A Simple Quantum Sketch With Applications to Graph Algorithms

John Kallaugher\textsuperscript{1}

\textsuperscript{1} Quantum Algorithms and Applications Collaboratory
Sandia National Labs
Suppose we have a set $S \subseteq U$, and we want a small sketch of it that solves:

- **Cardinality Given**
  
  Given $P \subseteq U$, how large is $S \cap P$? (to e.g. $\varepsilon |S| = O(|S|)$ error)

- **Sampling**
  
  If $|P \cap U| = \Omega(|S|)$, return a random element of $|P \cap U|$ with prob $\Omega(1)$.

Simple solution: uniformly sample $O(1)$ elements from $S$, using $O(\log |U|)$ bits.
Suppose we have a set $S \subseteq U$, and we want a small sketch of it that solves:

**Cardinality** Given $P \subseteq U$, how large is $S \cap P$? (to e.g. $\varepsilon |S| = O(\log |U|)$ error)

If $|P \cap U| = \Omega(|S|)$, return a random element of $|P \cap U|$ with prob $\Omega(1)$.

Simple solution: uniformly sample $O(1)$ elements from $S$, using $O(\log |U|)$ bits.
Suppose we have a set $S \subseteq U$, and we want a small sketch of it that solves:

**Cardinality** Given $P \subseteq U$, how large is $S \cap P$? (to e.g. $\varepsilon |S| = O(|S|)$ error)

**Sampling** If $|P \cap U| = \Omega(|S|)$, return a random element of $|P \cap U|$ with prob $\Omega(1)$. 

---

**Diagram:**

[Diagram of two overlapping circles, one larger and one smaller, with a point indicating the intersection.]
Sampling

Suppose we have a set $S \subseteq U$, and we want a small sketch of it that solves:

**Cardinality** Given $P \subseteq U$, how large is $S \cap P$? (to e.g. $\varepsilon|S| = O(|S|)$ error)

**Sampling** If $|P \cap U| = \Omega(|S|)$, return a random element of $|P \cap U|$ with prob $\Omega(1)$.

Simple solution: uniformly sample $O(1)$ elements from $S$, using $O(\log|U|)$ bits.
Sampling Pairs

What if $P$ is a set of (disjoint) pairs from $U$ instead?

Probability of a random sample hitting both elements of a pair is much lower.

Need $\Theta(\sqrt{|S|})$ samples even with $|\{p \in P : p \subseteq S\}| = \Omega(|P|)$.

This is optimal up to a log factor by reduction to Boolean Hidden Matching [Gavinsky, Kempe, Kerenidis, Raz, de Wolf].
Sampling Pairs

What if $P$ is a set of (disjoint) \textit{pairs} from $U$ instead?

- Probability of a random sample hitting both elements of a pair is much lower.

\[ \text{Need } \Theta(\sqrt{|S|}) \text{ samples even with } |\{p \in P : p \subseteq S\}| = \Omega(|P|). \]

This is optimal up to a log factor by reduction to Boolean Hidden Matching \cite{GavinskyK0RdW}. 
Sampling Pairs

What if $P$ is a set of (disjoint) pairs from $U$ instead?

- Probability of a random sample hitting both elements of a pair is much lower.
- Need $\Theta(\sqrt{|S|})$ samples even with $|\{p \in P : p \subseteq S\}| = \Omega(|P|)$. 

This is optimal up to a log factor by reduction to Boolean Hidden Matching [Gavinsky, Kempe, Kerenidis, Raz, de Wolf].
Sampling Pairs

What if $P$ is a set of (disjoint) pairs from $U$ instead?

- Probability of a random sample hitting both elements of a pair is much lower.
- Need $\Theta\left(\sqrt{|S|}\right)$ samples even with $|\{p \in P : p \subseteq S\}| = \Omega(|P|)$.
- This is optimal up to a log factor by reduction to Boolean Hidden Matching [Gavinsky, Kempe, Kerenidis, Raz, de Wolf].
Quantum Sketch

We can do better with a *quantum* sketch (inspired by a protocol for BHM).

\[
\frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle
\]
Quantum Sketch

We can do better with a quantum sketch (inspired by a protocol for BHM).

\[ \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle \]

With \( O(\log|U|) \) qubits:
Quantum Sketch

We can do better with a *quantum* sketch (inspired by a protocol for BHM).

