Solving fixed-point equations on semirings

Javier Esparza

Technical University of Munich

Joint work with

Stefan Kiefer, Michael Luttenberger, Maximilian Schlund

Fixed-point equations

We study systems of equations of the form

$$\begin{aligned} X_1 &= f_1(X_1, \ldots, X_n) \\ X_2 &= f_2(X_1, \ldots, X_n) \\ &\cdots \\ X_n &= f_n(X_1, \ldots, X_n) \end{aligned}$$

where the f_i's are polynomials over an ω-continuous semiring.

ω-continuous semirings

ω-continuity:

the relation $a \sqsubseteq b \Leftrightarrow \exists c : a + c = b$ is a partial order

$\sqsubseteq$-chains have limits

Examples: nonnegative integers and reals with ∞, tropical semiring, min-max semirings, complete lattices, Viterbi and Łukasiewicz semirings, language semiring . . .

In the rest of the talk:
semiring $\equiv$ ω-continuous semiring.

Research program

Develop generic solution or approximation methods, valid for all semirings or at least large classes.

Kleene iteration

Theorem [Kleene]: A system of fixed-point equations over a semiring has a least solution μf w.r.t. the natural order $\sqsubseteq$.

This least solution is the supremum of $\{k_i\}_{i \geq 0}$, where

$$k_0 = f(0)$$
$$k_{i+1} = f(k_i)$$

Basic algorithm for calculation of μf : compute $k_0, k_1, k_2, \ldots$ until either $k_i = k_{i+1}$ or the approximation is considered adequate.

The left-linear case

$$\begin{aligned}
X_1 &= a_{11}X_1 + \cdots + a_{1n}X_n + b_1 \\
X_2 &= a_{21}X_1 + \cdots + a_{2n}X_n + b_2 \\
&\cdots \\
X_n &= a_{n1}X_1 + \cdots + a_{nn}X_n + b_n
\end{aligned}$$

The left-linear case

(Loosely speaking!) Kleene iteration has linear convergence over the reals: k iterations give $\Theta(k)$ accurate bits.

The left-linear case

(Loosely speaking!) Kleene iteration has linear convergence over the reals: k iterations give $\Theta(k)$ accurate bits.

Gauss elimination:

Define $a^* := 1 + a + a \cdot a + a \cdot a \cdot a + \cdots$

Arden's Lemma: the least solution of $X = aX + b$ is $X := a^* b$

Algorithm: pick X_i and rewrite its equation as $X_i = aX_i + b$;
replace X_i in all other equations by $a^* b$

The left-linear case

Gauss elimination reduces solving a system of left-linear equations to computing a^* for a given semiring element a.

The left-linear case

Gauss elimination reduces solving a system of left-linear equations to computing a^* for a given semiring element a.

Real semiring: either $a^* = 0$, $a^* = 1/(1-a)$, or $a^* = \infty$

The left-linear case

Gauss elimination reduces solving a system of left-linear equations to computing a^* for a given semiring element a.

Real semiring: either $a^* = 0$, $a^* = 1/(1-a)$, or $a^* = \infty$

Language semiring: we use a^* as representation of $\sum_{i=0}^{\infty} a^i$

(What does it mean to “solve” an equation?)

The non-linear case

Real semiring: convergence of Kleene iteration can be very slow.

$$X = \frac{1}{2}X^2 + \frac{1}{2} \qquad \mu f = 1 = 0.99999\ldots$$

Logarithmic convergence: k iterations give $\Theta(\log k)$ accurate bits.

For example, $k_{2000} = 0.9990$

The non-linear case

Real semiring: convergence of Kleene iteration can be very slow.

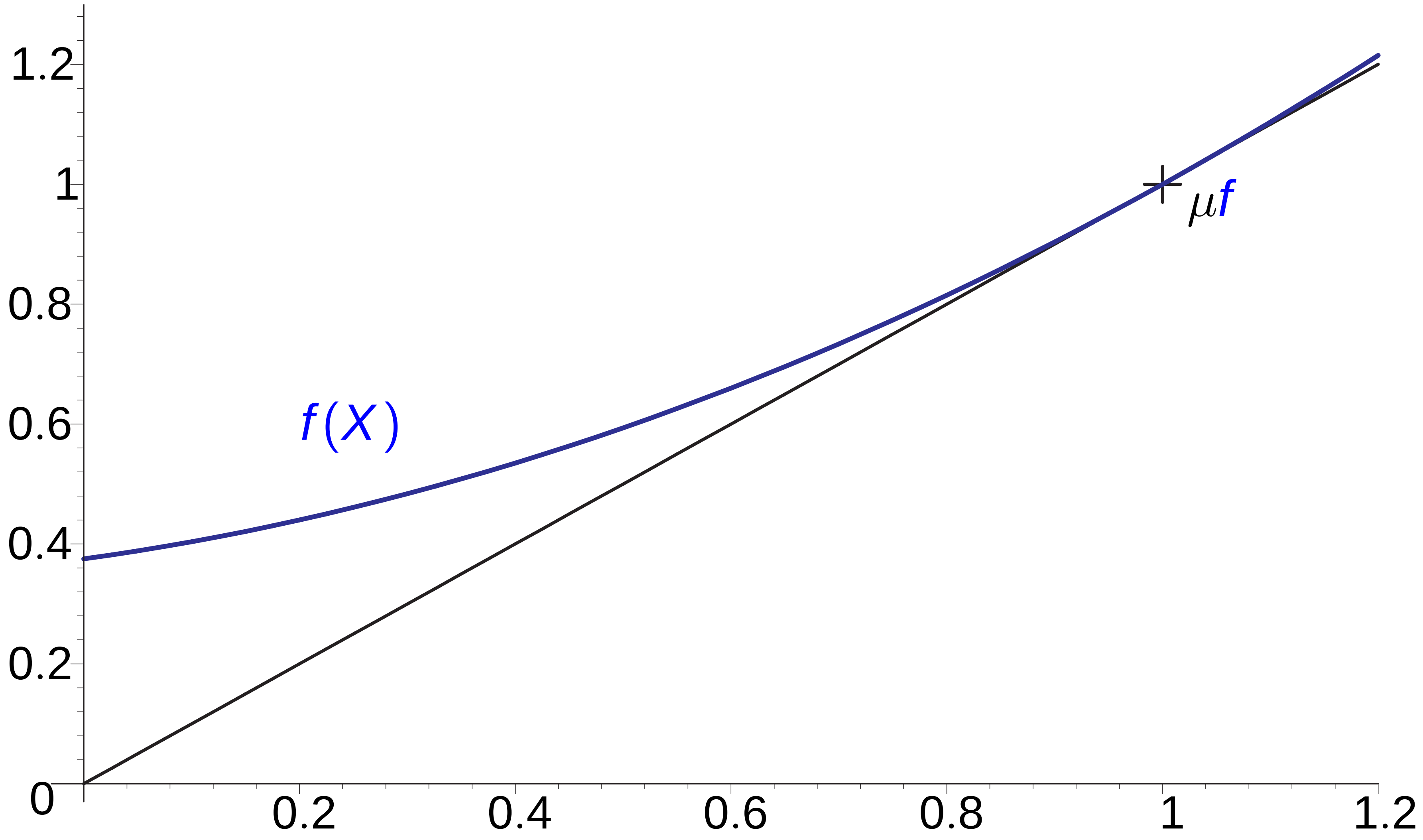
$$X = \frac{1}{2}X^2 + \frac{1}{2} \qquad \mu f = 1 = 0.99999\ldots$$

Logarithmic convergence: k iterations give $\Theta(\log k)$ accurate bits.

