Bosonic (and other) codes with interesting gate sets

Anthony Leverrier
Inria

Advances in Quantum Coding
16 February 2024

arXiv:2306.11621 [pdf, other]

Multimode bosonic cat codes with an easily implementable universal gate set

Aurélie Denys, Anthony Leverrier

A natural problem

- logical group $G \subseteq SU(d)$

  e.g., single-qubit Clifford group

- nice physical representation $g \mapsto \rho(g)$

  e.g., Gaussian unitaries, transversal gates $\rho(g) = g^{\otimes n}$

- design a code where logical g is implemented with $\rho(g)$?

If you don't care about bosonic codes, you can think about this for the next 10 minutes.

Story of the result

1. initial idea: design multimode bosonic cat codes
   - didn't really work, but found a somewhat okay 2-mode bosonic qutrit

     arXiv:2210.16188 [pdf, other]

     The $2T$-qutrit, a two-mode bosonic qutrit

     Aurélie Denys, Anthony Leverrier

     Comments: 24 pages, python code available at this https URL, v3 published version

     Journal-ref: Quantum 7, 1032 (2023)
   - inspired a very comprehensive generalization

     arXiv:2302.11593 [pdf, other] quant-ph cond-mat.mes-hall cs.IT math.MG

     Quantum spherical codes

     Authors: Shubham P. Jain, Joseph T. Iosue, Alexander Barg, Victor V. Albert
2. follow-up: same thing with nice logical gates
3. extension to arbitrary (non-bosonic) codes

Outline

1 initial idea: design multimode bosonic cat codes

- didn't really work, but found a somewhat okay 2-mode bosonic qutrit

arXiv:2210.16188 [pdf, other]

The $2T$-qutrit, a two-mode bosonic qutrit

Aurélie Denys, Anthony Leverrier

Comments: 24 pages, python code available at this https URL, v3 published version

Journal-ref: Quantum 7, 1032 (2023)

- inspired a very comprehensive generalization

arXiv:2302.11593 [pdf, other] quant-ph cond-mat.mes-hall cs.IT math.MG

Quantum spherical codes

Authors: Shubham P. Jain, Joseph T. Iosue, Alexander Barg, Victor V. Albert

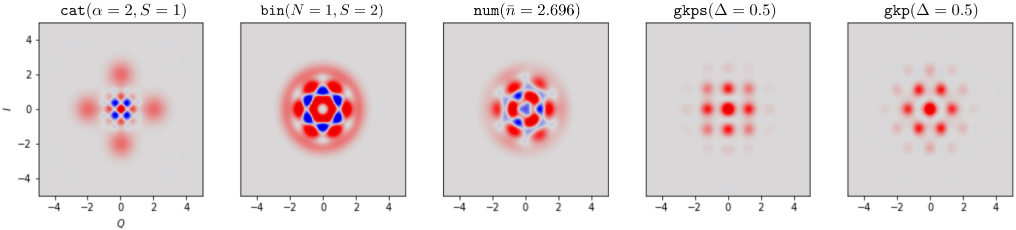
2 follow-up: same thing with nice logical gates

3 extension to arbitrary (non-bosonic) codes

The most-studied bosonic codes are single-mode

- cat code, GKP, binomial code

Code fingerprint: Wigner function of $\frac{1}{2}(|\bar{0}\rangle\langle\bar{0}| + |\bar{1}\rangle\langle\bar{1}|)$



V. Albert *et al*, PRA 2019

not clear whether there are any other super smart single-mode bosonic codes to be found

what about multimode bosonic codes? They should give better performance...

Codewords chosen as superpositions of coherent states

Natural choices of constellations: additive vs multiplicative group structure

- lattice (square, hexagonal...) in phase-space of dimension $2m$
  $\implies$ m-mode GKP code
- roots of unity in 2D $\implies$ cat codes

strategy

- pick a nice constellation of size N in $\mathbb{C}^2$ (for 2-mode codes)
  $\implies$ N-dimensional Hilbert space
- tricky part: find a good qubit/qudit in that space

Codewords chosen as superpositions of coherent states

Natural choices of constellations: additive vs multiplicative group structure

- lattice (square, hexagonal...) in phase-space of dimension $2m$
  $\implies$ m-mode GKP code
- roots of unity in 2D $\implies$ cat codes

strategy

- pick a nice constellation of size N in $\mathbb{C}^2$ (for 2-mode codes)
  $\implies$ N-dimensional Hilbert space
- tricky part: find a good qubit/qudit in that space

1st try: 2-mode generalization of cat codes

arXiv:2210.16188 [pdf, other]

The $2T$-qutrit, a two-mode bosonic qutrit

Aurélie Denys, Anthony Leverrier

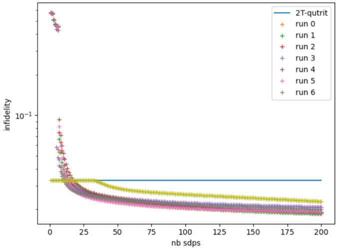
Comments: 24 pages, python code available at this https URL, v3 published version

Journal-ref: Quantum 7, 1032 (2023)

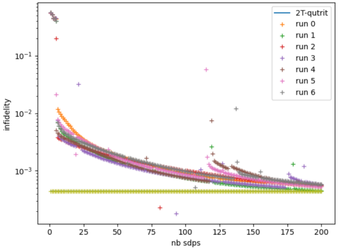
- constellation: 24 coherent states $|\alpha \mathrm{a}_\ell\rangle|\alpha \mathrm{b}_\ell\rangle$ where $\mathrm{a}_\ell, \mathrm{b}_\ell \in \mathbb{C}$ and $\{\mathrm{a}_\ell + \mathrm{j}\mathrm{b}_\ell\}$ form the binary tetrahedral group 2T (Pauli + Hadamard)
- it defines a 24-dim subspace of the 2-mode Fock space
- how to find a good code (qubit or qudit)?
- numerical optimization of encoding/decoding doesn't lead anywhere
- in the end, we defined a qutrit with nice symmetry properties $\implies$ 2T-qutrit

Fidelity of entanglement *vs* random codes

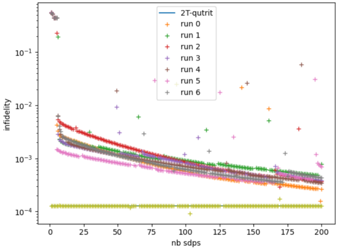
Performance for pure-loss channel, with loss parameter γ



