The Complexity of Dynamic Least-Squares Regression

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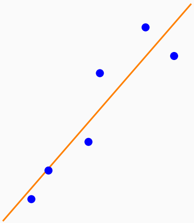
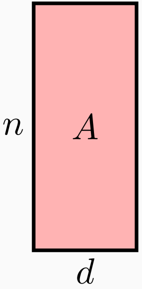
Least squares regression

- **Problem**: $$\min_{x \in \mathbb{R}^d} \|Ax - b\|_2$$
- **Applications** in high-dimensional statistical inference, signal processing, machine learning, etc.
- **Exact solution** (Normal equation): $x^* = (A^\top A)^{-1} A^\top b$
  - **Time complexity**: $O(nd^{\omega-1})$
  - Still too slow for many modern data-analysis applications.
- ϵ-**approximate solution**: $\|Ax - b\|_2 \le (1+\epsilon) \min_{x' \in \mathbb{R}^d} \|Ax' - b\|_2$
  - "Sketch and solve" paradigm [Woo14]
  - **Time complexity**: $\widetilde{O}\left((\mathrm{nnz}(A) + d^\omega) \log(1/\epsilon)\right)$ [CW17]



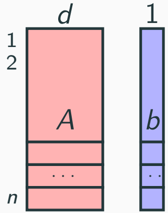


Dynamic least squares regression

- **Problem**: Dynamically maintain an ϵ-approximate LSR solution

$$\min_{x \in \mathbb{R}^d} \|A^{(i)}x - b^{(i)}\|_2,$$

  under insertion or deletion of rows $a^{(i)} \in \mathbb{R}^d$ and labels $\beta^{(i)} \in \mathbb{R}$.
  - Goal: minimize **amortized update time**.
  - In total n iterations, think of $n = \text{poly}(d)$.



- Models dynamic data applications, e.g., continual ML.
- **Incremental vs Fully dynamic**
  - Incremental: Only insertions of rows.
  - Fully dynamic: Both insertions and deletions of rows.
- **Oblivious updates vs Adaptive updates**
  - Oblivious updates: The sequence of updates are fixed in the beginning.
  - Adaptive updates: The next update is generated based on the previous outputs.

Algorithms for dynamic least squares regression

- **Exact solution**: Update the normal equation $x^{*,(i)} = (A^{(i)\top} A^{(i)})^{-1} A^{(i)\top} b^{(i)}$ using Woodbury identity. (Kalman filters [Kal60])
  - Works for fully dynamic and adaptive updates.
  - Time per update: $O(d^2)$.
- **Online row sampling** [CMP20]: Maintain an ϵ-approximate solution by sampling $O(d \log \kappa / \epsilon^2)$ number of rows, where $\kappa := \frac{\sigma_{\max}(A^{(n)})}{\sigma_{\min}(A^{(0)})}$.
  - Works for incremental and oblivious updates.
  - Time per update: $O(d^2)$ (to compute sampling probability).
- **Adaptive online row sampling** [BHM+21]: Sample $O(d^2 \kappa \log \kappa / \epsilon^2)$ number of rows, where $\kappa := \frac{\sigma_{\max}(A^{(n)})}{\sigma_{\min}(A^{(0)})}$.
  - Works for incremental and adaptive updates.
  - Time per update: $O(d^2)$.
- **Question:** Can we achieve $O(d)$ time per update / $O(nd)$ total time?

Our results: Upper bound

Theorem (Upper bound). There is a dynamic data structure that maintains an ϵ-approximate LSR solution under *oblivious incremental* updates, with total time $\widetilde{O}\big(nd + d^3\,\mathrm{poly}(\epsilon^{-1})\big)$. The data structure can be made to work against *adaptive incremental* updates with total time $\widetilde{O}\big(nd + d^5\,\mathrm{poly}(\epsilon^{-1})\big)$.