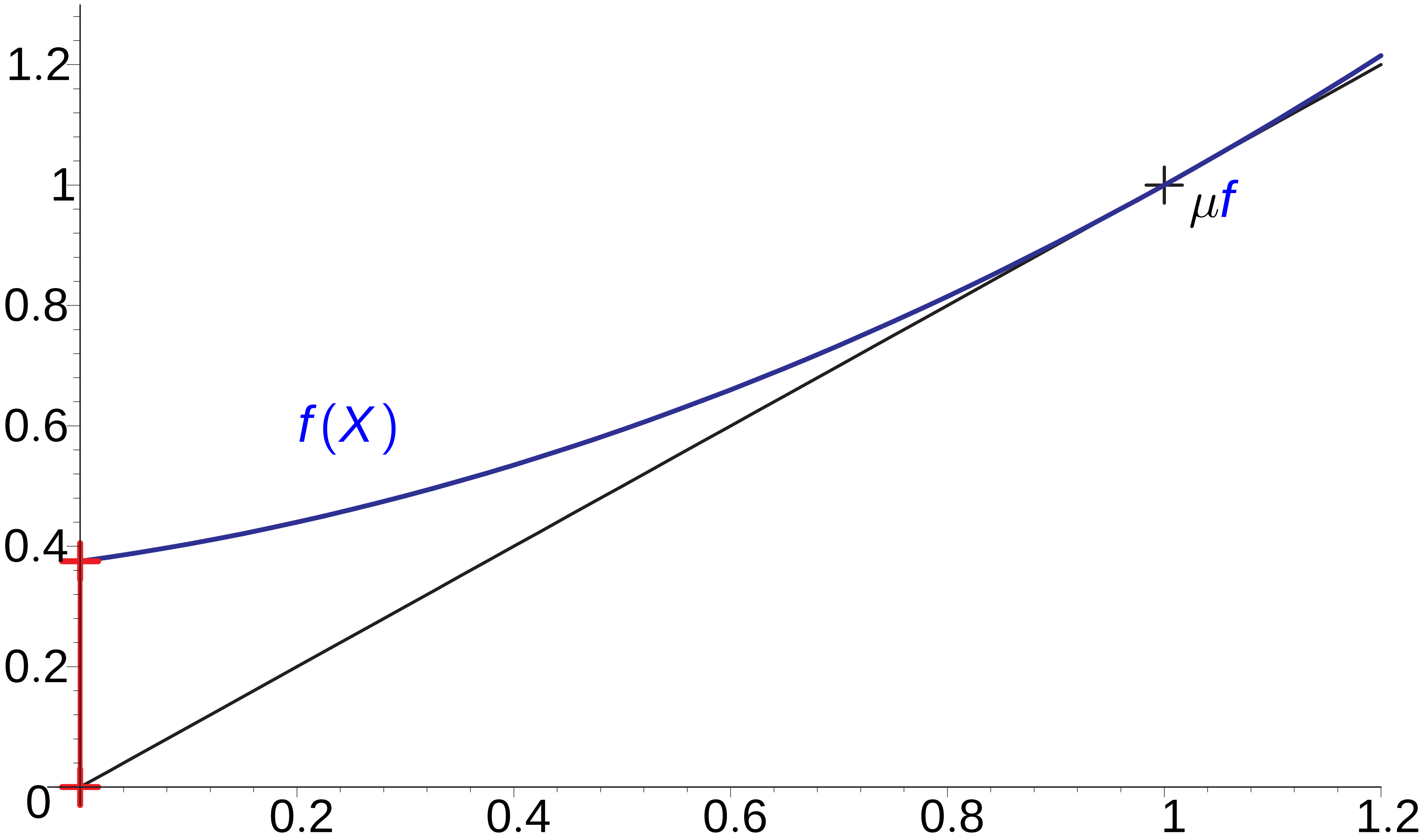
For example, $k_{2000} = 0.9990$

No reduction to computing Kleene stars

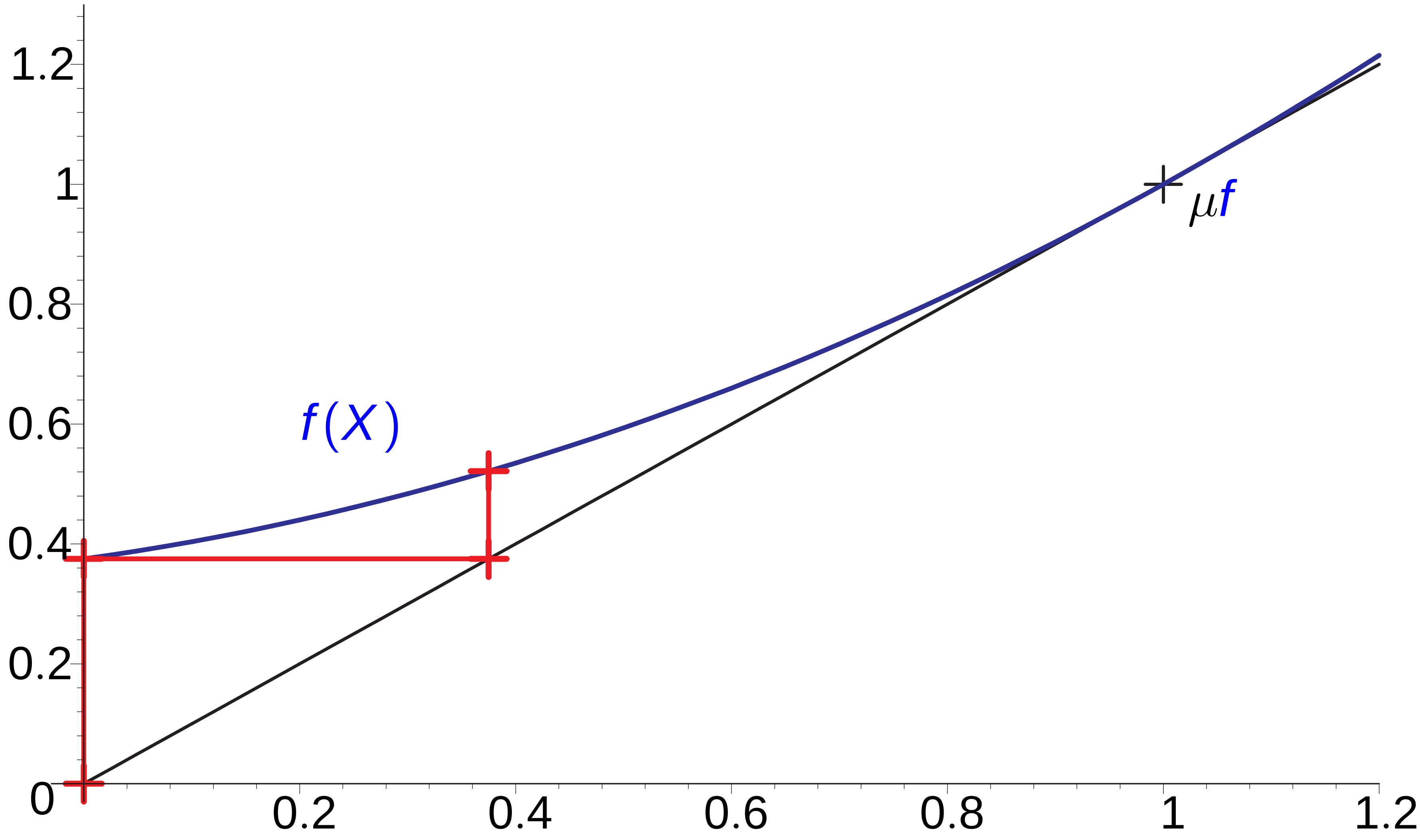
Kleene Iteration for $X = f(X)$ (univariate case)



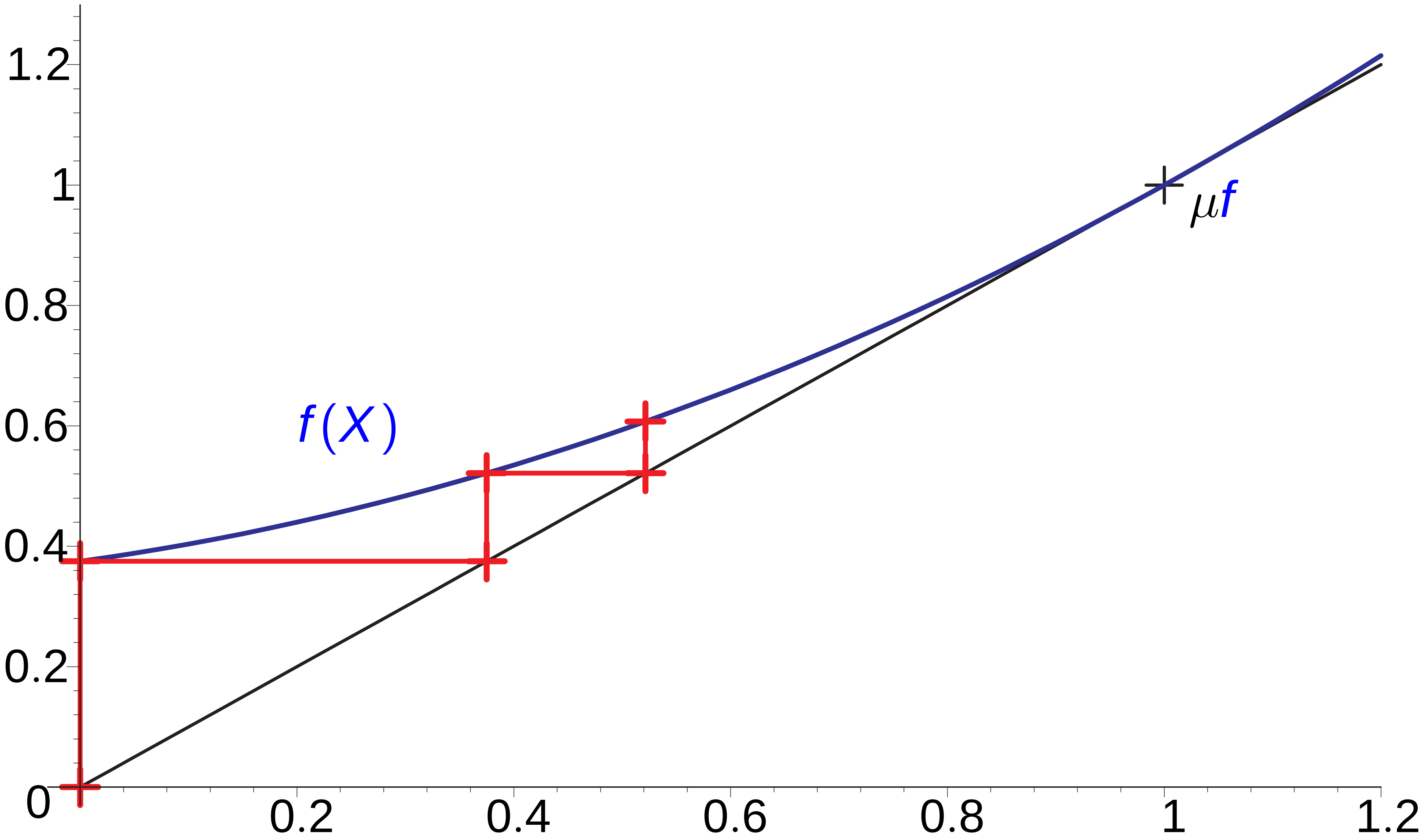
Kleene Iteration for $X = f(X)$ (univariate case)



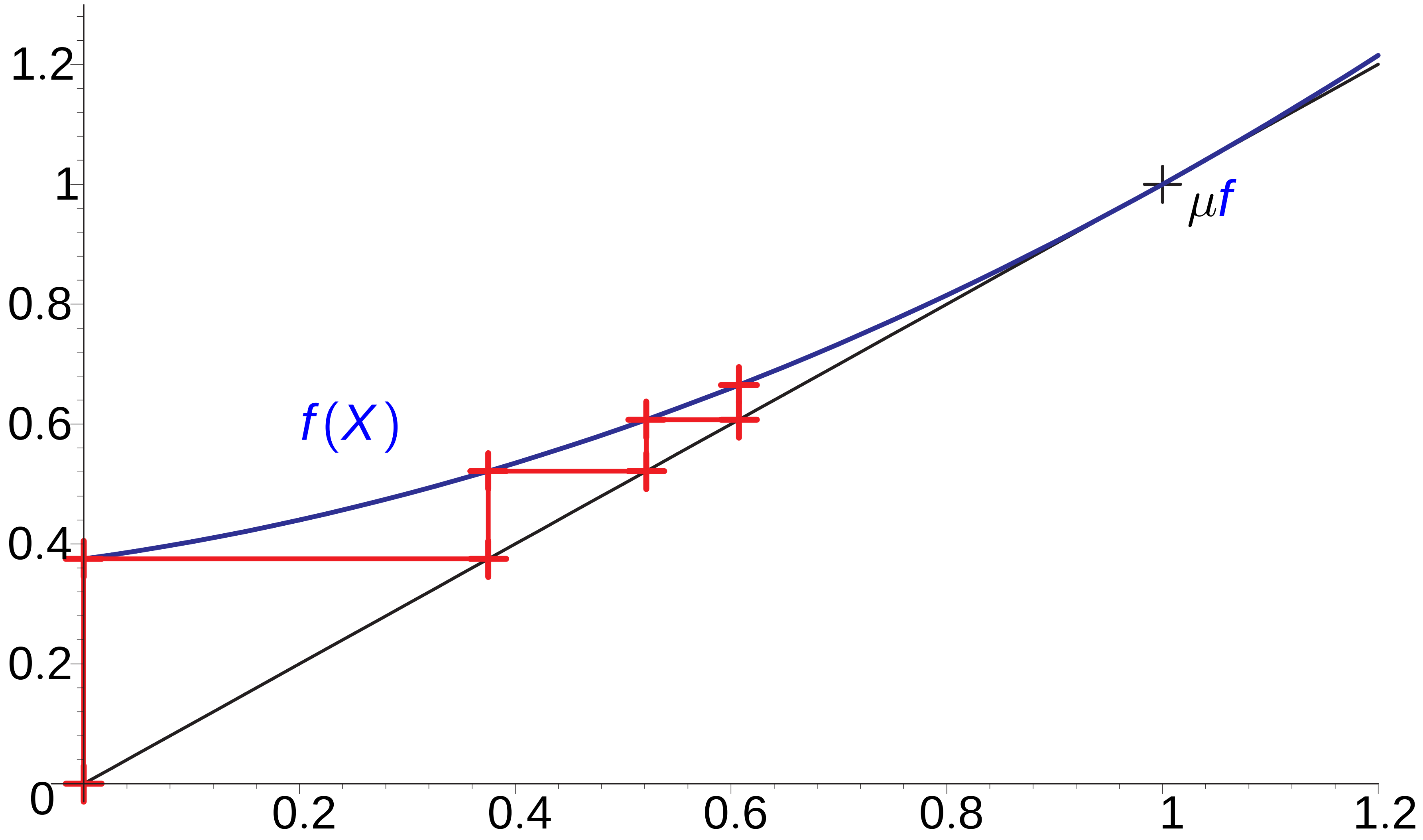
Kleene Iteration for $X = f(X)$ (univariate case)