\[
\frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle
\]

With \(O(\log |U|)\) qubits:

**Cardinality Estimate** \(|\{p \in P : p \subseteq S\}|\) to \(\varepsilon|S| = O(|S|)\) error.
Quantum Sketch

We can do better with a *quantum* sketch (inspired by a protocol for BHM).

\[ \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle \]

With \( O(\log|U|) \) qubits:

- **Cardinality**: Estimate \( |\{ p \in P : p \subseteq S \}| \) to \( \varepsilon|S| = O(|S|) \) error.
- **Sampling**: If \( |\{ p \in P : p \subseteq S \}| = \Omega(|S|) \), return a random element of \( \{ p \in P : p \subseteq S \} \) with probability \( \Omega(1) \).
Quantum Sketch

We can do better with a quantum sketch (inspired by a protocol for BHM).

$$\Rightarrow \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$$

With $O(\log |U|)$ qubits:

- **Cardinality** Estimate $|\{p \in P : p \subseteq S\}|$ to $\varepsilon|S| = O(|S|)$ error.
- **Sampling** If $|\{p \in P : p \subseteq S\}| = \Omega(|S|)$, return a random element of $\{p \in P : p \subseteq S\}$ with probability $\Omega(1)$.
  - Otherwise: can return $p$ s.t. $|p \cap S| = 1$, but with $\frac{1}{2}$ chance of $-1$ label.
Constructing the Quantum Sketch
Two Quantum Primitives

Superposition

A superposition \( \sum_{x \in U} \alpha_x |x\rangle \) over \( U \) is a unit-length vector in \( \mathbb{C}^U \).

Write \( |x\rangle \) for the basis element corresponding to \( x \in U \).
### Superposition

A superposition $\sum_{x \in U} \alpha_x |x\rangle$ over $U$ is a unit-length vector in $\mathbb{C}^U$.

Write $|x\rangle$ for the basis element corresponding to $x \in U$.

- Can express a superposition over $U$ with $\log |U|$ qubits.
Two Quantum Primitives

Superposition

A superposition $\sum_{x \in U} \alpha_x |x\rangle$ over $U$ is a unit-length vector in $\mathbb{C}^U$.

Write $|x\rangle$ for the basis element corresponding to $x \in U$.

- Can express a superposition over $U$ with $\log|U|$ qubits.

Projective Measurement

Using labeled projectors $(\Lambda_i)$ s.t. $\sum_i \Lambda_i = I$, measure state $\psi$. With probability $\|\Lambda_i \psi\|_2^2$, get result $i$ and transform $\psi$ to $\Lambda_i \psi / \|\Lambda_i \psi\|_2$. 
Constructing the Quantum Sketch

The sketch will be $O(1)$ copies of the *uniform superposition* over $S$.

$$S \implies \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$$
Constructing the Quantum Sketch

The sketch will be $O(1)$ copies of the uniform superposition over $S$.

$$S \Rightarrow \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$$

To query: measure each copy with projectors onto $|x\rangle + |y\rangle$ and $|x\rangle - |y\rangle$ for each $\{x, y\} \in P$. 
Constructing the Quantum Sketch

The sketch will be $O(1)$ copies of the *uniform superposition* over $S$.

\[ S \Rightarrow \frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle \]

To query: measure each copy with projectors onto $|x\rangle + |y\rangle$ and $|x\rangle - |y\rangle$ for each \{x, y\} \in P.

- $P$ being *disjoint* pairs is necessary here as it makes these projectors orthogonal.
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \( |x\rangle + |y\rangle \).
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \( |x\rangle + |y\rangle \).
- If \(|\{x, y\} \cap S| = 1\): \( \frac{1}{2|S|} \) chance of returning each of \( |x\rangle \pm |y\rangle \).
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \(|x\rangle + |y\rangle\).
- If \(|\{x, y\} \cap S| = 1\): \( \frac{1}{2|S|} \) chance of returning each of \(|x\rangle \pm |y\rangle\).
- \( \{x, y\} \cap S = \emptyset \): neither of \(|x\rangle \pm |y\rangle\) will ever be returned.
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \( |x\rangle + |y\rangle \).
- If \( |\{x, y\} \cap S| = 1 \): \( \frac{1}{2|S|} \) chance of returning each of \( |x\rangle \pm |y\rangle \).
- \( \{x, y\} \cap S = \emptyset \): neither of \( |x\rangle \pm |y\rangle \) will ever be returned.