- When $n \gg d$ and ϵ is a small constant, the amortized cost per iteration is $\widetilde{O}(d)$.
- The *nd* term is in fact $\mathrm{nnz}(A^{(n)})$.
- For *adaptive* incremental updates, we improve the number of sampled rows from $O(d^2 \kappa \log \kappa/\epsilon^2)$ [BHM+21] to $O(d^2 \log^2 \kappa/\epsilon^2)$.
- **Question**: Can we improve $\mathrm{poly}(\epsilon^{-1})$ dependence to $\log(\epsilon^{-1})$ as the static case?
- **Question**: Algorithms for fully dynamic updates?

Our results: Lower bound

Theorem (Lower bound). Under the OMv conjecture: [HKNS15]

- **High vs low accuracy.** Any dynamic data structure that maintains an $\epsilon = 1/\operatorname{poly}(n)$-approximate LSR solution under *oblivious incremental* updates requires $\Omega(d^{2-o(1)})$ amortized cost per iteration.
- **Fully vs partially dynamic.** If the data structure supports *adaptive fully dynamic* updates, then maintaining $\epsilon = 0.01$-approximate LSR solution requires $\Omega(d^{2-o(1)})$ amortized cost per iteration.

– Impossible to improve $\operatorname{poly}(\epsilon^{-1})$ dependence to $\log(\epsilon^{-1})$.

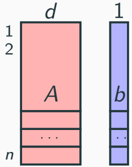
– Impossible to make the algorithm work for fully dynamic updates.

I. Upper Bound: Incremental Oblivious Setting

Exact solution for dynamic LSR

- **Notations**: In the i-th iteration, given a new row $a^{(i)} \in \mathbb{R}^d$ and a new label $\beta^{(i)} \in \mathbb{R}$, solve for

$$\min_{x \in \mathbb{R}^d} \|A^{(i)} x - b^{(i)}\|_2.$$



- **Exact solution** (Kalman filters [Kal60]): Compute $x^{*,(i)} = (A^{(i)\top} A^{(i)})^{-1} A^{(i)\top} b^{(i)}$.
  - Inverse $(A^{(i)\top} A^{(i)})^{-1} = (\underbrace{A^{(i-1)\top} A^{(i-1)}}_{M} + a^{(i)} a^{(i)\top})^{-1}$.
  - Woodbury identity: $(M + a^{(i)} a^{(i)\top})^{-1} = M^{-1} - \frac{M^{-1} a^{(i)} a^{(i)\top} M^{-1}}{1 + a^{(i)\top} a^{(i)}}$.

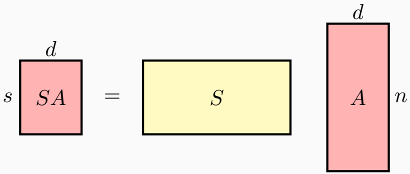
$$\left(\begin{bmatrix} M \end{bmatrix} + \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} a^\top \end{bmatrix} \right)^{-1} = \begin{bmatrix} M^{-1} \end{bmatrix} - \frac{1}{1 + a^\top a} \cdot \begin{bmatrix} M^{-1} \end{bmatrix} \begin{bmatrix} a \end{bmatrix} \begin{bmatrix} a^\top \end{bmatrix} \begin{bmatrix} M^{-1} \end{bmatrix}$$

  - Time per update: $O(d^2)$.

Subspace embedding and approximate LSR

- **Subspace embedding** (See survey [Woo14]):
  Given a matrix $A \in \mathbb{R}^{n \times d}$, matrix $S \in \mathbb{R}^{s \times n}$ is a $(1 \pm \epsilon)$ subspace embedding for A if

$$\|SAx\|_2 = (1 \pm \epsilon)\|Ax\|_2 \text{ for all } x.$$



- **Approx LSR**: Let S be a $(1 \pm \epsilon)$ subspace embedding of matrix $[A, b]$.