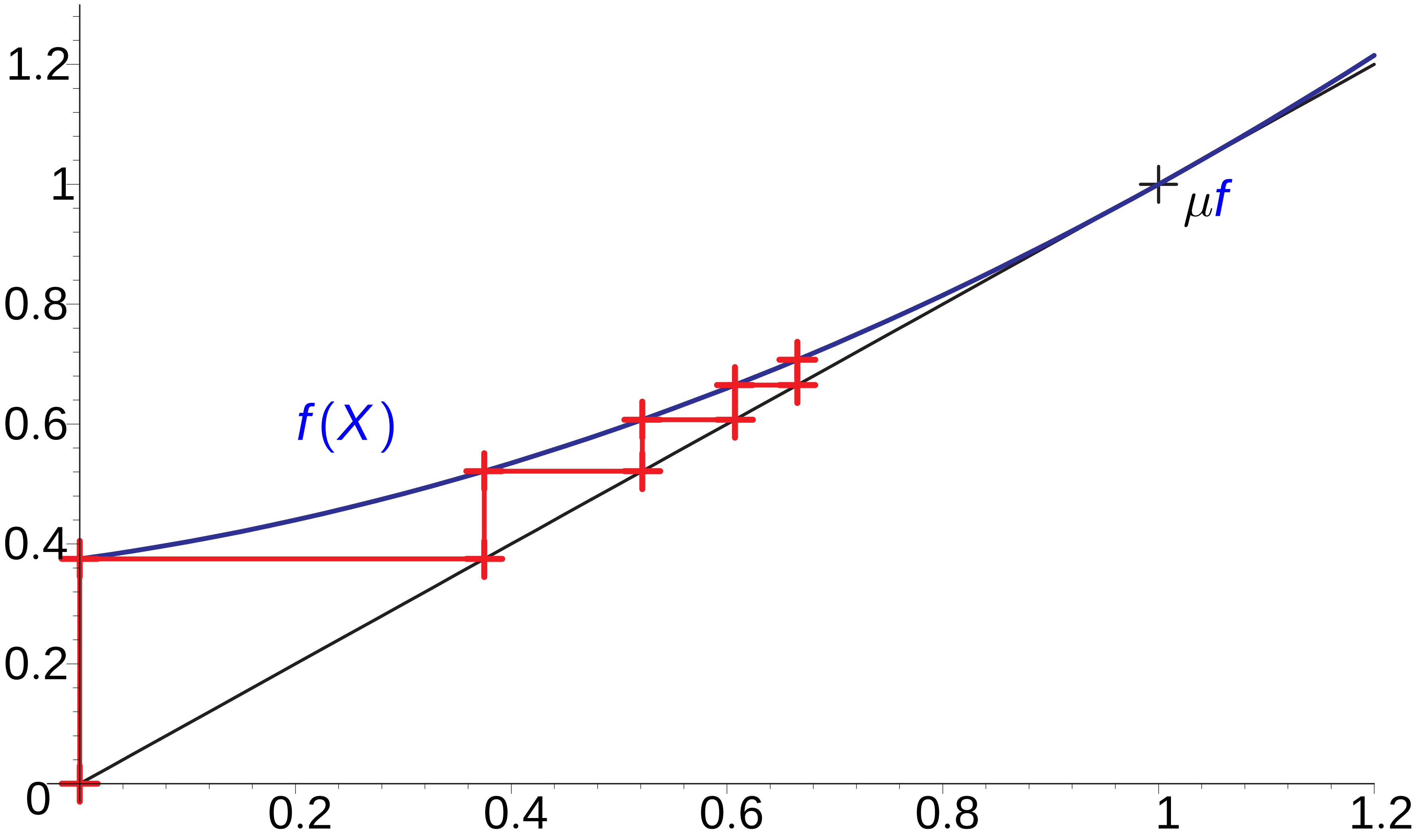
Kleene Iteration for $X = f(X)$ (univariate case)



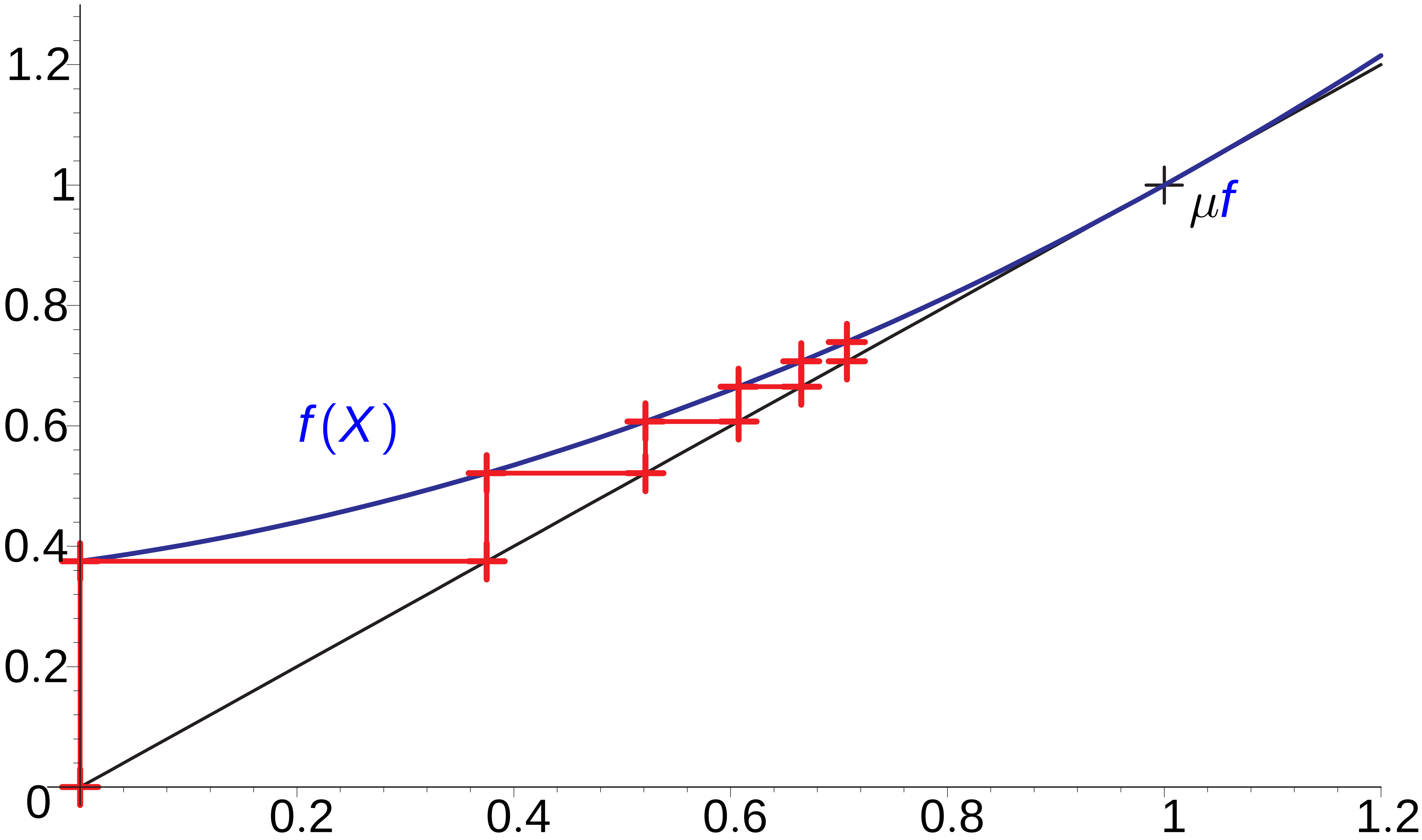
Kleene Iteration for $X = f(X)$ (univariate case)



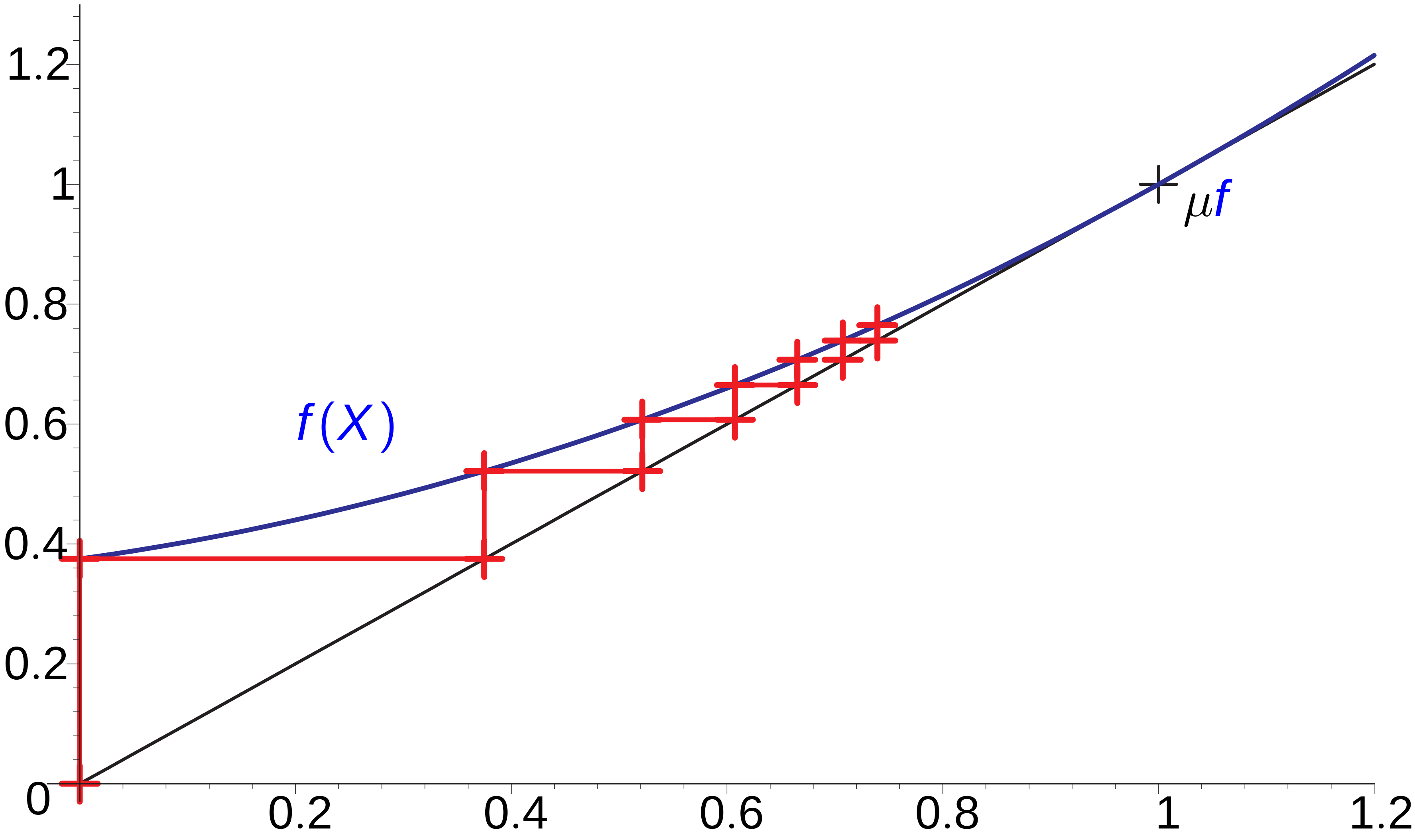
Kleene Iteration for $X = f(X)$ (univariate case)