(a) $\gamma = 0.1$



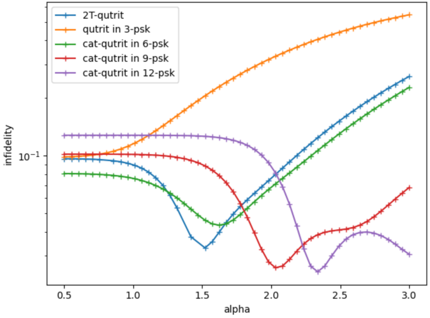
(b) $\gamma = 0.01$



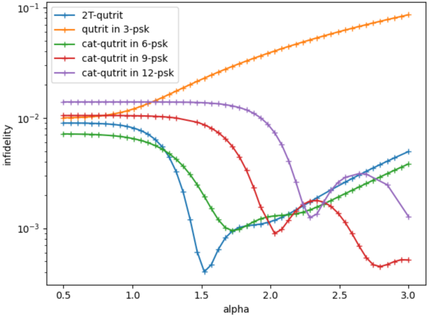
(c) $\gamma = 0.005$

- iterative numerical optimization (SDP) of decoding and encoding
- 2T-qutrit = fixed point $\implies$ local optimum
- pretty competitive for low loss

Fidelity of entanglement *vs* single-mode cat qutrits



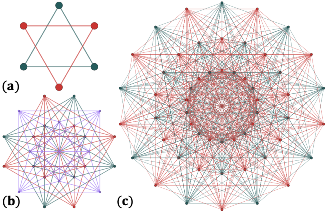
(a) $\gamma = 0.1$



(b) $\gamma = 0.01$

- free parameter of the code: amplitude $\alpha > 0$ of the coherent states
- sweet spot for specific value
- again pretty competitive for low loss
- additional feature: some nice logical gates (thanks to group structure)

Generalization: quantum spherical codes



Shubham Jain, Joseph Iosue, Alexander Barg, Victor Albert

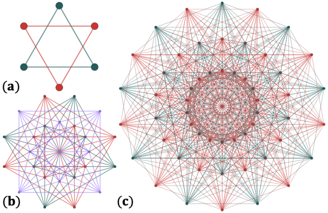
arXiv:2302.11593

- idea: replace the group 2T by spherical designs
- nice error protection, also work out some logical gates
- even a variant of the 2T-qutrit with better performance against loss $\implies$ a bit depressing
- but the set of nice logical operations is a bit larger for the 2T-qutrit

new question: can we design bosonic codes with nice logical gate sets?

answer: YES! with a nice systematic construction

Generalization: quantum spherical codes



Shubham Jain, Joseph Iosue, Alexander Barg, Victor Albert

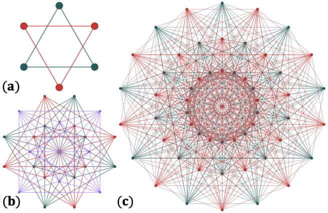
arXiv:2302.11593

- idea: replace the group 2T by spherical designs
- nice error protection, also work out some logical gates
- even a variant of the 2T-qutrit with better performance against loss $\implies$ a bit depressing
- but the set of nice logical operations is a bit larger for the 2T-qutrit

new question: can we design bosonic codes with nice logical gate sets?

answer: YES! with a nice systematic construction

Generalization: quantum spherical codes



Shubham Jain, Joseph Iosue, Alexander Barg, Victor Albert

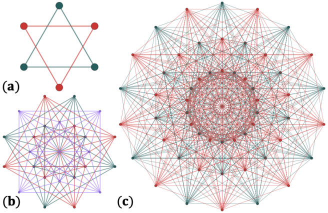
arXiv:2302.11593

- idea: replace the group 2T by spherical designs
- nice error protection, also work out some logical gates
- even a variant of the 2T-qutrit with better performance against loss $\implies$ a bit depressing
- but the set of nice logical operations is a bit larger for the 2T-qutrit

new question: can we design bosonic codes with nice logical gate sets?

answer: YES! with a nice systematic construction

Generalization: quantum spherical codes



Shubham Jain, Joseph Iosue, Alexander Barg, Victor Albert

arXiv:2302.11593

- idea: replace the group 2T by spherical designs
- nice error protection, also work out some logical gates
- even a variant of the 2T-qutrit with better performance against loss $\implies$ a bit depressing
- but the set of nice logical operations is a bit larger for the 2T-qutrit

new question: can we design bosonic codes with nice logical gate sets?

answer: YES! with a nice systematic construction

Outline

1. initial idea: design multimode bosonic cat codes
   - didn't really work, but found a somewhat okay 2-mode bosonic qutrit

     arXiv:2210.16188 [pdf, other]
     The $2T$-qutrit, a two-mode bosonic qutrit
     Aurélie Denys, Anthony Leverrier
     Comments: 24 pages, python code available at this https URL, v3 published version
     Journal-ref: Quantum 7, 1032 (2023)
   - inspired a very comprehensive generalization

     arXiv:2302.11593 [pdf, other] quant-ph cond-mat.mes-hall cs.IT math.MG
     Quantum spherical codes
     Authors: Shubham P. Jain, Joseph T. Iosue, Alexander Barg, Victor V. Albert
2. follow-up: same thing with nice logical gates
3. extension to arbitrary (non-bosonic) codes

General idea

Standard strategy for designing quantum codes

1. find a code with good parameters (rate, distance)
2. understand how to perform gate fault-tolerantly

our appproach

1. find a code family with nice logical gate set (easy)
2. optimize code distance / tolerance to noise (less easy?)

General idea

Standard strategy for designing quantum codes

1. find a code with good parameters (rate, distance)
2. understand how to perform gate fault-tolerantly

our appproach

1. find a code family with nice logical gate set (easy)
2. optimize code distance / tolerance to noise (less easy?)

Codes with nice logical gates

input

- group of logical gates $\mathrm{G} \subseteq \mathrm{SU}(2)$ (for a single logical qubit)
- nice physical representation on physical Hibert space: $\rho : \mathrm{g} \in \mathrm{G} \mapsto \rho(\mathrm{g})$

output: encoding: $\mathcal{E} : \mathbb{C}^2 \to \mathcal{H}_\mathrm{P}$ such that

$$\mathcal{E}(\mathrm{g}|\psi\rangle) = \rho(\mathrm{g})\,\mathcal{E}(|\psi\rangle)$$

- this is always possible! Simple general construction
- main open question: how to get protection against noise

Previous work on this question (apologies to missing references!)