$$x' := \arg\min_{x \in \mathbb{R}^d} \|SAx - Sb\|_2$$

  is an $O(\epsilon)$-approximate solution for the original problem:

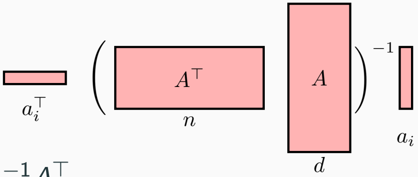
$$\|Ax' - b\|_2 \leq (1 + \epsilon) \min_{x \in \mathbb{R}^d} \|Ax - b\|_2.$$

- Subspace embedding technique that is easy to dynamize: leverage score sampling

Leverage score sampling

- **Leverage scores**: For a fixed matrix A, the leverage score of its i-th row a_i is

$$\tau_i(A) := a_i^\top (A^\top A)^{-1} a_i$$

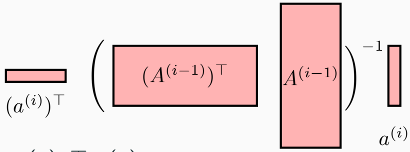


  Diagonal entries of the projection matrix $A(A^\top A)^{-1}A^\top$.
- Measures how important the row a_i is for the row space of A.
  - If $\tau_i(A) = 1$: removing row i will decrease the rank of A by 1.
  - If all rows are the same, they all have $\tau_i(A) = d/n$.
- **Main properties**: (i) $0 \le \tau_i(A) \le 1$. (ii) $\sum_{i=1}^n \tau_i(A) = d$.
- **Leverage score sampling**: Sample the i-th row with probability $p_i = \tau_i(A)/\epsilon^2$. Let $D_{ii} = 1/\sqrt{p_i}$ if the i-th row is sampled, and 0 otherwise. Then with high probability D is a $(1 \pm \epsilon)$ subspace embedding for A.
- In expectation sample $\sum_{i=1}^n p_i = O(d/\epsilon^2)$ rows.

Online leverage score sampling [CMP20]

- **Online leverage scores**:

$$\overline{\tau}_i := (a^{(i)})^\top ((A^{(i-1)})^\top A^{(i-1)})^{-1} a^{(i)}$$



- **Overestimates**: $\overline{\tau}_i \geq \tau_i$ since $(A^{(i-1)})^\top A^{(i-1)} \preceq (A^{(n)})^\top A^{(n)}$.
- **Online leverage score sampling**: When the i-th row arrives, sample it with probability $p_i = \overline{\tau}_i/\epsilon^2$. Let $D_{ii} = 1/\sqrt{p_i}$ if the i-th row is sampled, and 0 otherwise. Then whp D is a $(1 \pm \epsilon)$ subspace embedding for $A^{(i)}$.
- **Sum of online leverage scores**: $\sum_{i=1}^n \overline{\tau}_i \leq d \log(d\kappa)$, where $\kappa := \frac{\sigma_{\max}(A^{(n)})}{\sigma_{\min}(A^{(0)})}$.
  - Fact: $\log\det(M + aa^\top) \geq \log\det(M) + a^\top M^{-1} a$.
  - Apply this fact to the rows:

$$\log\det((A^{(n)})^\top A^{(n)}) \geq \log\det((A^{(n-1)})^\top A^{(n-1)}) + \overline{\tau}_n \geq \cdots \geq \log\det((A^{(0)})^\top A^{(0)}) + \sum_{i=1}^n \overline{\tau}_i$$

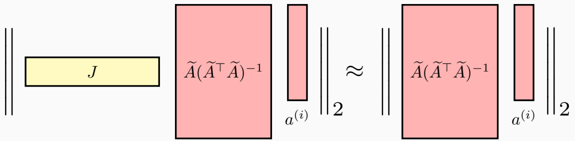
- In expectation sample $\sum_{i=1}^n p_i = \widetilde{O}(d \log(\kappa)/\epsilon^2)$ rows.