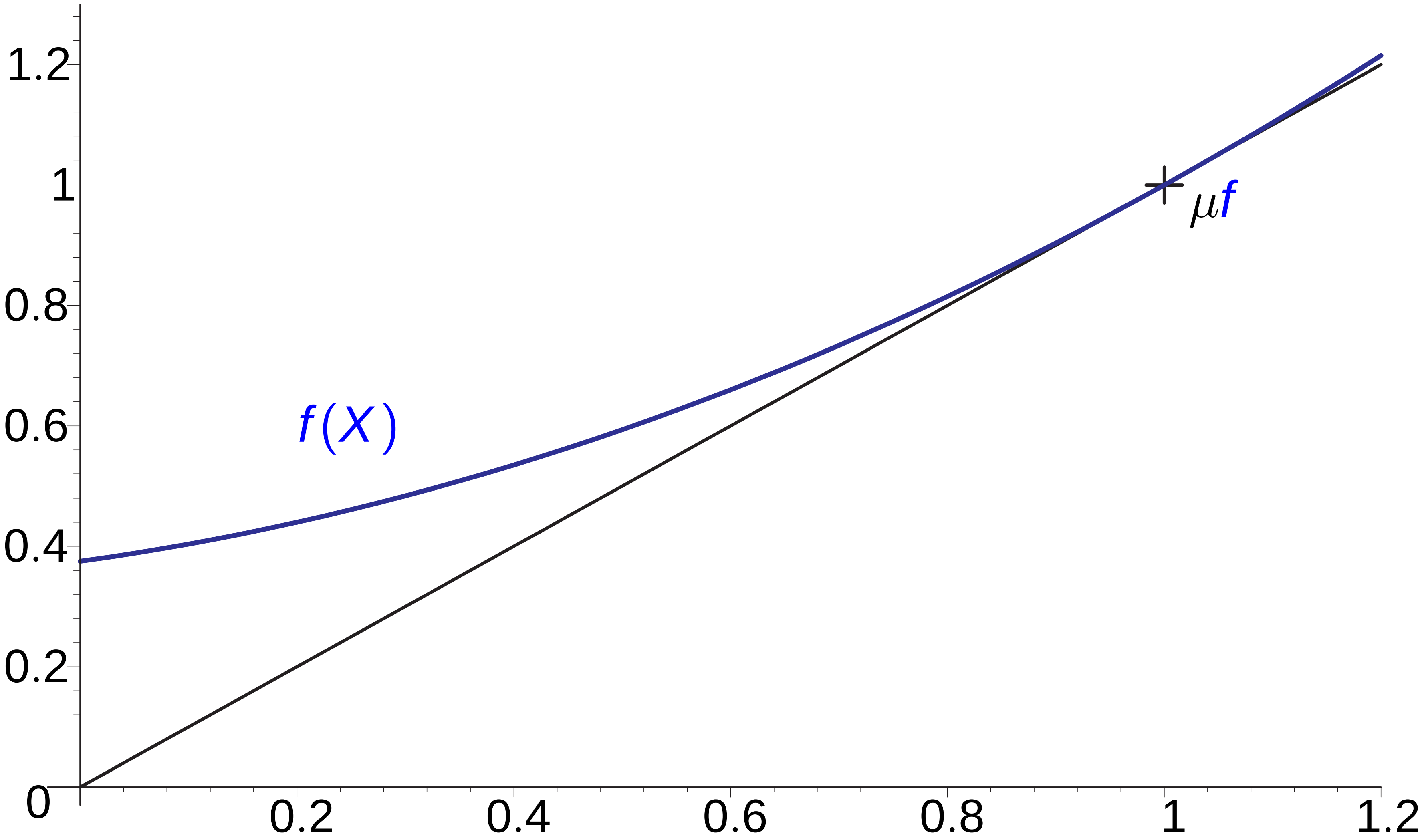
Kleene Iteration for $X = f(X)$ (univariate case)



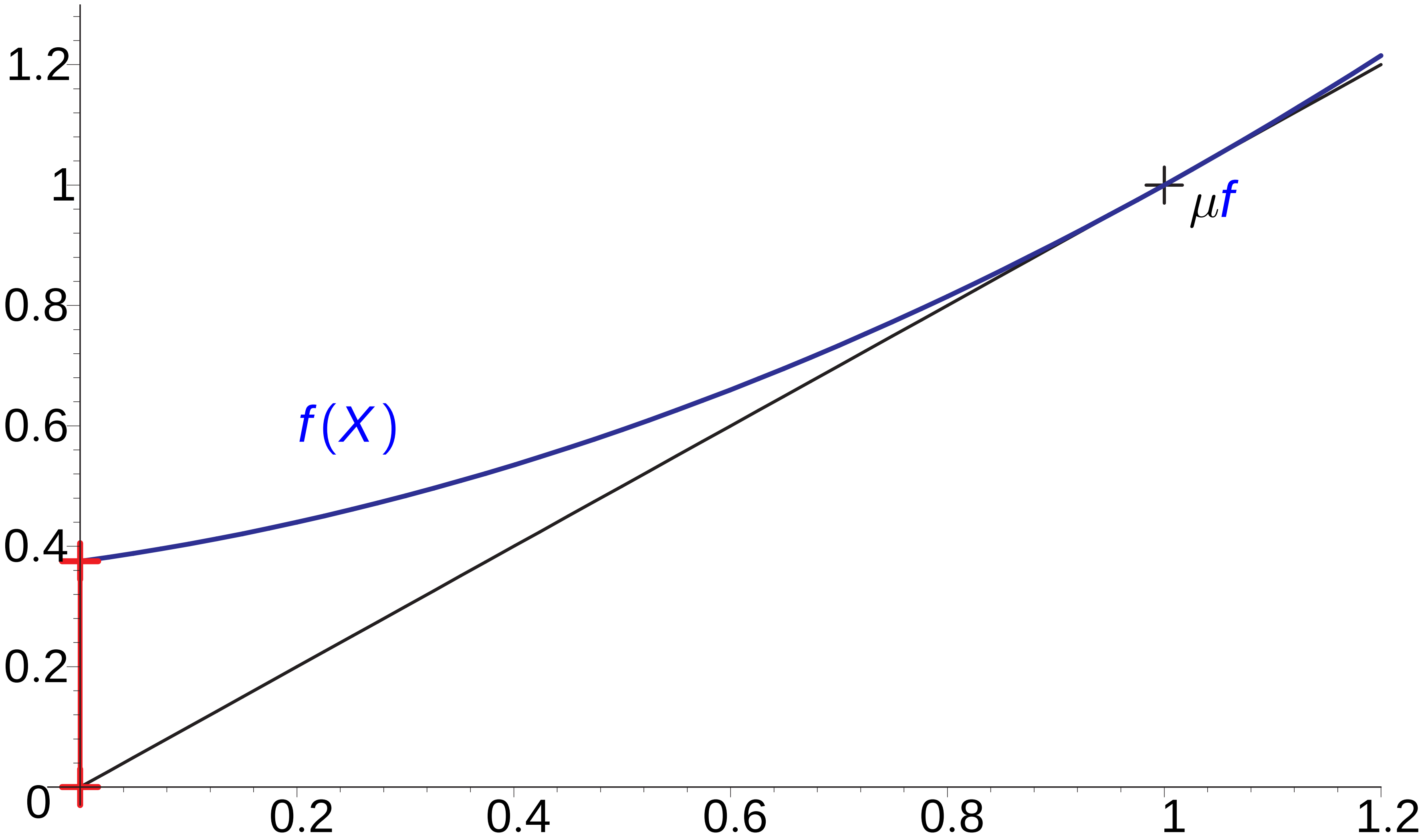
Kleene Iteration for $X = f(X)$ (univariate case)



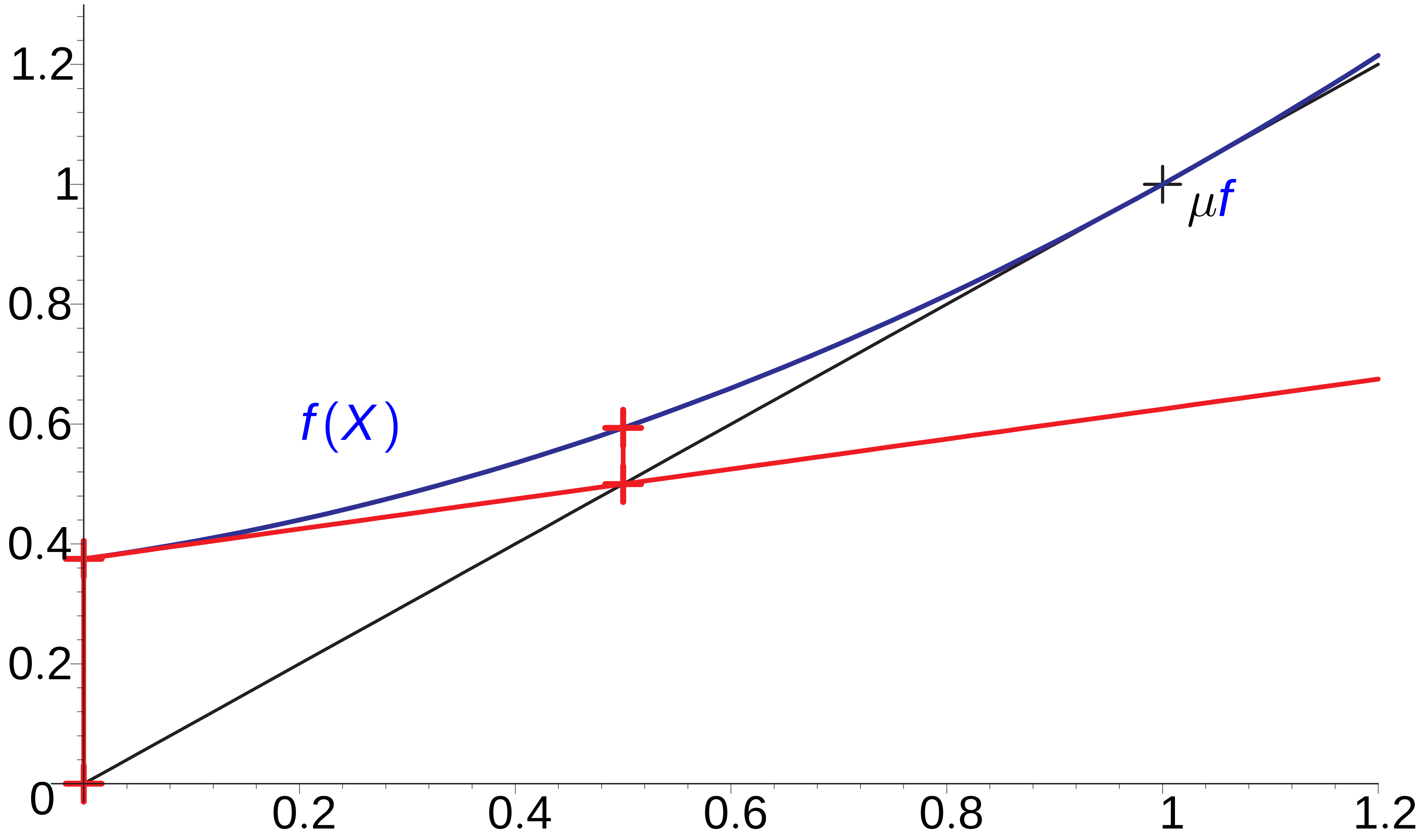
Newton's Method for $X = f(X)$ (univariate case)



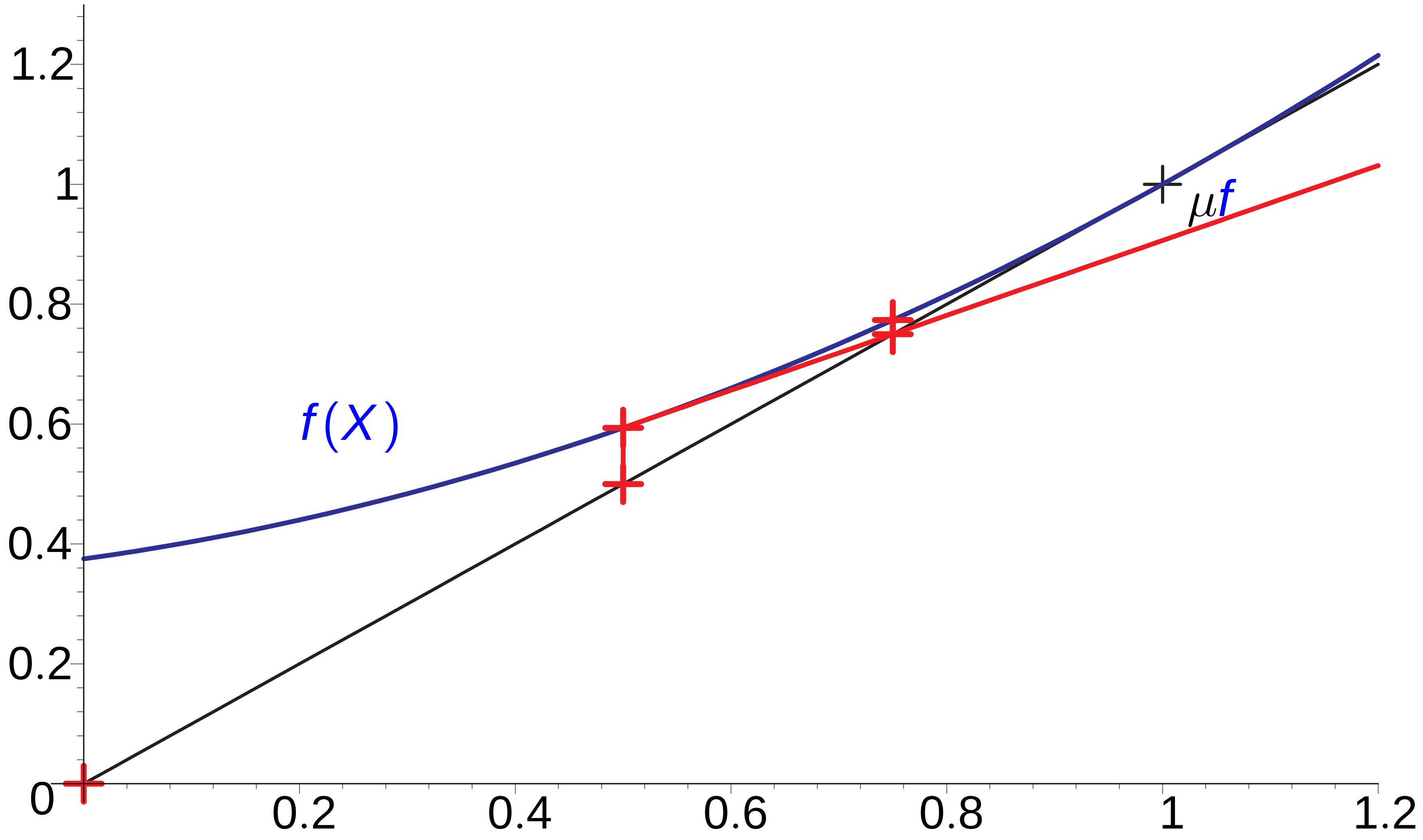
Newton's Method for $X = f(X)$ (univariate case)