- encoding qubits in spins

arXiv:2005.10910 [pdf, other] quant-ph doi 10.1103/PhysRevLett.127.010504
Encoding a qubit in a spin
Authors: Jonathan A. Gross

arXiv:2304.08611 [pdf, ps, other] quant-ph doi 10.1103/PhysRevA.108.022424
Multispin Clifford codes for angular momentum errors in spin systems
Authors: Sivaprasad Omanakuttan, Jonathan A. Gross

- qubit codes with transversal gates

arXiv:2305.07023 [pdf, other] quant-ph doi 10.1103/PhysRevLett.131.240601
A Family of Quantum Codes with Exotic Transversal Gates
Authors: Eric Kubischta, Ian Teixeira

arXiv:2310.17652 [pdf, other] quant-ph
The Not-So-Secret Fourth Parameter of Quantum Codes
Authors: Eric Kubischta, Ian Teixeira

arXiv:2402.01638 [pdf, ps, other] quant-ph
Free Quantum Codes from Twisted Unitary t-groups
Authors: Eric Kubischta, Ian Teixeira

- codes with continuous symmetries

arXiv:1902.07725 [pdf, other] quant-ph math-ph doi 10.22331/q-2020-03-23-245
Continuous groups of transversal gates for quantum error correcting codes from finite clock reference frames
Authors: Mischa P. Woods, Álvaro M. Alhambra

arXiv:1902.07714 [pdf, other] quant-ph cond-mat.stat-mech hep-th doi 10.1103/PhysRevX.10.041018
Continuous symmetries and approximate quantum error correction
Authors: Philippe Faist, Sepehr Nezami, Victor V. Albert, Grant Salton, Fernando Pastawski, Patrick Hayden, John Preskill

General recipe

- group of logical gates $\mathrm{G} \subseteq \mathrm{SU(d)}$
- nice physical representation ρ on physical space $\mathcal{H}_\mathrm{P}$
- pick any logical state $|\Sigma\rangle \in \mathbb{C}^\mathrm{d}$ and any physical state $|\Phi\rangle \in \mathcal{H}_\mathrm{P}$ (e.g. vacuum state)

Encoding map

$$\begin{aligned} \mathcal{E} : \quad \mathbb{C}^\mathrm{d} &\rightarrow \mathcal{H}_\mathrm{P} \\ |\psi\rangle &\mapsto \frac{\mathrm{d}}{|\mathrm{G}|} \sum_{\mathrm{g}\in\mathrm{G}} \langle\Sigma|\mathrm{g}^\dagger|\psi\rangle \, \rho(\mathrm{g}) |\Phi\rangle \end{aligned}$$

slightly more general (useful for GKP, cat qudits):

- arbitrary group G
- d-dim representation ρ_L: replace $\langle\Sigma|\mathrm{g}^\dagger|\psi\rangle$ by $\langle\Sigma|\rho_\mathrm{L}(\mathrm{g})^\dagger|\psi\rangle$
- need a bit of care if ρ_L is not irreducible

General recipe

- group of logical gates $\mathrm{G} \subseteq \mathrm{SU(d)}$
- nice physical representation ρ on physical space $\mathcal{H}_\mathrm{P}$
- pick any logical state $|\Sigma\rangle \in \mathbb{C}^\mathrm{d}$ and any physical state $|\Phi\rangle \in \mathcal{H}_\mathrm{P}$ (e.g. vacuum state)

Encoding map

$$\begin{aligned} \mathcal{E}: \quad \mathbb{C}^\mathrm{d} &\to \mathcal{H}_\mathrm{P} \\ |\psi\rangle &\mapsto \tfrac{\mathrm{d}}{|\mathrm{G}|} \textstyle\sum_{\mathrm{g}\in\mathrm{G}} \langle\Sigma|\mathrm{g}^\dagger|\psi\rangle\, \rho(\mathrm{g})|\Phi\rangle \end{aligned}$$

slightly more general (useful for GKP, cat qudits):

- arbitrary group G
- d-dim representation ρ_L: replace $\langle\Sigma|\mathrm{g}^\dagger|\psi\rangle$ by $\langle\Sigma|\rho_\mathrm{L}(\mathrm{g})^\dagger|\psi\rangle$
- need a bit of care if ρ_L is not irreducible

Elementary facts of representation theory

Representation $\cong$ sum of irreducible representations

$$\rho(g) = U \left(\bigoplus_i \rho_i(g) \otimes \mathbb{1}_{M_i} \right) U^\dagger$$

- ρ_i: irreducible representations of G
- M_i: multiplicity of ρ_i in $\rho(g)$

Orthogonality of irreps

$$\frac{d}{|G|} \sum_{g \in G} \rho_i(g)^\dagger \otimes \rho_j(g) = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{p,q=0}^{d-1} |p\rangle\langle q| \otimes |q\rangle\langle p| = \mathrm{SWAP} & \text{if } i = j \end{cases}$$

Projector onto (isotypic component) of ρ_i

$$\Pi = \frac{2}{|G|} \sum_g \mathrm{tr}(g^\dagger) \otimes \rho(g)$$

Elementary facts of representation theory

Representation $\cong$ sum of irreducible representations

$$\rho(g) = U \left(\bigoplus_i \rho_i(g) \otimes \mathbb{1}_{M_i} \right) U^\dagger$$

- ρ_i: irreducible representations of G
- M_i: multiplicity of ρ_i in $\rho(g)$

Orthogonality of irreps

$$\frac{d}{|G|} \sum_{g \in G} \rho_i(g)^\dagger \otimes \rho_j(g) = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{p,q=0}^{d-1} |p\rangle\langle q| \otimes |q\rangle\langle p| = \mathrm{SWAP} & \text{if } i = j \end{cases}$$

Projector onto (isotypic component) of ρ_i

$$\Pi = \frac{2}{|G|} \sum_g \mathrm{tr}(g^\dagger) \otimes \rho(g)$$

Elementary facts of representation theory

Representation $\cong$ sum of irreducible representations

$$\rho(g) = U \left(\bigoplus_i \rho_i(g) \otimes \mathbb{1}_{M_i} \right) U^\dagger$$