Algorithm for oblivious updates

- **Algorithm**: We maintain a subsampled matrix $\widetilde{A} = DA^{(i)}$. In each iteration:
  - When $a^{(i)}$ arrives, compute $\overline{\tau}_i = a^{(i)\top} \cdot (\widetilde{A}^\top \widetilde{A})^{-1} \cdot a^{(i)}$. ①
  - Flip a coin with probability $p_i = \overline{\tau}_i / \epsilon^2$:
    - If 1: Add $a^{(i)}/\sqrt{p_i}$ as a new row to $\widetilde{A}$. Update $(\widetilde{A}^\top \widetilde{A})^{-1}$ and solution. ②
    - If 0: Ignore $a^{(i)}$. Output the same solution.
- **Update time** ②:
  - One update takes $O(d^2)$ time by using Woodbury identity.
  - The total number of updates is $\sum_{i=1}^{n} \overline{\tau}_i / \epsilon^2 = \widetilde{O}(d \log(\kappa)/\epsilon^2)$.
  - Total time is $\widetilde{O}(d^3 \log(\kappa)/\epsilon^2)$.
  - Amortized cost is $d^{o(1)}$ when $n \gg d$.

Computing leverage scores more efficiently

- Recall: We want to compute $\overline{\tau}_i = a^{(i)\top}(\widetilde{A}^\top \widetilde{A})^{-1} a^{(i)}$ ① in each iteration. Direct computation takes $O(d^2)$ time in [CMP20].
- **Johnson-Lindenstrauss lemma**: There exists JL matrix J that compresses dimension from d to $O(\log n)$ and guarantees $\|Jx\|_2^2 \approx_{0.01} \|x\|_2^2$ for fixed n vectors.
- $a^\top \cdot (A^\top A)^{-1} \cdot a = \|A(A^\top A)^{-1} \cdot a\|_2^2$. [SS08].
- The algorithm also maintains $J \cdot \widetilde{A}(\widetilde{A}^\top \widetilde{A})^{-1}$.
- We have $\overline{\tau}_i = \|\widetilde{A}(\widetilde{A}^\top \widetilde{A})^{-1} \cdot a^{(i)}\|_2^2 \approx_{0.01} \|J\widetilde{A}(\widetilde{A}^\top \widetilde{A})^{-1} \cdot a^{(i)}\|_2^2$



- This estimate can be computed in $O(d \log n)$ time.

  $\implies$ Total time is $O(nd \log n)$.

Algorithm for oblivious updates

Theorem (Upper bound in oblivious setting). There is a dynamic data structure that maintains an ϵ-approximate LSR solution under *oblivious incremental* updates, with total time $O\big(nd \log n + d^3 \operatorname{poly}(\epsilon^{-1})\big)$.

II. Upper Bound: Incremental Adaptive Setting

Adaptive updates

- Adaptive updates are inherent in many iterative algorithms.
- To make our algorithm work against adaptive updates:
  - Make JL trick work against adaptive updates.
    - Make the JL estimate an over-estimate.
    - Renew the JL sketch whenever a row is sampled.
  - Make online leverage score sampling work against adaptive updates.

Proof of oblivious leverage score sampling

- **Leverage score sampling**: Sample the i-th row with probability $p_i = \tau_i(A)/\epsilon^2$. Let $D_{ii} = 1/\sqrt{p_i}$ if the i-th row is sampled, and 0 otherwise. Then whp D is a $(1 \pm \epsilon)$ subspace embedding for A.
- **Matrix Chernoff bound:** Given independently random PSD matrices $X_1, \cdots, X_n \in \mathbb{R}^{d \times d}$ s.t. $X_i \preceq R \cdot I$. Let $W = \mathbb{E}[\sum_{i=1}^n X_i]$. Then

$$\Pr[\lambda_{\min}(\sum_{i=1}^n X_i) \leq (1-\epsilon)\lambda_{\min}(W)] \leq d \cdot 2^{-\epsilon^2 \lambda_{\min}(W)/R},$$

$$\Pr[\lambda_{\max}(\sum_{i=1}^n X_i) \geq (1+\epsilon)\lambda_{\max}(W)] \leq d \cdot 2^{-\epsilon^2 \lambda_{\max}(W)/R}.$$