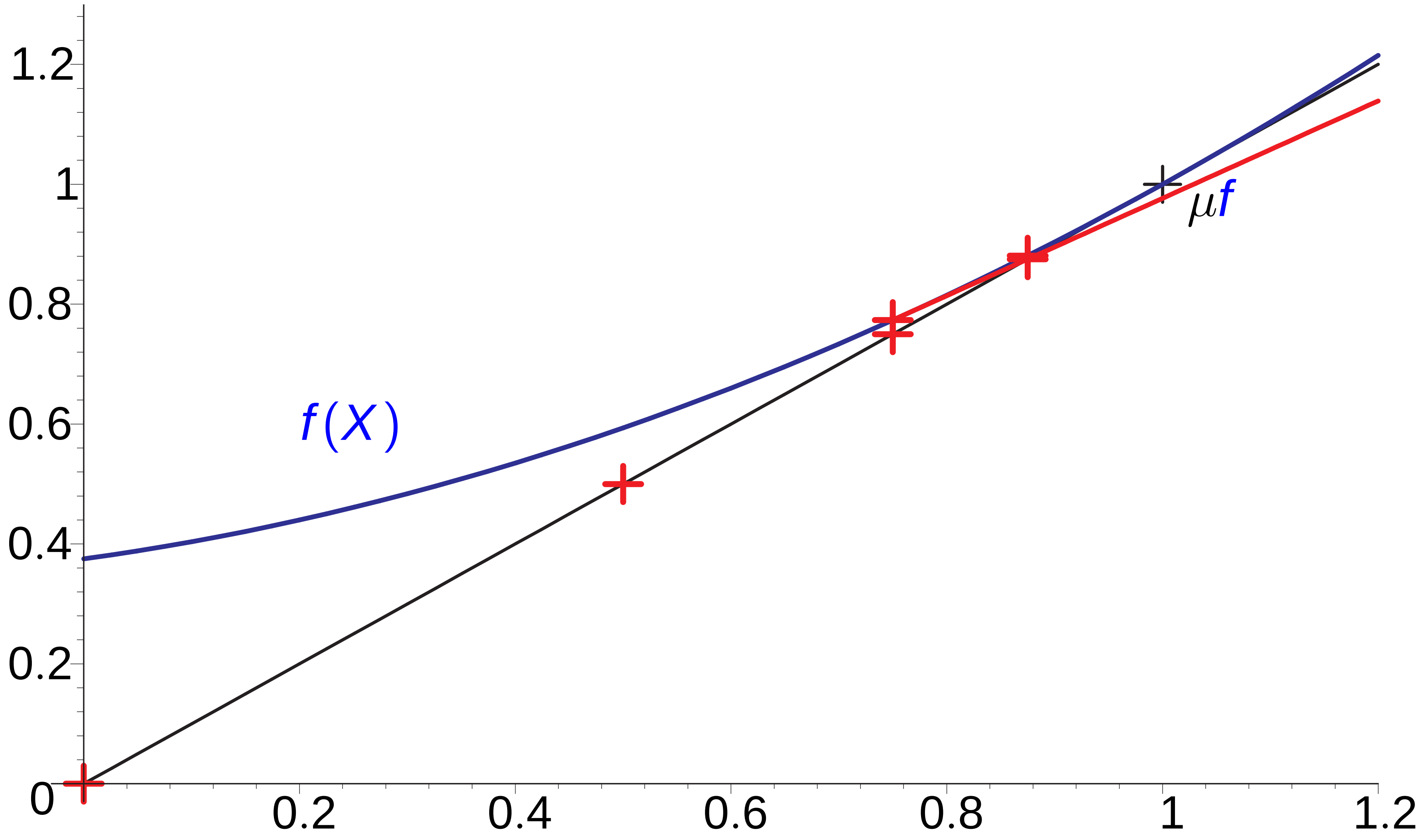
Newton's Method for $X = f(X)$ (univariate case)



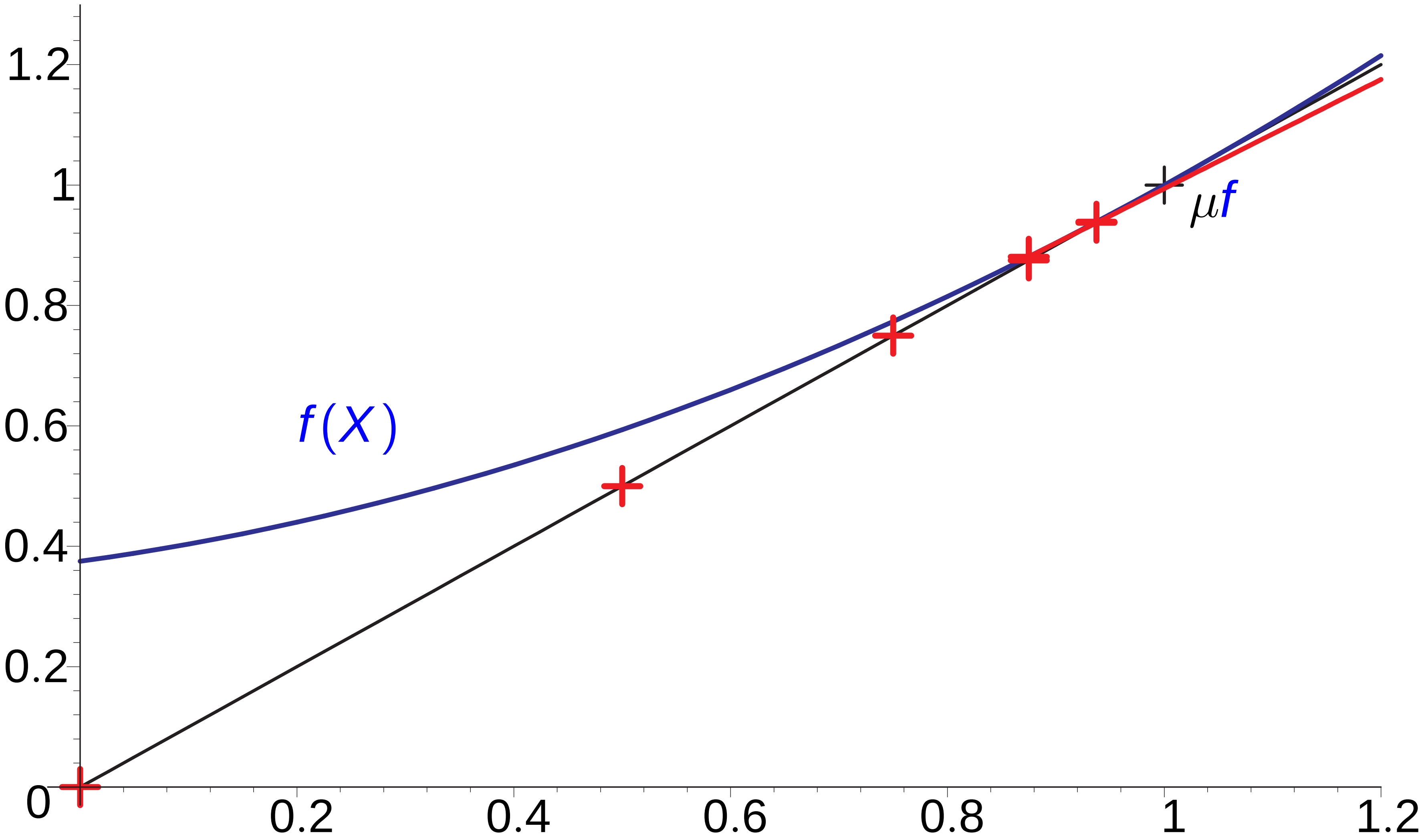
Newton's Method for $X = f(X)$ (univariate case)



Newton's Method for $X = f(X)$ (univariate case)



Newton's Method for $X = f(X)$ (univariate case)



Newton's method is usually very efficient

Often exponential convergence*: k iterations give $\Theta(2^k)$ bits.

but not robust.

May not converge, converge only locally (in some neighborhood of the least fixed-point), or converge very slowly.

* Called quadratic convergence in numerical mathematics.

A frustrating mismatch

- Kleene Iteration is robust and applicable to every semiring, but converges slowly.
- Newton's Method may converge very fast, but is not robust and can only be applied to the reals.

A frustrating mismatch and its solution

- Kleene Iteration is robust and applicable to every semiring, but converges slowly.
- Newton's Method may converge very fast, but is not robust and can only be applied to the reals.

Main results:

- Newton's Method can be generalized to arbitrary semirings, and becomes as robust as Kleene's method (our work).
- Newton's method converges at least linearly and often exponentially over the real semiring (some work by us + work by Etessami, Stewart and Yannakakis).