- ρ_i: irreducible representations of G
- M_i: multiplicity of ρ_i in $\rho(g)$

Orthogonality of irreps

$$\frac{d}{|G|} \sum_{g \in G} \rho_i(g)^\dagger \otimes \rho_j(g) = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{p,q=0}^{d-1} |p\rangle\langle q| \otimes |q\rangle\langle p| = \mathrm{SWAP} & \text{if } i = j \end{cases}$$

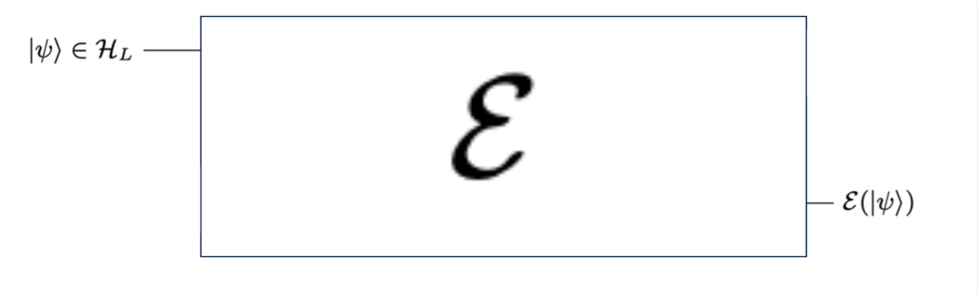
Projector onto (isotypic component) of ρ_i

$$\Pi = \tfrac{2}{|G|} \textstyle\sum_g \mathrm{tr}(g^\dagger) \otimes \rho(g)$$

Rewriting the encoding map

$$\mathcal{E}(|\psi\rangle) = \frac{d}{|G|}\sum_{g\in G}\langle\Sigma|g^\dagger|\psi\rangle\,\rho(g)|\Phi\rangle \qquad \rho(g) = U\left(\bigoplus_i \rho_i(g)\otimes \mathbb{1}_{M_i}\right)U^\dagger$$

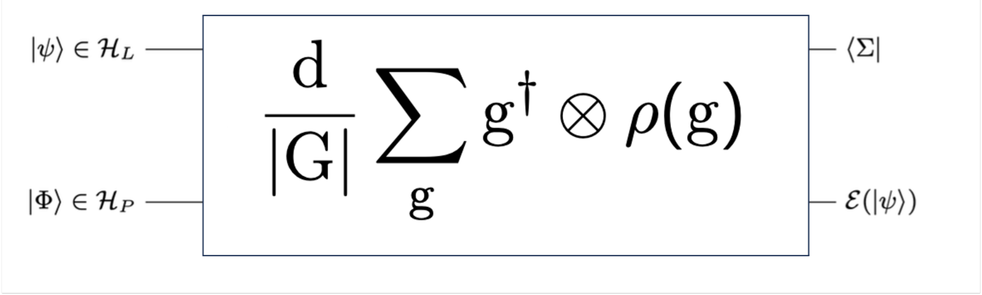
$$\Pi = \frac{2}{|G|}\sum_g \mathrm{tr}(g^\dagger)\otimes\rho(g)$$ projector onto isotypic component



Rewriting the encoding map

$$\mathcal{E}(|\psi\rangle) = \frac{d}{|G|}\sum_{g\in G}\langle\Sigma|g^\dagger|\psi\rangle\,\rho(g)|\Phi\rangle \qquad\qquad \rho(g) = U\left(\bigoplus_i \rho_i(g)\otimes \mathbb{1}_{M_i}\right)U^\dagger$$

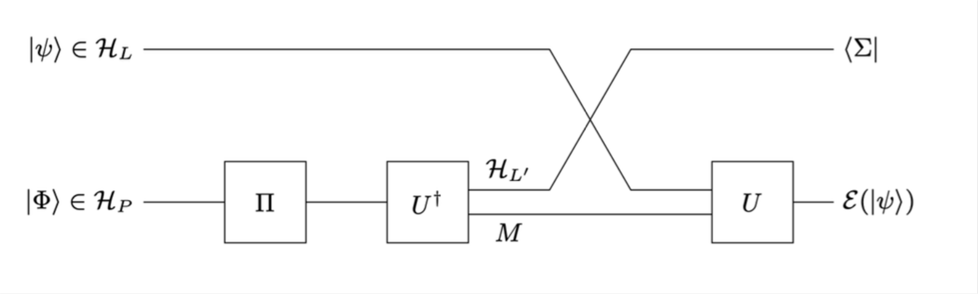
$$\Pi = \frac{2}{|G|}\sum_g \mathrm{tr}(g^\dagger)\otimes\rho(g)$$ projector onto isotypic component



Rewriting the encoding map

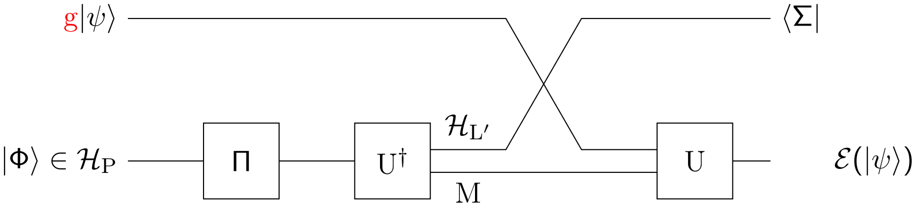
$$\mathcal{E}(|\psi\rangle) = \frac{d}{|G|} \sum_{g \in G} \langle \Sigma | g^\dagger | \psi \rangle \, \rho(g) | \Phi \rangle \qquad \rho(g) = U \left(\bigoplus_i \rho_i(g) \otimes \mathbb{1}_{M_i} \right) U^\dagger$$

$$\Pi = \frac{2}{|G|} \sum_g \mathrm{tr}(g^\dagger) \otimes \rho(g)$$ projector onto isotypic component



Proof of covariance

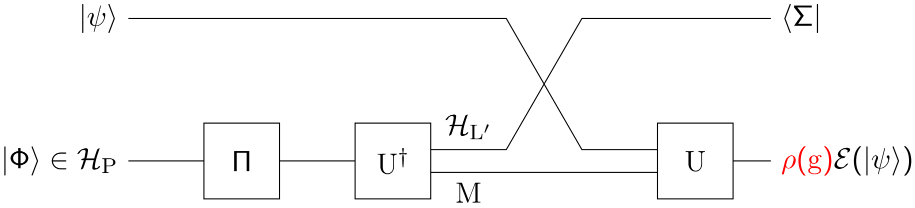
$$\rho(g)\Pi = U(g \otimes \mathbb{1}_M)U^\dagger$$



$$\mathcal{E}(g|\psi\rangle) = \rho(g)\,\mathcal{E}(|\psi\rangle) \qquad \forall g \in G$$