- **Proof of leverage score sampling**: Define $X_i := \begin{cases} \frac{1}{p_i} \cdot a^{(i)}(a^{(i)})^\top & \text{w.p. } p_i \\ 0 & \text{otherwise} \end{cases}.$

  Apply Matrix Chernoff bound to **scaled version**: $\overline{X}_i = W^{-1/2} X_i W^{-1/2}$.

Adaptive online leverage score sampling

- **Adaptive Matrix Chernoff bound.** Given **adaptive** random PSD matrices $X_1, \cdots, X_n \in \mathbb{R}^{d \times d}$ s.t. $X_i \preceq R \cdot I$. Let $W = \sum_{i=1}^n \mathbb{E}[X_i | X_1, \cdots, X_{i-1}]$. Then we have that for any μ:

$$\Pr[\lambda_{\min}(\sum_{i=1}^n X_i) \le (1-\epsilon)\mu \text{ and } \lambda_{\min}(W) \ge \mu] \le d \cdot 2^{-\epsilon^2 \mu / R},$$

$$\Pr[\lambda_{\max}(\sum_{i=1}^n X_i) \ge (1+\epsilon)\mu \text{ and } \lambda_{\max}(W) \le \mu] \le d \cdot 2^{-\epsilon^2 \mu / R}.$$

- W is a random variable.
- Cannot use **scaled version** $\overline{X}_i = W^{-1/2} X_i W^{-1/2}$ anymore!
- By "guessing" the matrix W, and use a union bound over all "guesses", we can prove ϵ-approximation when $p_i = C \cdot \overline{\tau}_i / \epsilon^2$, where $C = \widetilde{O}(d^2 \log(\kappa))$.
- Using scalar concentration bounds, only lose a factor of $C = \widetilde{O}(d \log(\kappa))$.

Adaptive online leverage score sampling

Lemma (Adaptive online leverage score sampling)
Let $a^{(1)}, \cdots, a^{(n)}$ be a sequence of **adaptive** *updates. Sample the i-th row with probability $p_i = C \cdot \overline{\tau}_i/\epsilon^2$, where $C = \widetilde{O}(d \log(\kappa))$. Let $D_{ii} = 1/\sqrt{p_i}$ if the i-th row is sampled, and 0 otherwise. Then whp D is a $(1 \pm \epsilon)$ subspace embedding for A.*

Proof ideas of [BHM$^+$21]

- Instead of proving $DA \approx_\epsilon A$, prove the scalar case that $\|DAv\|_2 \approx_\epsilon \|Av\|_2$
- Need to prove this for all vector v's in an ϵ-net of size $(\kappa/\epsilon)^{\widetilde{O}(d)}$.
- Need $\delta < (\epsilon/\kappa)^{\widetilde{O}(d)}$ to use union bound.
  $\implies$ Lose a factor of $d \log(\kappa)$ in $\log \frac{1}{\delta}$.

Proof ideas of [BHM+21] (continued)

- Define $x_i := (D_{ii}^2 - 1) \cdot v^\top a^{(i)} (a^{(i)})^\top v$.
- Goal is to prove $|\sum_{i=1}^n x_i| \le \epsilon \cdot \|A^{(n)} v\|_2^2$.
- Use concentration bound for scalar adaptive sequences:

  Freedman's inequality (simplified for talk). Let $x_1, \cdots, x_n \in \mathbb{R}$ be an adaptive sequence such that $\mathbb{E}[x_i \mid x_1, \cdots, x_{i-1}] = 0$, and $|x_i| \le R$. Then for any μ,

$$\Pr[|\sum_{i=1}^n x_i| \ge \mu] \le e^{-\mu/R}.$$

- Would like to set $\mu = \epsilon \cdot \|A^{(n)} v\|_2^2$. However, $\|A^{(n)} v\|_2^2$ is a random variable!
- [BHM+21]: Use $\sigma_{\min} \le \|A^{(n)} v\|_2 \le \sigma_{\max}$. $\implies$ lose a factor of $\kappa = \frac{\sigma_{\max}}{\sigma_{\min}}$.