Generalizing Newton’s Method

Derivation trees I

An equation $X = f(X)$ over a semiring induces a context-free grammar G

Examples: $X = 0.3\,X^2 + 0.5$ induces $X \;\rightarrow\; 0.3\;X\;X \;\mid\; 0.5$

$X = 0.2\,XY + 0.3$ induces $X \;\rightarrow\; 0.2\;X\;Y \;\mid\; 0.8$

$Y = 0.7\,XY + 0.1$ $\quad Y \;\rightarrow\; 0.7\;X\;Y \;\mid\; 0.1$

Running example with arbitrary semiring elements a, b, c:

$X \;=\; a\,X^2 \;+\; b\,X \;+\; c$ and $G:\; X \;\rightarrow\; a\,X\,X \;\mid\; b\,X \;\mid\; c$

Derivation trees II

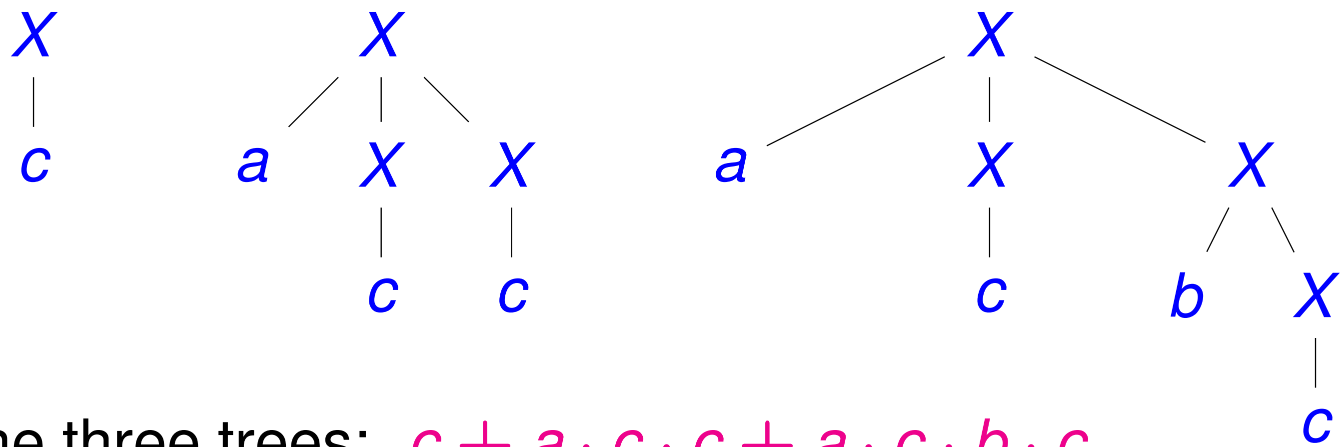
Assign to a derivation tree t its yield

$$Y(t) := \text{(ordered) product of the leaves of } t$$

Assign to a set T of derivation trees its yield

$$Y(T) := \text{sum of the yields of the elements of } T$$

$G: \; X \rightarrow aXX \mid bX \mid c$

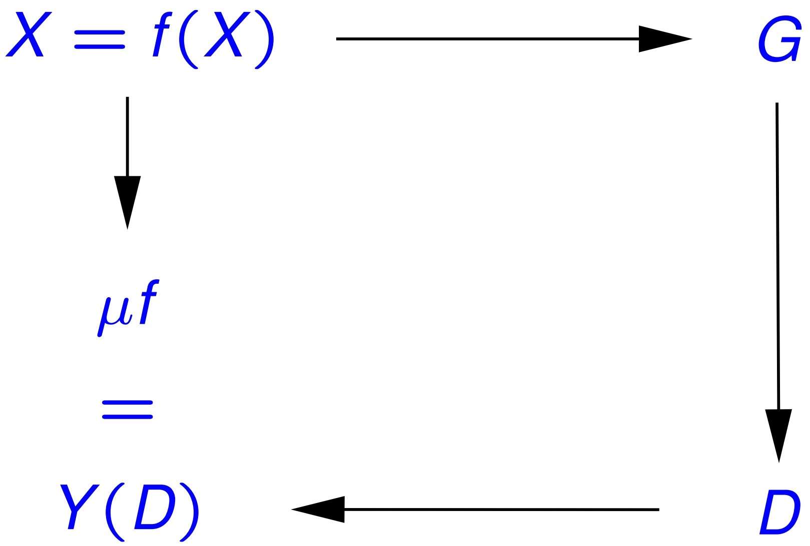


Yield of the three trees: $c + a \cdot c \cdot c + a \cdot c \cdot b \cdot c$

Derivation trees III

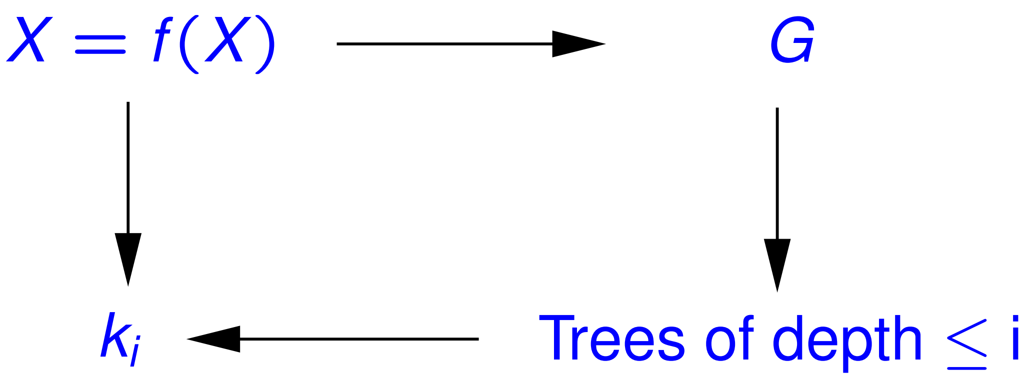
Proposition: Let D be the set of all derivation trees of G. Then

$$\mu f = Y(D)$$



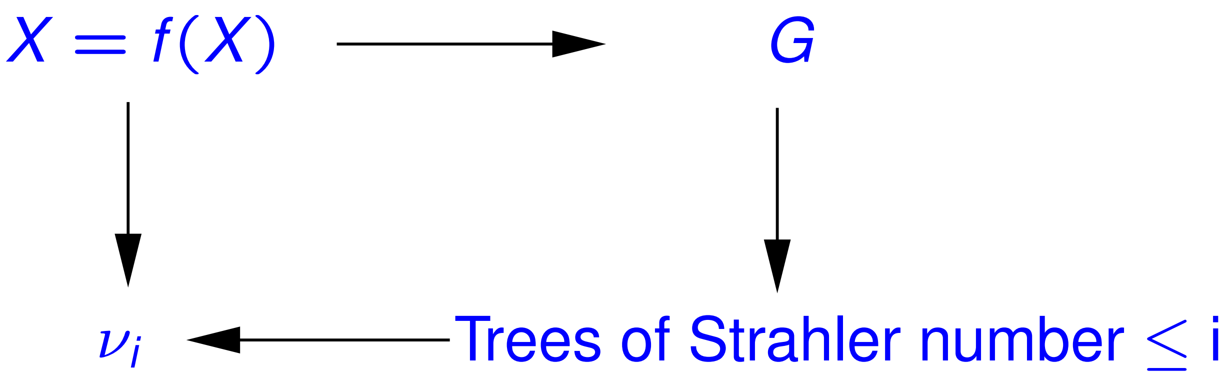
Approximants as yields: Kleene iteration

Proposition: The i-th Kleene approximant k_i is the yield of all derivation trees of height at most i.



Approximants as yields: Newton iteration

Theorem: The *i*-th Newton approximant ν_i is the yield of all derivation trees of Strahler number at most *i*.



Arthur N. Strahler (1952)

BULLETIN OF THE GEOLOGICAL SOCIETY OF AMERICA

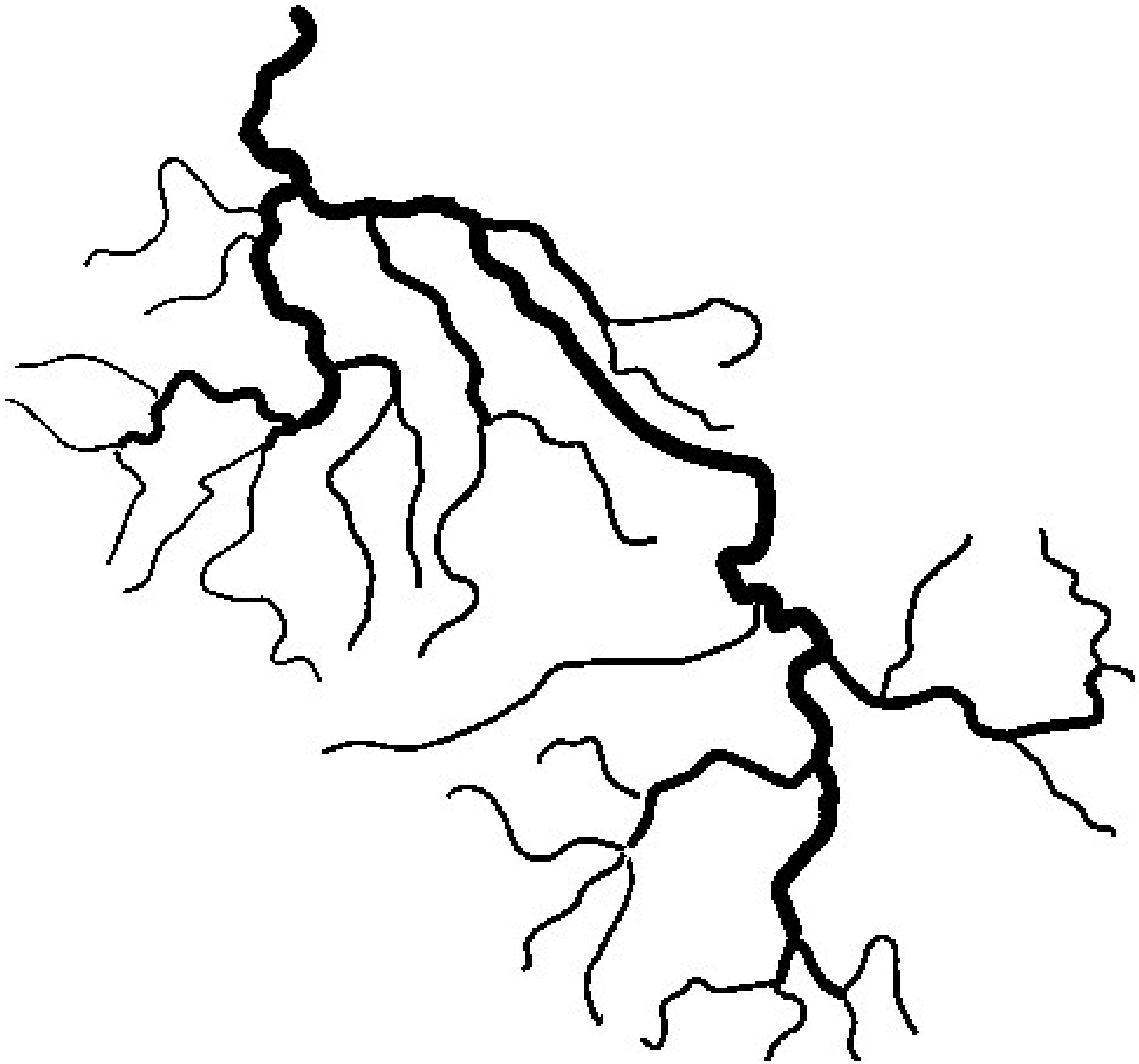
VOL. 63, PP. 1117-1142, 23 FIGS., 1 PL. NOVEMBER 1952