Proof of covariance

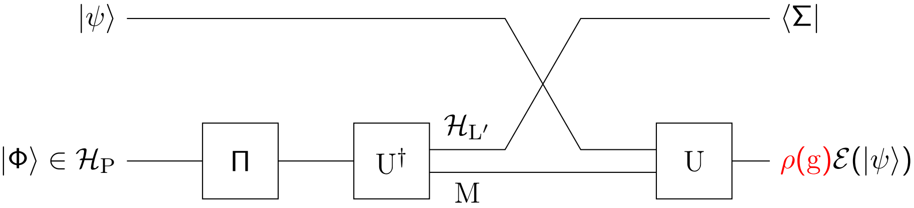
$$\rho(g)\Pi = U(g \otimes \mathbb{1}_M)U^\dagger$$



$$\mathcal{E}(g|\psi\rangle) = \rho(g)\,\mathcal{E}(|\psi\rangle) \qquad \forall g \in G$$

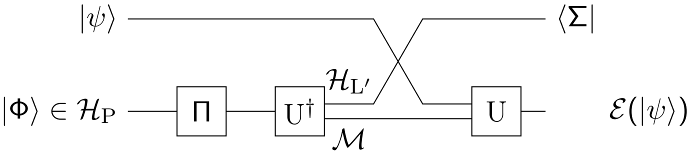
Proof of covariance

$$\rho(g)\Pi = U(g \otimes \mathbb{1}_M)U^\dagger$$



$$\mathcal{E}(g|\psi\rangle) = \rho(g)\,\mathcal{E}(|\psi\rangle) \qquad \forall g \in G$$

Alternative encoding map



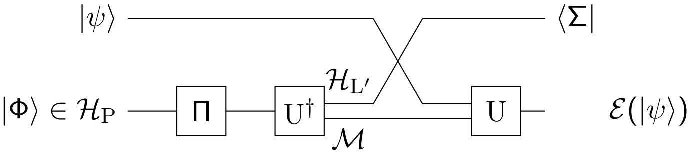
A code for any $|\phi\rangle \in \mathcal{M}$:

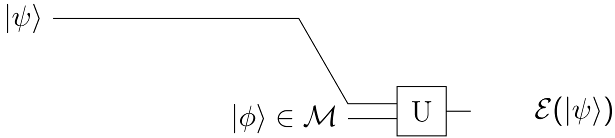
$$\mathcal{E}_\phi(|\psi\rangle) = U|\psi\rangle|\phi\rangle$$

$U : \mathbb{C}^d \otimes \mathcal{M} \to \mathcal{H}_P$ isometry, given by G and ρ

challenge: find the states $|\phi\rangle$ that give good protection against noise

Alternative encoding map





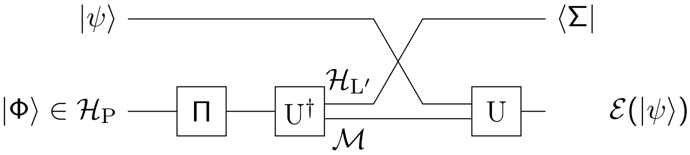
A code for any $|\phi\rangle \in \mathcal{M}$:

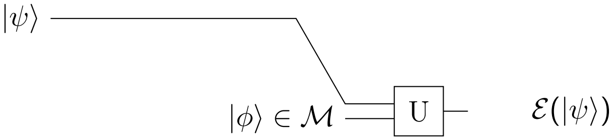
$$\mathcal{E}_\phi(|\psi\rangle) = U|\psi\rangle|\phi\rangle$$

$U : \mathbb{C}^d \otimes \mathcal{M} \to \mathcal{H}_P$ isometry, given by G and ρ

challenge: find the states $|\phi\rangle$ that give good protection against noise

Alternative encoding map





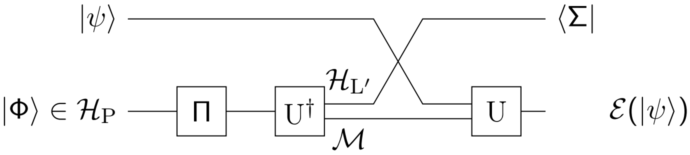
A code for any $|\phi\rangle \in \mathcal{M}$:

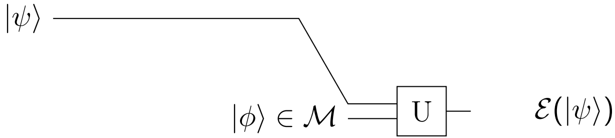
$$\mathcal{E}_\phi(|\psi\rangle) = U|\psi\rangle|\phi\rangle$$

$U : \mathbb{C}^d \otimes \mathcal{M} \to \mathcal{H}_P$ isometry, given by G and ρ

challenge: find the states $|\phi\rangle$ that give good protection against noise

Alternative encoding map





A code for any $|\phi\rangle \in \mathcal{M}$:

$$\mathcal{E}_\phi(|\psi\rangle) = U|\psi\rangle|\phi\rangle$$

$U : \mathbb{C}^d \otimes \mathcal{M} \to \mathcal{H}_P$ isometry, given by G and ρ

challenge: find the states $|\phi\rangle$ that give good protection against noise

Application to bosonic codes

The case of bosonic codes

Pick

- $|\Phi\rangle = |\vec{\alpha}\rangle$ a coherent state
- $\rho(\mathrm{g})$ Gaussian unitary: $\rho(\mathrm{g})|\vec{\alpha}\rangle = |\mathrm{g}\vec{\alpha}\rangle$

$$\mathcal{E}(|\psi\rangle) = \frac{2}{|\mathrm{G}|} \sum_{\mathrm{g} \in \mathrm{G}} \langle \Sigma | \mathrm{g}^{\dagger} | \psi \rangle \, |\mathrm{g}\vec{\alpha}\rangle$$

is a superposition of coherent states.

