmitted and protected against errors.
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- A constrained word represents the data to be transmitted and protected against errors.
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**But in QCC:**

- We consider the embedding process of information in the constrained media as an irreversible *quantization* process rather than a coding procedure.
- The constrained word is considered as a corrupted version of the input message, obtained by a quantization procedure.
Let’s get mathematical

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- We fix a constrained system $X$ over an alphabet $\Sigma$, a block length $n$, and the set of constrained words $\mathcal{B}_n(X)$ of length $n$.

- Assume that $r \in \mathbb{N}$ is a number such that for any word $\bar{y} \in \Sigma^n$ there exists a constrained word $\bar{x} \in \mathcal{B}_n(X)$ such that $d(\bar{x}, \bar{y}) \leq r$. 

- Let $C \subseteq \Sigma^n$ be an error-correcting code, capable of correcting $t > r$ errors. Assume that we have an ECC encoder and an ECC decoder.

\[
\begin{array}{cccc}
\text{Encoder} & c \in C & \text{Quantizer} & x \in \mathcal{B}_n(X) \\
\text{Channel} & x' \text{ECC} & \text{Decoder} & u' \text{ECC}
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\overline{u} \xrightarrow{\text{ECC Encoder}} \overline{c} \in C \xrightarrow{\text{Quantizer}} \overline{x} \in \mathcal{B}_n \xrightarrow{\text{Channel}} \overline{x}' \xrightarrow{\text{ECC Decoder}} \overline{u}'
```
Quantized-Constrain Concatenation – The big picture

The Procedure

• Encoding: Given an information word $u$, use an encoder for an error-correcting code to map it to a codeword $c \in C$.

• Quantization: Given $c \in C$, find a constrained word $x \in B_n (X)$ such that $d(c, x) \leq r$, and transmit $x$.

• Channel: At the channel output, $x' \in \Sigma_n$, a corrupted version of $x$, is observed.

• Decoding: Use the decoder for $C$ on $x'$ and obtain $u'$. 

Diagram:

- $\bar{u}$
- ECC Encoder
- $\bar{c} \in C$
- Quantizer
- $\bar{x} \in B_n$
- Channel
- $\bar{x}'$
- ECC Decoder
- $\bar{u}'$
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Performance analysis

\[ \overline{u} \rightarrow \text{ECC Encoder} \rightarrow \overline{c} \in C \rightarrow \text{Quantizer} \rightarrow \overline{x} \in B_n \rightarrow \text{Channel} \rightarrow \overline{x}' \rightarrow \text{ECC Decoder} \rightarrow \overline{u}' \]

Error-Correcting Capabilities

- If the channel does not introduce more than \( t - r \) errors, i.e., \( d(x, x') \leq t - r \), then \( d(c, x') \leq t \).

- Since \( C \) can correct \( t \) errors, we have \( u = u' \), namely, it is possible to correct \( t - r \) channel errors.

Conclusion

The error correction capability of QCC is lower bounded by the minimal number \( r \) to satisfy that for any \( y \in \Sigma^n \) there is \( x \in B_n(x) \) with \( d(x, y) \leq r \). This is exactly the covering radius of \( B_n(x) \):

Correcting \( \Theta(n) \) Errors

Assume that \( R(B_n(x)) / n \) converges to some number \( \rho \) and assume that \( (C_n)_{n \in \mathbb{N}} \) is a sequence of codes capable of correcting \( \lceil \delta n \rceil \) errors, where \( \delta > \rho \). In that case, we can correct \( (\delta - \rho) n = \Theta(n) \) errors!
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Assume that $R(\mathcal{B}_n(X))/n$ converges to some number $\rho$ and assume that $(C_n)_{n \in \mathbb{N}}$ is a sequence of codes capable of correcting $\lceil \delta n \rceil$ errors, where $\delta > \rho$. 

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- The asymptotic rate of our scheme is determined by the rates of the codes $(C_n)_{n \in \mathbb{N}}$.
Performance analysis

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Conclusion

If $\rho < \frac{1}{2}(1 - \frac{1}{q})$ it is possible to correct $\Theta(n)$ errors with a non-vanishing rate.
The Covering Radius of a Constrained System
### Definition

Let $X, Y$ be constrained systems over $\Sigma$. 

---

**Remark**

Typically, $Y = \Sigma_Z$, hence, $B_n(Y) = \Sigma_n$ for all $n$ and $R(B_n(X), B_n(Y))$ is the usual covering radius of $B_n(X)$. 

The covering radius of a constrained system

**Definition**

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- For a fixed $n$, the covering radius of $\mathcal{B}_n(X)$ relatively to $\mathcal{B}_n(Y)$ is defined as

  \[
  R(\mathcal{B}_n(X), \mathcal{B}_n(Y)) \triangleq \min \left\{ r \in \mathbb{N} \mid \mathcal{B}_n(Y) \subseteq \bigcup_{\bar{x} \in \mathcal{B}_n(X)} \text{Ball}_r(\bar{x}) \right\}.
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- The (combinatorial) covering radius of $X$ relatively to $Y$ is

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Example

Let $X_{0,k}$ be the system of all binary words that do not contain $k + 1$ consecutive zeros, and let $Y = \{0, 1\}^\mathbb{Z}$ be the system of all binary words.