Better dependence on κ

- **Idea**: "Guess" the value of $\|A^{(n)}v\|_2$.
- Build an ϵ-net of the line segment $[\sigma_{\min}, \sigma_{\max}]$.
- For any s in the ϵ-net (s is a guess of $\|A^{(n)}v\|_2$), define a truncated sequence $x_{s,1}, \cdots, x_{s,n}$:

$$x_{s,i} := \begin{cases} x_i & \text{if } \|A^{(i)}v\|_2 \leq s, \\ 0 & \text{otherwise.} \end{cases}$$

- Now can prove $|\sum_{i=1}^{n} x_{s,i}| \leq \epsilon \cdot s^2$ by setting $\mu = \epsilon \cdot s^2$.
- Since the size of the ϵ-net is $\propto \kappa$, we only lose another additive $\log(\kappa)$ factor.

Algorithm for adaptive updates

Theorem (Upper bound in adaptive setting). There is a dynamic data structure that maintains an ϵ-approximate LSR solution under *adaptive incremental* updates, with total time $O\big(nd \log n + d^5 \operatorname{poly}(\epsilon^{-1}) \log \kappa\big)$.

III. Lower Bounds

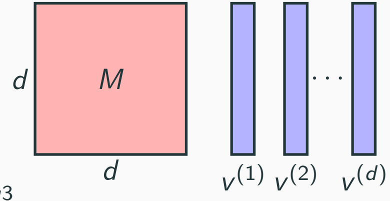
Conditional lower bounds

Theorem (Lower bound). Under the **OMv conjecture**:

- **High vs low accuracy.** Any dynamic data structure that maintains an $\epsilon = 1/\operatorname{poly}(n)$-approximate LSR solution under *oblivious incremental* updates requires $\Omega(d^{2-o(1)})$ amortized cost per iteration.
- **Fully vs partially dynamic.** If the data structure supports *adaptive fully dynamic* updates, then maintaining 0.01-approximate LSR solution requires $\Omega(d^{2-o(1)})$ amortized cost per iteration.

OMv conjecture

OMv conjecture. [HKNS15] In the *online matrix vector multiplication* (OMv) problem, initially a matrix $M \in \{0,1\}^{d \times d}$ is given, then a sequence of vectors $v^{(1)}, v^{(2)}, \cdots, v^{(d)} \in \{0,1\}^d$ are revealed one by one, and the algorithm needs to output $M \cdot v^{(i)}$ in the i-th round. The conjecture states that there is no algorithm for OMv with poly(d) preprocessing time, and $\boldsymbol{O(d^{2-\epsilon})}$ **amortized query time**.

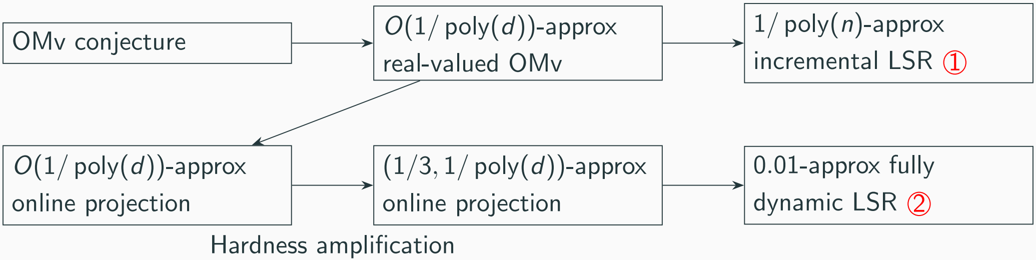


- Offline: d^ω. Online: d^3.
- Only way to speed up matrix vector multiplication is **batching**.
- A unified approach to prove conditional lower bound for dynamic problems.
- Also holds when there are $n = \text{poly}(d)$ queries.