$\implies$ generalization of quantum spherical codes, but with nice gate sets.

one can recover the usual suspects:

- GKP: Pauli group and displacements
- cat codes: $\langle \sigma_X \rangle$ and dephasing

and define new codes: for $\mathrm{G} =$ Pauli or Clifford group

The case of bosonic codes

Pick

- $|\Phi\rangle = |\vec{\alpha}\rangle$ a coherent state
- $\rho(\mathrm{g})$ Gaussian unitary: $\rho(\mathrm{g})|\vec{\alpha}\rangle = |\mathrm{g}\vec{\alpha}\rangle$

$$\mathcal{E}(|\psi\rangle) = \frac{2}{|\mathrm{G}|}\sum_{\mathrm{g}\in\mathrm{G}} \langle \Sigma|\mathrm{g}^\dagger|\psi\rangle\, |\mathrm{g}\vec{\alpha}\rangle$$

is a superposition of coherent states.

$\implies$ generalization of quantum spherical codes, but with nice gate sets.

one can recover the usual suspects:

- GKP: Pauli group and displacements
- cat codes: $\langle \sigma_X \rangle$ and dephasing

and define new codes: for $\mathrm{G} =$ Pauli or Clifford group

Example 1: the GKP code

- encode a qubit in single-mode Fock space
- physical representation displacement operators:

  $G_P = \langle D(\alpha), D(\beta) \rangle$ with $D(\alpha) = e^{\alpha^* a^\dagger - \alpha a}$

$$D(\alpha)D(\beta) = -D(\beta)D(\alpha) \quad \text{if} \quad \beta\alpha^* - \beta^*\alpha = i\pi$$

  standard square GKP lattice: $\alpha = \sqrt{\frac{\pi}{2}}, \beta = i\sqrt{\frac{\pi}{2}}$

- logical representation $\rho_L(D(\alpha)) = \sigma_X,\ \rho_L(D(\beta)) = \sigma_Z \quad \Longrightarrow \quad$ Pauli group
- pick $|\Sigma\rangle = |0\rangle \in \mathbb{C}^2, \quad |\Phi\rangle = |0\rangle \in \mathcal{H}_P$ (vacuum state)

Example 1: the GKP code

Let's compute

$$|\bar{0}\rangle \propto \sum_{g \in G_P} \langle 0|\rho_L(g)^\dagger|0\rangle \; g|0\rangle$$

The only nonzero coefficients $\langle 0|\rho_L(g)^\dagger|0\rangle \neq 0$ are for $\rho_L(g) \in \{\pm\mathbb{1}, \pm\sigma_Z\}$:
i.e. $g = D(2p\alpha)D(q\beta) = (-1)^{pq}D(2p\alpha + q\beta)$ with $p, q \in \mathbb{Z}$

$$|\bar{0}\rangle \propto \sum_{p,q \in \mathbb{Z}} (-1)^{pq}|2p\alpha + q\beta)\rangle$$

$$\begin{aligned} |\bar{1}\rangle &\propto \rho(\sigma_X)|\bar{1}\rangle \\ &\propto D(\alpha)|\bar{0}\rangle \\ &\propto \sum_{p,q \in \mathbb{Z}} (-1)^{pq}(-i)^q|(2p+1)\alpha + \beta q\rangle \end{aligned}$$

$\Longrightarrow$ recover GKP code without any fine-tuning

Example 2: the $2N$-legged cat qubit

- $G = \langle e^{i\pi/N} \rangle$, cyclic group of order $2N$
- logical representation $\rho_L(e^{i\pi/N}) = \sigma_X$
- physical representation with dephasing: $\rho(e^{i\pi/N}) = e^{i\pi\hat{n}/N}$
- pick $|\Sigma\rangle = |0\rangle \in \mathbb{C}^2, \quad |\Phi\rangle = |\alpha\rangle \in \mathcal{H}_P$ (arbitrary coherent state)

$$
\begin{aligned}
|\overline{0}\rangle &\propto \sum_{g \in G_P} \langle 0|\rho_L(g)^\dagger|0\rangle \rho(g)|\alpha\rangle \propto \sum_{k=0}^{2N-1} \delta_{k,\mathrm{even}} e^{ki\pi\hat{n}/N}|\alpha\rangle \\
&\propto \sum_{k=0}^{N-1} |e^{2\pi i k/N}\alpha\rangle
\end{aligned}
$$

$$
|\overline{1}\rangle = \rho(e^{i\pi/N})|\overline{0}\rangle \propto \sum_{k=0}^{N-1} |e^{\pi i(2k+1)k/N}\alpha\rangle
$$

$\Longrightarrow$ this is the cat qubit

New code 1: G = Pauli group with Gaussian unitaries

- logical group: $G_L = \langle \sigma_X, \sigma_Z \rangle$
- $\mathcal{H}_P$: 2-mode Fock space
- physical representation: Gaussian unitary (beamsplitters and phase-shifts)

$$\rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : |\alpha\rangle\beta\rangle \mapsto |a\alpha + b\beta\rangle|c\alpha + d\beta\rangle$$

- pick $|\Sigma\rangle = |0\rangle \in \mathbb{C}^2, \quad |\Phi\rangle = |\alpha\rangle|\beta\rangle \in \mathcal{H}_P$ (arbitrary coherent state)

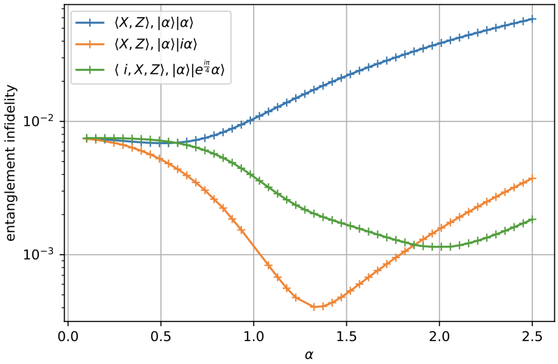
$$|\overline{0}\rangle \propto |c_1(\alpha)\rangle|c_0(\beta)\rangle$$
$$|\overline{1}\rangle \propto |c_0(\beta)\rangle|c_1(\alpha)\rangle$$

with $|c_0(\alpha)\rangle = |\alpha\rangle + |-\alpha\rangle, \quad |c_1(\alpha)\rangle = |\alpha\rangle - |-\alpha\rangle$

- recover the dual-rail encoding in the limit $\alpha \to 0$:

a single photon in 2 modes: $\quad |\overline{0}\rangle = |1\rangle|0\rangle, \qquad |\overline{1}\rangle = |0\rangle|1\rangle$

New code 1: G = Pauli group with Gaussian unitaries



entanglement infidelity for pure-loss channel $\gamma = 10^{-2}$

dual-rail encoding: $\alpha = 0$

$\implies$ need to optimize the initial state $|\Phi\rangle \in \mathcal{H}_\mathrm{P}$ (maximize the distance between the points of the constellation, same as for quantum spherical codes)