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The case of \((0, k) - RLL\)

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- On the other hand if take $\bar{y} = \bar{0}$, then $d(\bar{x}, \bar{0}) \geq \left\lceil \frac{n}{k+1} \right\rceil$ for all $\bar{x} \in B_n(X_{0,k})$. 

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# The case of \((0, k) – RLL\)

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## Solution

Let us evaluate \(R(\mathcal{B}_n(X_{0,k}), \{0, 1\}^n)\):

- For any \((y_1, \ldots, y_n) = \overline{y} \in \{0, 1\}^n\), consider \(\overline{x}\) obtained by setting the values in the coordinates \(k + 1, 2(k + 1), \ldots\) to 1. Clearly \(\overline{x}\) does not contain a run of \(k + 1\) zeros, and \(d(\overline{x}, \overline{y}) \leq \left\lfloor \frac{n}{k+1} \right\rfloor\). This proves that \(R(\mathcal{B}_n(X_{0,k}), \{0, 1\}^n) \leq \left\lfloor \frac{n}{k+1} \right\rfloor\).

- On the other hand if take \(\overline{y} = \overline{0}\), then \(d(\overline{x}, \overline{0}) \geq \left\lfloor \frac{n}{k+1} \right\rfloor\) for all \(\overline{x} \in \mathcal{B}_n(X_{0,k})\). This proves that \(R(\mathcal{B}_n(X_{0,k}), \{0, 1\}^n) \geq \left\lfloor \frac{n}{k+1} \right\rfloor\).

Taking limits: \(R(X_{0,k}, Y) = \liminf_{n \to \infty} \frac{R(\mathcal{B}_n(X_{0,k}), \{0, 1\}^n)}{n} = \frac{1}{k + 1}\).
An intriguing phenomenon

Example

- For $k = 1$ we have $R(X_{0,1}, \{0, 1\}^\mathbb{Z}) = \frac{1}{2}$. 
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- We have two systems, one which has strictly positive capacity ($\text{Cap}(X_{0,1}) \approx 0.694$) and the other with zero capacity ($\text{Cap}(X_{\text{rep}}) = 0$), with the same covering radius!
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We need an alternative definition for the covering radius which ignores such rare patterns.
The Essential Covering Radius
A Trade-off Between Quantization-Error and Rate

**Question:** What happens to the covering radius if we allow to drop an \( \varepsilon \in (0, 1) \) fraction of the words in \( B_n(Y) \) to be covered?
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**Definition**

Let $X$ and $Y$ be constrained systems, $\mu$ be an invariant ergodic measure on $Y$. For $\varepsilon \in (0, 1)$ we define $R_\varepsilon(\mathcal{B}_n(X), \mathcal{B}_n(Y), \mu_n)$ by:

$$
\min \left\{ r \in \mathbb{N} \left| \mu_n \left( \mathcal{B}_n(Y) \cap \left( \bigcup_{\bar{x} \in \mathcal{B}_n(X)} \text{Ball}_r(\bar{x}) \right) \right) \geq 1 - \varepsilon \right\}.
$$
The essential covering radius

**Remark**

In the typical case, where $Y$ is the trivial (non) constrained system, taking $\mu$ to be the i.i.d uniform measure, $R_\varepsilon(B_n(X), B_n(Y), \mu_n)$ is the minimal radius for covering a fraction of $(1 - \varepsilon)$ of $\Sigma^n$. 
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An Asymptotic Definition
For a fixed $\varepsilon \in (0, 1)$ define

$$R_\varepsilon(X, Y, \mu) \triangleq \liminf_{n \to \infty} \frac{R_\varepsilon(\mathcal{B}_n(X), \mathcal{B}_n(Y), \mu_n)}{n}.$$
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Taking the uncovered-fraction of $Y$ to 0 we define the essential covering radius of $X$ with respect to $(Y, \mu)$ as

$$R_0(X, Y, \mu) \triangleq \lim_{\varepsilon \to 0} R_\varepsilon(X, Y, \mu).$$
Do we get improved results?

The Case of $(0, k)$-RLL

We revisit the case where $Y = \{0, 1\}^\mathbb{Z}$ is non-constrained and $X_{0,k}$ is the $(0, k)$-RLL system. Let $\mu$ be the $\text{Ber}(\frac{1}{2})$ i.i.d measure on $Y$. 

Question: is the essential covering radius strictly smaller than the combinatorial covering radius in that case?

Theorem

$R_{0}(X_{0,k}, Y, \mu) = \frac{1}{2}(2^{k+1} - 1) \ll \frac{1}{k+1} = R(X_{0,k}, Y)$.

In the Context of QCC

For a sequence of ECCs capable of correcting $\delta_n$ errors:

• Using the combinatorial covering radius – it is possible to correct up to $(\delta - 1)k + 1$ errors.
• Using the essential covering radius – with vanishing loss of rate, it is possible to correct $(\delta - 1)\frac{1}{2}(2^{k+1} - 1)$ errors!
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For a sequence of ECCs capable of correcting \(\delta n\) errors:

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Results
The Combinatorial Covering Radius

- We prove that under the assumption of primitive $X$ or $Y$, the lim inf in the definition is a limit.

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- We find an equivalent characterization of the essential covering radius using ergodic theory.
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**Possible Directions for The Future**

- The algorithmic aspect of QCC - developing quantization algorithms.
- Studying the covering radii of well-known constrained systems.
- Providing general bounds and methods to study the covering radius for studying constrained systems.
Thank you for your attention!