Roadmap

Theorem (Lower bound). Under the OMv conjecture:

- **High vs low accuracy.** Any dynamic data structure that maintains an $\epsilon = 1/\operatorname{poly}(n)$-approximate LSR solution under *oblivious incremental* updates requires $\Omega(d^{2-o(1)})$ amortized cost per iteration. ①
- **Fully vs partially dynamic.** If the data structure supports *adaptive fully dynamic* updates, then maintaining 0.01-approximate LSR solution requires $\Omega(d^{2-o(1)})$ amortized cost per iteration. ②



$1/\operatorname{poly}(n)$-approximate incremental LSR

OMv conjecture → $O(1/\operatorname{poly}(d))$-approx real-valued OMv → $1/\operatorname{poly}(n)$-approx incremental LSR ①

$O(1/\operatorname{poly}(d))$-approx OMv:

- Matrix $M \in \mathbb{R}^{d \times d}$ has constant eigenvalues.
- Query vectors all have unit norm.
- Allow $O(1/\operatorname{poly}(d))$ additive error in output: $\|y^{(i)} - M \cdot v^{(i)}\|_2 \le O(1/\operatorname{poly}(d))$

Proof:

- Assume we have a $1/(nd^{10})$-approx incremental LSR oracle.
- Construct LSR instance: Initially set $(A^{(0)\top} A^{(0)})^{-1} = M$. Add row $a^{(i)} = \frac{v^{(i)}}{nd^5}$.
- Since $\|a^{(i)}\|_2$ is small, we always maintain $(A^{(i)\top} A^{(i)})^{-1} \approx M$.
- By Woodbury identity, $x^{(i)} = x^{(i-1)} + M \cdot a^{(t)} \pm O(\frac{1}{nd^{10}})$.
- Output $y^{(i)} = (x^{(i)} - x^{(i-1)}) \cdot nd^5$ for OMv problem.

0.01-approximate fully dynamic LSR

$O(1/\text{poly}(d))$-approx online projection → $(1/3, 1/\text{poly}(d))$-approx online projection → 0.01-approx fully dynamic LSR ②

Hardness amplification

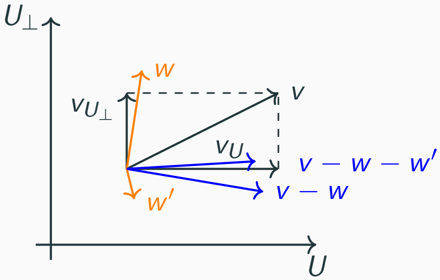
Proof ideas:

- Assume we have a 0.01-approx fully dynamic LSR oracle.
- Fully dynamic LSR oracle is more powerful:
  - Again add row $a^{(i)} \propto v^{(i)}$ in i-th round.
  - Delete the row $a^{(i)}$ after this round!
- Similar as before, compute output using $x^{(i)} - x^{(0)} = Mv^{(i)} \pm 0.01$.
- Need to reduce from a hardness result with **constant error**.