More interesting code: G = single-qubit Clifford group

- 2O group: binary octahedral group (aka single-qubit Clifford group)

$$2O = \langle S, H \rangle, \qquad |2O| = 48$$

$$S = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}, \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta & \eta \\ -\eta^{-1} & \eta^{-1} \end{bmatrix} \in SU(2)$$

$$\rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : |\alpha\rangle|\beta\rangle \mapsto |a\alpha + b\beta\rangle|c\alpha + d\beta\rangle$$

$\implies$ $|\overline{0}\rangle, |\overline{1}\rangle$: superpositions of 40 coherent states in 2 modes

- "relatively" easy to get a universal gate set with quartic Hamiltonians*

$$\overline{T} = \exp\left(i\frac{\pi}{16}(\hat{n}_1 - \hat{n}_2 - 1)^2\right), \qquad \overline{CZ} = \exp\left(i\frac{\pi}{4}(\hat{n}_1 - \hat{n}_2 - 1)(\hat{n}_3 - \hat{n}_4 - 1)\right)$$

- measurement in $\{|\overline{0}\rangle, |\overline{1}\rangle\}$ basis is easy
- state preparation and error correction??

(*) similar to CROT. e.g. rotation-symmetric bosonic codes (Grimsmo, Combes, Baragiola)

Outline

1. initial idea: design multimode bosonic cat codes

   - didn't really work, but found a somewhat okay 2-mode bosonic qutrit

     arXiv:2210.16188 [pdf, other]

     The $2T$-qutrit, a two-mode bosonic qutrit
     Aurélie Denys, Anthony Leverrier
     Comments: 24 pages, python code available at this https URL, v3 published version
     Journal-ref: Quantum 7, 1032 (2023)

   - inspired a very comprehensive generalization

     arXiv:2302.11593 [pdf, other] quant-ph cond-mat.mes-hall cs.IT math.MG

     Quantum spherical codes
     Authors: Shubham P. Jain, Joseph T. Iosue, Alexander Barg, Victor V. Albert

2. follow-up: same thing with nice logical gates

3. extension to arbitrary (non-bosonic) codes

Beyond bosonic codes

The construction is very general:

$$\mathcal{H}_L = \mathbb{C}^d, \qquad \mathcal{H}_P = (\mathbb{C}^{d'})^{\otimes n}$$

Natural choices for the physical representation $\rho(g)$:

- transversal gates $\rho(g) = g^{\otimes n}$
- $\rho(g) = (g^\dagger)^{\otimes n}$
- $\rho(g) = g^{\otimes p} \otimes (g^\dagger)^{\otimes (n-p)}$

Codes $[\![n,k]\!]$ with G = Pauli group and physical Pauli gates

$$G = \mathcal{P}_k \qquad \text{and} \qquad \rho(g) \in \mathcal{P}_n$$
$$|\mathcal{P}_k| = 2 \times 4^k \qquad (\text{only} \pm 1 \text{ phases})$$

Projector onto the irrep:

$$\Pi = \frac{2^k}{|G|} \sum_g \mathrm{tr}(g^\dagger)\rho(g) = \frac{2^k}{2 \times 4^k} \sum_{g \in \{\pm \mathbb{1}\}} \mathrm{tr}(g^\dagger)\rho(g) = \frac{2^k}{2 \times 4^k} \times \left(2 \times 2^k \mathbb{1}\right) = \mathbb{1}$$

- multiplicity space of maximal dimension
- unitary $U : (\mathbb{C}^2)^{\otimes k} \otimes (\mathbb{C}^2)^{\otimes (n-k)} \to (\mathbb{C}^2)^{\otimes n}$ can be chosen Clifford
- recover stabilizer codes: $|\psi\rangle \mapsto U|\psi\rangle|0\rangle^{\otimes (n-k)}$

Code $[\![5,1]\!]$ with transversal $2\mathrm{T}$

- 2T group: binary tetrahedral group, $2\mathrm{T} = \langle \mathrm{Z}, \mathrm{H} \rangle = (\text{Paulis } + \text{ Hadamard})$, $\quad |2\mathrm{T}| = 24$
- 3 irreps of dimension 2: ρ_4, ρ_4^*, ρ_5
- pick $\rho_\mathrm{L} = \rho_5$: $\quad \rho_5(\mathrm{Z}) = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \rho_5(\mathrm{H}) = \frac{e^{\mathrm{i}\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}$
- $\rho(g) = \rho_5(g)^{\otimes 5}$
- easy to compute that:

$$\rho = \rho_4^{\oplus 5} \oplus \rho_4^{*\oplus 5} \oplus \rho_5^{\oplus 6}$$

$$\mathrm{U} : \mathbb{C}^2 \otimes \mathbb{C}^6 \to (\mathbb{C}^2)^{\otimes 5}$$

- can find $|\phi\rangle \in \mathbb{C}^6$ such that

$$\mathrm{span}(\mathrm{U}|0\rangle|\phi\rangle, \mathrm{U}|1\rangle|\phi\rangle) = [\![5,1,3]\!]$$

Recover the 5-qubit code, but need to choose $|\phi\rangle$ carefully.

Code $[\![5,1]\!]$ with transversal $2\mathrm{T}$

- $2\mathrm{T}$ group: binary tetrahedral group, $2\mathrm{T} = \langle \mathrm{Z}, \mathrm{H} \rangle = (\text{Paulis} + \text{Hadamard})$, $\quad |2\mathrm{T}| = 24$
- 3 irreps of dimension 2: ρ_4, ρ_4^*, ρ_5
- pick $\rho_{\mathrm{L}} = \rho_5$: $\quad \rho_5(\mathrm{Z}) = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \rho_5(\mathrm{H}) = \frac{e^{\mathrm{i}\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}$
- $\rho(\mathrm{g}) = \rho_5(\mathrm{g})^{\otimes 5}$
- easy to compute that:

$$\rho = \rho_4^{\oplus 5} \oplus \rho_4^{*\oplus 5} \oplus \rho_5^{\oplus 6}$$

$$\mathrm{U} : \mathbb{C}^2 \otimes \mathbb{C}^6 \to (\mathbb{C}^2)^{\otimes 5}$$

- can find $|\phi\rangle \in \mathbb{C}^6$ such that

$$\mathrm{span}(\mathrm{U}|0\rangle|\phi\rangle, \mathrm{U}|1\rangle|\phi\rangle) = [\![5,1,3]\!]$$

Recover the 5-qubit code, but need to choose $|\phi\rangle$ carefully.