HYPSOMETRIC (AREA-ALTITUDE) ANALYSIS OF EROSIONAL TOPOGRAPHY

BY ARTHUR N. STRAHLER

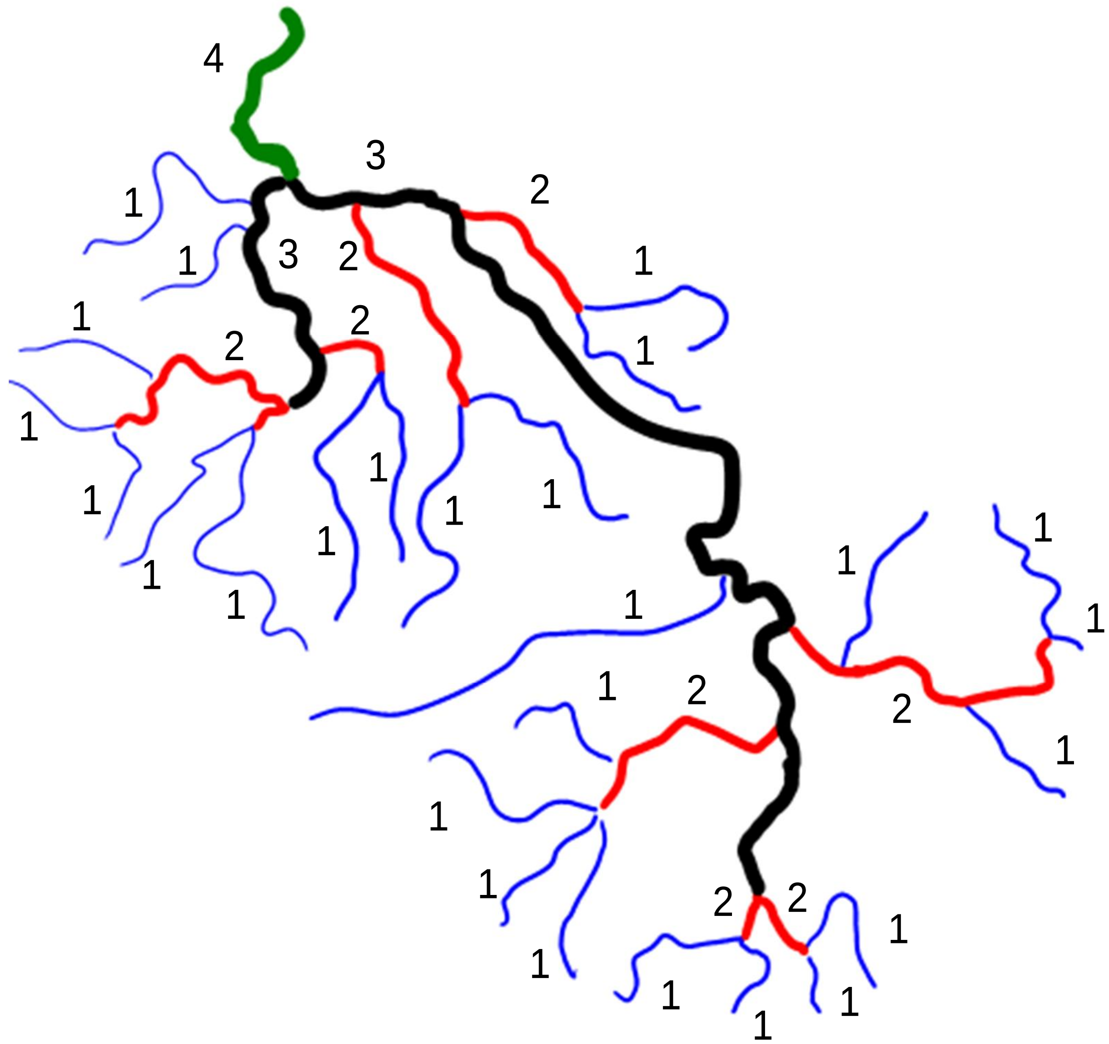
Arthur N.Strahler (1952)

Which is the main stream?



Arthur N.Strahler (1952)

The "finger-tip" channels constitute the first-order segments. [. . .].
A second-order segment is formed by the junction of any two first-order streams; a third-order segment is formed by the joining of any two second order streams, etc.
Streams of lower order joining a higher order stream do not change the order of the higher stream

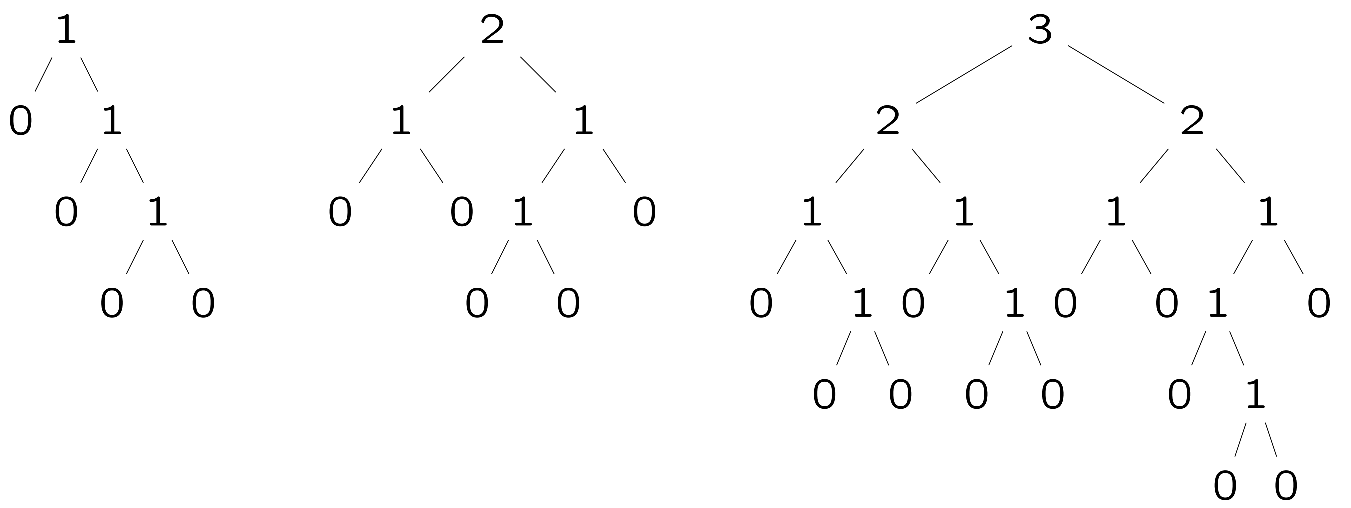


Strahler number of a tree

Definition: Strahler number $S(t)$ of a tree t:

If t has no subtrees (t has only one node), then $S(t) := 0$.

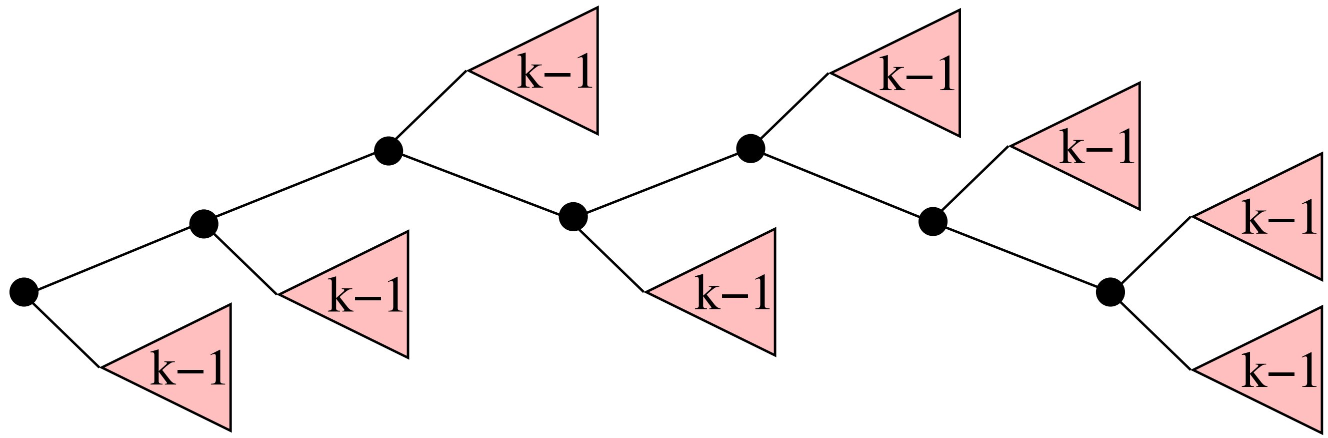
If t has subtrees $t_1, \ldots, t_n$, then let $k := \max\{S(t_1), \ldots, S(t_n)\}$.
If exactly one subtree of t has Strahler number k, then $S(t) := k$;
otherwise, $S(t) := k + 1$.



Understanding Strahler numbers

A tree has Strahler number $k > 0$ if it consists of a spine

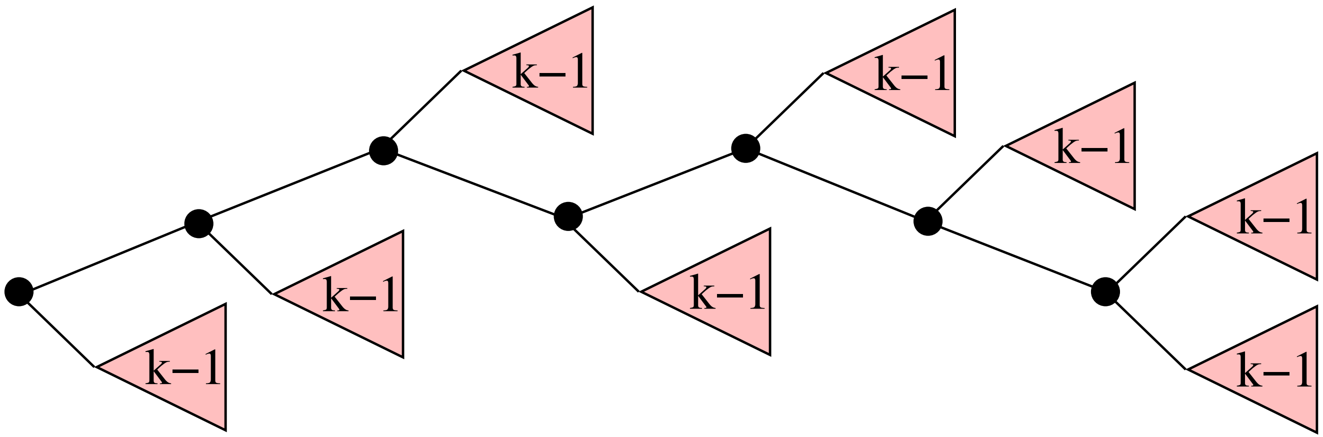
- with subtrees of Strahler number at most $k - 1$
- ending at a node with two subtrees of dimension exactly $k - 1$.



Understanding Strahler numbers

A binary tree tree has Strahler number $k > 0$ if it consists of a spine

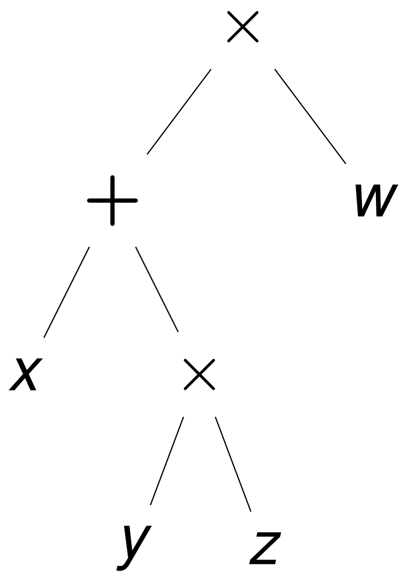
- with subtrees of Strahler number at most $k - 1$
- ending at subtrees of dimension exactly $k - 1$.



Characterizations of the Strahler number

Fact: The Strahler number of a tree is the height of the largest minor that is a full binary tree.

Fact: The Strahler number of an arithmetic expression is the minimal number of registers needed to evaluate it.