Hardness amplification

- **Online projection problem**: Initially a **projection matrix** $UU^\top \in \mathbb{R}^{d\times d}$ is given, then a sequence of unit vectors $v^{(1)}, v^{(2)}, \cdots, v^{(d)} \in \mathbb{R}^d$ are revealed one by one. Let $v_U^{(i)} = UU^\top \cdot v^{(i)}$. The algorithm needs to output:
  - $O(1/\operatorname{poly}(d))$-approx solution $\|y^{(i)} - v_U^{(i)}\|_2 \le O(\frac{1}{\operatorname{poly}(d)})$.
  - $(1/3, 1/\operatorname{poly}(d))$-approx solution $\|y^{(i)} - v_U^{(i)}\|_2 \le \frac{1}{3} \cdot \|v_U^{(i)}\|_2 + O(\frac{1}{\operatorname{poly}(d)})$.
- **Hardness amplification:** No $O(d^{2-\epsilon})$ time algorithm for $O(1/\operatorname{poly}(d))$-approx online projection problem. $\Longrightarrow$ No $O(d^{2-\epsilon})$ time algorithm for $(1/3, 1/\operatorname{poly}(d))$-approx online projection problem.
- **Proof:** Given an online projection instance $UU^\top$ and $v^{(1)}, \cdots, v^{(n)}$. We have two $O(1/3, 1/\operatorname{poly}(d))$-approximate projection oracles:
  - $\mathbb{P}_U$ that outputs $y^{(i)}$ s.t. $\|y^{(i)} - v_U^{(i)}\|_2 \le \frac{1}{3} \cdot \|v_U^{(i)}\|_2 + O(\frac{1}{\operatorname{poly}(d)})$.
  - $\mathbb{P}_{U_\perp}$ that outputs $w^{(i)}$ s.t. $\|w^{(i)} - v_{U_\perp}^{(i)}\|_2 \le \frac{1}{3} \cdot \|v_{U_\perp}^{(i)}\|_2 + O(\frac{1}{\operatorname{poly}(d)})$.
  - **Goal**: Use $\operatorname{poly}\log d$ oracle calls to compute $y^{(i)}$: $\|y^{(i)} - v_U^{(i)}\|_2 \le O(\frac{1}{\operatorname{poly}(d)})$.

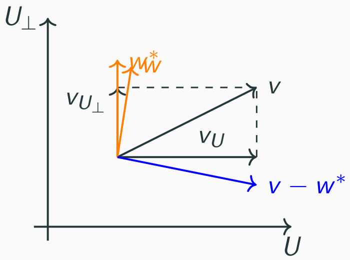
Hardness amplification (continued)



First attempt:

- Call the projection oracle $\mathbb{P}_{U_\perp}(v)$ to compute $w \approx v_{U_\perp}$.
- Remove the component in $U_\perp$: compute $v - w$.
- Repeat for $O(\log d)$ times: the component in $U_\perp$ is at most $1/\operatorname{poly}(d)$.
- **Problem**: Introduce error in the component in U.

Hardness amplification (continued)



Final algorithm:

- We've shown: How to compute $y \approx v_U$ s.t. y has nearly zero component in $U_\perp$.
- Use this algorithm to compute w^* s.t. its component in U is nearly zero.
- Again remove the component in $U_\perp$: compute $v - w^*$.
- This time we don't introduce extra error in U.
- Repeat for $O(\log d)$ times: reduce $1/3$ relative error to $1/\operatorname{poly}(d)$ additive error.

Summary and Open problems

- ϵ-approximate dynamic least squares regression
- **Upper bound.** $O(d)$ amortized time when (1) ϵ is constant, (2) incremental updates, (3) either oblivious or adaptive.
- **Lower bounds.** Under the OMv conjecture:
  - **High vs low accuracy.** If $\epsilon = 1/\operatorname{poly}(n)$, need $\Omega(d^{2-o(1)})$ amortized time.
  - **Fully vs partially dynamic.** If updates are *fully dynamic* and adaptive, then even constant approximation needs $\Omega(d^{2-o(1)})$ amortized time.

Open problems:

- Improve the $O(d^5)$ term in the total time of *adaptive* incremental setting?
- Dynamic ℓ_p regression?
- Lower bound in fully dynamic and *oblivious* setting?
- Other reductions from "$(1/3, 1/d^3)$-approximate online projection"?

Thank you!

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