Code $[\![7,1]\!]$ with transversal Clifford group

2O group: binary octahedral group (aka single-qubit Clifford group)

$$2\mathrm{O} = \langle \mathrm{S}, \mathrm{H} \rangle, \qquad |2\mathrm{O}| = 48$$

$$\rho_7(\mathrm{S}) = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}, \qquad \rho_7(\mathrm{H}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta & \eta \\ -\eta^{-1} & \eta^{-1} \end{bmatrix}$$

$$\rho_{\mathrm{L}} = \rho_7 \qquad \rho_7^{\otimes 7} = \rho_6^{\oplus 7} \oplus \rho_7^{\oplus 15} \oplus \rho_8^{\oplus 21}$$

ρ_6, ρ_7: dimension 2, ρ_8: dimension 8

$$\rho(\mathrm{g}) = \rho_7(\mathrm{g})^{\dagger \otimes 7} \quad \Longrightarrow \quad \text{standard Steane code} \quad [\![7,1,3]\!]$$

$$\rho(\mathrm{g}) = \rho_7(\mathrm{g})^{\otimes 7} \quad \Longrightarrow \quad \text{Steane code with different labeling of the logical states}$$

again, the state in the multiplicity space $\mathbb{C}^{15}$ should be chosen with care

Code $[\![7,1]\!]$ with transversal Clifford group

2O group: binary octahedral group (aka single-qubit Clifford group)

$$2\mathrm{O} = \langle \mathrm{S}, \mathrm{H} \rangle, \qquad |2\mathrm{O}| = 48$$

$$\rho_7(\mathrm{S}) = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}, \qquad \rho_7(\mathrm{H}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta & \eta \\ -\eta^{-1} & \eta^{-1} \end{bmatrix}$$

$$\rho_\mathrm{L} = \rho_7 \qquad \rho_7^{\otimes 7} = \rho_6^{\oplus 7} \oplus \rho_7^{\oplus 15} \oplus \rho_8^{\oplus 21}$$

ρ_6, ρ_7: dimension 2, ρ_8: dimension 8

$\rho(\mathrm{g}) = \rho_7(\mathrm{g})^{\dagger \otimes 7} \quad \Longrightarrow \quad$ standard Steane code $\quad [\![7,1,3]\!]$

$\rho(\mathrm{g}) = \rho_7(\mathrm{g})^{\otimes 7} \quad \Longrightarrow \quad$ Steane code with different labeling of the logical states

again, the state in the multiplicity space $\mathbb{C}^{15}$ should be chosen with care

What about the distance?

- not completely clear at the moment
- recent preprint by Kubischta, Teixeira (arXiv:2402.01638) constructs codes with distance $t + 1$ from twisted unitary t-groups
- one can write the Knill-Laflamme conditions for a code of distance d

$$|E| < d \qquad \Longrightarrow \quad \Pi_C E \Pi_C = c_E \Pi_C$$

$$\Pi_C = U(\mathbb{1}_2 \otimes \phi)U^\dagger \qquad \text{with} \qquad \phi := |\phi\rangle\langle\phi|$$

KL conditions become:

$$\text{find} \quad |\phi\rangle \in \mathbb{C}^M \quad \text{s.t.} \quad \{(\mathbb{1}_2 \otimes \phi)U^\dagger E U(\mathbb{1}_2 \otimes \phi) = c_E(\mathbb{1}_2 \otimes \phi) \ : \ |E| < d\}$$

(For the 5-qubit code, there exists a canonical choice of $|\phi\rangle$. The corresponding code satisfies 90 out of the 105 KL conditions for $d = 3$.)

Code with universal set of transversal gates?

- The same construction works for $G = SU(2)$.
- Eastin-Knill theorem: a code of distance > 1 has a finite set of transvsersal gates
- For bosonic codes, each irrep of $SU(2)$ has multiplicity 1

$\implies$ there's a single code, and this is the dual-rail encoding

- what about multiqubit codes?
  - multiplicity of 2-dim irrep in tensor product representation is very large!

$$n = 1 \implies M = 1, \quad n = 3 \implies M = 2, \quad n = 5 \implies M = 5$$
$$n = 7 \implies M = 14, \quad n = 2p + 1 \implies M \approx \frac{2^n}{\sqrt{2\pi n}}$$

  - the multiplicity isn't sufficient to say that a code with good distance exists
  - $U =$ Schur transform
  - the error $\sum_i P_i$ acts trivially on the multiplicity space

Code with universal set of transversal gates?

- The same construction works for $G = SU(2)$.
- Eastin-Knill theorem: a code of distance > 1 has a finite set of transvsersal gates
- For bosonic codes, each irrep of $SU(2)$ has multiplicity 1

  $\implies$ there's a single code, and this is the dual-rail encoding
- what about multiqubit codes?
  - multiplicity of 2-dim irrep in tensor product representation is very large!

$$n = 1 \implies M = 1, \qquad n = 3 \implies M = 2, \qquad n = 5 \implies M = 5$$
$$n = 7 \implies M = 14, \qquad n = 2p + 1 \implies M \approx \frac{2^n}{\sqrt{2\pi n}}$$

  - the multiplicity isn't sufficient to say that a code with good distance exists
  - $U =$ Schur transform
  - the error $\sum_i P_i$ acts trivially on the multiplicity space

Summary

- general formalism to design "codes" with specific physical representation of logical gates
- recovers the standard bosonic codes (GKP, cat codes) without fine tuning
- new multimode bosonic codes with reasonably nice universal gate set
- for qubit codes: can recover the standard (small) codes, but if you know where to look
- very general: qudits, oscillators, rotors for both logical and physical systems

Many questions

- is this formalism a curiosity or can it be useful?
- how to find the codes with good parameters?
- logical state preparation? error correction?

Thanks!

Summary

- general formalism to design "codes" with specific physical representation of logical gates
- recovers the standard bosonic codes (GKP, cat codes) without fine tuning
- new multimode bosonic codes with reasonably nice universal gate set
- for qubit codes: can recover the standard (small) codes, but if you know where to look
- very general: qudits, oscillators, rotors for both logical and physical systems

Many questions

- is this formalism a curiosity or can it be useful?
- how to find the codes with good parameters?
- logical state preparation? error correction?

Thanks!

FRANCE 2030