$R_1 \leftarrow y$

$R_2 \leftarrow z$

$R_2 \leftarrow R_1 \times R_2$

$R_2 \leftarrow x$

$R_1 \leftarrow R_1 + R_2$

$R_2 \leftarrow w$

$R_1 \leftarrow R_1 \times R_2$

Computing the k-th Newton approximant

$$G: \; X \rightarrow aXX \mid bX \mid c$$

Define grammars $G_0, G_1, G_2 \ldots$ such that:

trees of G_k $=$ trees of G of Strahler number $\leq k$

Computing the k-th Newton approximant

$$G: \; X \rightarrow a\,X\,X \;\mid\; b\,X \;\mid\; c$$

Define grammars $G_0, G_1, G_2 \ldots$ such that:

$$\text{trees of } G_k \;=\; \text{trees of } G \text{ of Strahler number} \leq k$$

Idea: $X_{\langle k \rangle}$ generates the trees of Strahler number $= k$

$X_{[k]}$ generates the trees of Strahler number $\leq k$

$$\begin{array}{llll}
G_k: & X_{[k]} & \rightarrow & X_{\langle k \rangle} \mid X_{[k-1]} \\
 & X_{\langle k \rangle} & \rightarrow & aX_{\langle k-1 \rangle}X_{\langle k-1 \rangle} \mid aX_{[k-1]}X_{\langle k \rangle} \mid aX_{\langle k \rangle}X_{[k-1]} \mid bX_{\langle k \rangle} \\
 & \ldots & & \\
G_1: & X_{[1]} & \rightarrow & X_{\langle 1 \rangle} \mid X_{[0]} \\
 & X_{\langle 1 \rangle} & \rightarrow & aX_{[0]}X_{[0]} \mid aX_{[1]}X_{[0]} \mid aX_{[0]}X_{[1]} \mid bX_{\langle 1 \rangle} \\
G_0: & X_{[0]} & \rightarrow & c
\end{array}$$

Computing the k-th Newton approximant

$$G: \ X \ \rightarrow \ a\,X\,X \ \mid \ b\,X \ \mid \ c$$

Define grammars $G_0, G_1, G_2 \ldots$ such that:

$$\text{trees of } G_k \ = \ \text{trees of } G \text{ of Strahler number} \leq k$$

Idea: $X_{\langle k \rangle}$ generates the trees of Strahler number $= k$

$X_{[k]}$ generates the trees of Strahler number $\leq k$

$$\begin{aligned}
X_{[k]} &= X_{\langle k \rangle} + X_{[k-1]} \\
X_{\langle k \rangle} &= aX_{\langle k-1 \rangle}X_{\langle k-1 \rangle} + aX_{[k-1]}X_{\langle k \rangle} + aX_{\langle k \rangle}X_{[k-1]} + bX_{\langle k \rangle} \\
\ldots \\
X_{[1]} &= X_{\langle 1 \rangle} + X_{[0]} \\
X_{\langle 1 \rangle} &= aX_{[0]}X_{[0]} + aX_{[1]}X_{[0]} + aX_{[0]}X_{[1]} + bX_{\langle 1 \rangle} \\
X_{[0]} &= c
\end{aligned}$$

Computing the k-th Newton approximant

The equation

$$X_{\langle k\rangle} \;=\; aX_{\langle k-1\rangle}X_{\langle k-1\rangle} + aX_{[k-1]}X_{\langle k\rangle} + aX_{\langle k\rangle}X_{[k-1]} + bX_{\langle k\rangle}$$

is linear in $X_{\langle k\rangle}$.

Computing the k-th Newton approximant

The equation

$$X_{\langle k \rangle} = aX_{\langle k-1 \rangle}X_{\langle k-1 \rangle} + aX_{[k-1]}X_{\langle k \rangle} + aX_{\langle k \rangle}X_{[k-1]} + bX_{\langle k \rangle}$$

is linear in $X_{\langle k \rangle}$.

Computing the k-th Newton approximant

The equation

$$X_{\langle k \rangle} \;=\; aX_{\langle k-1 \rangle}X_{\langle k-1 \rangle} + aX_{[k-1]}X_{\langle k \rangle} + aX_{\langle k \rangle}X_{[k-1]} + bX_{\langle k \rangle}$$

is linear in $X_{\langle k \rangle}$.

> Newton's method approximates the solution of non-linear equations
> assuming
> one can solve (or approximate) linear equations.

Computing the k-th Newton approximant

The equation

$$X_{\langle k\rangle} \;=\; aX_{\langle k-1\rangle}X_{\langle k-1\rangle} + aX_{[k-1]}X_{\langle k\rangle} + aX_{\langle k\rangle}X_{[k-1]} + bX_{\langle k\rangle}$$

is linear in $X_{\langle k\rangle}$.

Newton's method approximates the solution of non-linear equations
assuming
one can solve (or approximate) linear equations.

Over commutative semirings:

$$\begin{aligned} X_{\langle k\rangle} &= (2aX_{[k-1]} + b)X_{\langle k\rangle} + aX_{\langle k-1\rangle}X_{\langle k-1\rangle} \\ X_{\langle k\rangle} &= (2aX_{[k-1]} + b)^{*}\, a\, X_{\langle k-1\rangle}^{2} \end{aligned}$$

Some applications

Termination proofs

Theorem: Let $X = f(X)$ be a system of n polynomial fixpoint equations over an idempotent and commutative semiring. Then Newton's method terminates after at most $n + 1$ iterations.

Idempotent: number of copies of each tree is irrelevant

Commutative: trees with different order of leaves have the same yield.

Prove that for every derivation tree there is a derivation tree of Strahler number at most $n + 1$ with the same leaves, possibly in different order

Parikh's theorem

Theorem [Parikh '66]: For every context-free language there is a regular language with the same commutative image.

Problem: Given a CFG G, construct an automaton A such that $L(G)$ and $L(A)$ have the same commutative image.

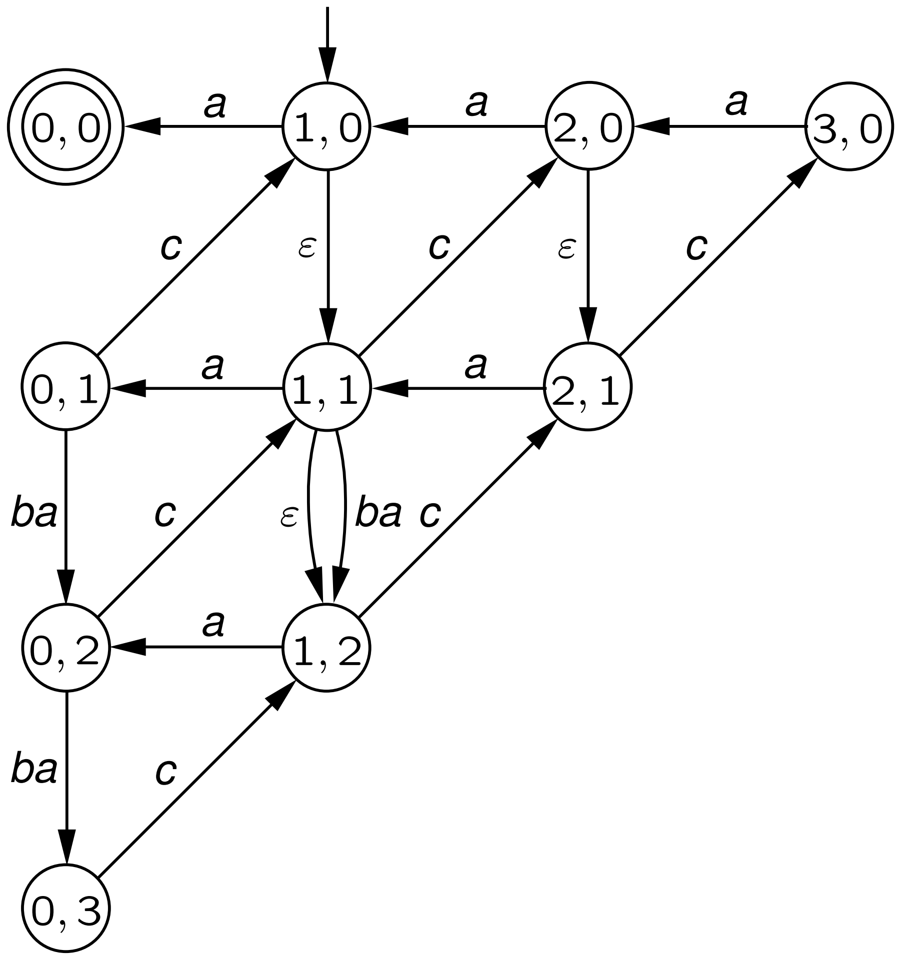
Solution: Use that $L(G)$ and $L(G_n)$ have the same commutative image.

Construct A whose runs "simulate" the derivations of G_n.

Parikh's theorem

Example: $A_1 \to A_1A_2 \mid a \qquad A_2 \to bA_2aA_2 | cA_1$

$$
\begin{array}{llcl}
 & A_1 & & (0,1) \\
\Rightarrow & A_1A_2 & \xrightarrow{\epsilon} & (1,1) \\
\Rightarrow & A_1bA_2aA_2 & \xrightarrow{ba} & (1,2) \\
\Rightarrow & A_1bcA_1aA_2 & \xrightarrow{c} & (2,1) \\
\Rightarrow & abcA_1aA_2 & \xrightarrow{a} & (1,1) \\
\Rightarrow & abcaaA_2 & \xrightarrow{a} & (0,1) \\
\Rightarrow & abcaacA_1 & \xrightarrow{c} & (1,0) \\
\Rightarrow & abcaaca & \xrightarrow{a} & (0,0)
\end{array}
$$



A recommendation system

Idea: design a recommendation system in which:

- individuals can recommend other individuals or groups;
- membership in groups is defined in a fuzzy quantitative way;
- groups can be defined recursively (the friends of my friends are my friends)

Participants: researchers, universities, conferences, papers . . .

Relations: researcher-of, professor-at, student-of, author-of, . . .

- Notation: $p.r$
- Meaning: group of participants that are in relation r with p.

Rules

Information about group membership expressed through membership rules

$$\text{Aachen.professor} \longleftarrow \text{Grädel}$$

$$\text{Aachen.researcher} \longleftarrow \text{Aachen.professor}$$

$$\text{Aachen.researcher} \longleftarrow \text{Aachen.researcher.phd-student}$$

$$\text{Grädel.phd-student} \longleftarrow \text{Naaf}$$

Inference

To establish that Naaf is a researcher at Aachen:

$$\text{Aachen.researcher} \longleftarrow \text{Aachen.researcher.phd-student}$$

Inference

To establish that Naaf is a researcher at Aachen:

$$\text{Aachen.researcher} \longleftarrow \text{Aachen.researcher.phd-student}$$
$$\longleftarrow \text{Aachen.professor.phd-student}$$

Inference

To establish that Naaf is a researcher at Aachen:

$$\text{Aachen.researcher} \longleftarrow \text{Aachen.researcher.phd-student}$$
$$\longleftarrow \text{Aachen.professor.phd-student}$$
$$\longleftarrow \text{Grädel.phd-student}$$

Inference

To establish that Naaf is a researcher at Aachen:

$$
\begin{aligned}
\text{Aachen.researcher} &\longleftarrow \text{Aachen.researcher.phd-student} \\
&\longleftarrow \text{Aachen.professor.phd-student} \\
&\longleftarrow \text{Grädel.phd-student} \\
&\longleftarrow \text{Naaf}
\end{aligned}
$$

Weights

Membership rules carry weights (degree of membership)

$$\text{LICS.author} \xleftarrow{14/1000} \text{Grädel}$$

$$\text{LICS.author} \xleftarrow{1/1000} \text{Naaf}$$

Weights

Membership rules carry weights (degree of membership)

$$\text{LICS.author} \xleftarrow{14/1000} \text{Grädel}$$

$$\text{LICS.author} \xleftarrow{1/1000} \text{Naaf}$$

$$\text{Grädel.co-author} \xleftarrow{1/30} \text{Naaf}$$

$$\text{Naaf.co-author} \xleftarrow{1/2} \text{Grädel}$$

Weights

Membership rules carry weights (degree of membership)

$$\text{LICS.author} \xleftarrow{14/1000} \text{Grädel}$$

$$\text{LICS.author} \xleftarrow{1/1000} \text{Naaf}$$

$$\text{Grädel.co-author} \xleftarrow{1/30} \text{Naaf}$$

$$\text{Naaf.co-author} \xleftarrow{1/2} \text{Grädel}$$

Recursive group definitions with damping factors.

$$\text{Grädel.community} \xleftarrow{1} \text{Grädel.co-author}$$

$$\text{Grädel.community} \xleftarrow{0.5} \text{Grädel.community.co-author}$$

Recommending individuals and groups

Recommendations

$$\text{Grädel} \xleftarrow{\;10\;} \text{Naaf}$$

$$\text{Grädel} \xleftarrow{\;8\;} \text{LICS.author}$$

Grädel recommends Naaf through two paths

$$\text{Grädel} \xleftarrow{\;10\;} \text{Naaf}$$

$$\text{Grädel} \xleftarrow{\;8\;} \text{LICS.author} \xleftarrow{\;1/1000\;} \text{Naaf}$$

Semiring operations $\oplus$ and $\odot$ to aggregate values:

$$\text{Grädel} \xleftarrow{\;10 \oplus (8 \odot 1/30)\;} \text{Naaf}$$

Computing reputation

Assume a given set of weighted rules and recommendations

Reputation of an individual: total weight with which participants recommend the individual (through all possible paths)

Theorem: The reputation of the individuals is the least fixpoint of a system of non-linear equations

Thank you for your attention