So with a large enough constant number of copies we can 1) estimate \( |\{p \in P : p \subseteq S\}| \) and 2) sample, provided \( |p \in P : p \subseteq S| = \Omega(|S|) \).
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \(|x\rangle + |y\rangle\).
- If \(|\{x, y\} \cap S| = 1\): \( \frac{1}{2|S|} \) chance of returning each of \(|x\rangle \pm |y\rangle\).
- \( \{x, y\} \cap S = \emptyset \): neither of \(|x\rangle \pm |y\rangle\) will ever be returned.

So with a large enough constant number of copies we can 1) estimate \(|\{p \in P : p \subseteq S\}|\) and 2) sample, provided \(|p \in P : p \subseteq S| = \Omega(|S|)\).

Cardinality \( \sum_{\{x, y\} \in P} (\mathbb{P}[|x\rangle + |y\rangle] - \mathbb{P}[|x\rangle - |y\rangle]) = 2|\{p \in P : p \subseteq S\}|. \)
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \( |x\rangle + |y\rangle \).
- If \( |\{x, y\} \cap S| = 1 \): \( \frac{1}{2|S|} \) chance of returning each of \( |x\rangle \pm |y\rangle \).
- \( \{x, y\} \cap S = \emptyset \): neither of \( |x\rangle \pm |y\rangle \) will ever be returned.

So with a large enough constant number of copies we can 1) estimate \( |\{p \in P : p \subseteq S\}| \) and 2) sample, provided \( |p \in P : p \subseteq S| = \Omega(|S|) \).

**Cardinality** \( \sum_{\{x,y\} \in P} (\mathbb{P} [ |x\rangle + |y\rangle ] - \mathbb{P} [ |x\rangle - |y\rangle ]) = 2|\{p \in P : p \subseteq S\}|. \)

**Sampling** Return \((x, y)\) with sign \((-1)^b\) on seeing \( |x\rangle + (-1)^b|y\rangle \).
Measurement Outcomes

- If \( \{x, y\} \subseteq S \): \( \frac{2}{|S|} \) chance of returning \( |x\rangle + |y\rangle \).
- If \( |\{x, y\} \cap S| = 1 \): \( \frac{1}{2|S|} \) chance of returning each of \( |x\rangle \pm |y\rangle \).
- \( \{x, y\} \cap S = \emptyset \): neither of \( |x\rangle \pm |y\rangle \) will ever be returned.

So with a large enough constant number of copies we can 1) estimate \( |\{p \in P : p \subseteq S\}| \) and 2) sample, provided \( |p \in P : p \subseteq S| = \Omega(|S|) \).

Cardinality \( \sum_{\{x,y\} \in P} (\mathbb{P}[|x\rangle + |y\rangle] - \mathbb{P}[|x\rangle - |y\rangle]) = 2|\{p \in P : p \subseteq S\}| \).

Sampling Return \((x, y)\) with sign \((-1)^b\) on seeing \( |x\rangle + (-1)^b |y\rangle \).

- So when we return \( p \) with size-1 overlap, it comes with a random sign.
Application: Quantum Advantage for Counting Triangles in the Stream
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received \textit{one edge at a time}. 
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received *one edge at a time*. 

- 

- 

- 

- 

- 

-
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received *one edge at a time*.
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received *one edge at a time.*
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received \textit{one edge at a time}.

\[
\begin{array}{c}
\text{graph} \\
\text{problem}
\end{array} = \begin{array}{c}
\text{problem 1} \\
\text{problem 2}
\end{array} + \begin{array}{c}
\text{problem 3} \\
\text{problem 4}
\end{array}
\]
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received *one edge at a time*.

\[
\begin{align*}
&\quad = \quad + \quad + \quad + \\
&\begin{array}{c}
\text{Diagram of graphs}
\end{array}
\end{align*}
\]
Graph Streaming

Remainder of this talk: two applications of the sketch to problems in graph streaming.

- Graph problems where the input is received one edge at a time.

We will be concerned with space complexity.
Are there streaming problems that require asymptotically fewer qubits than bits?
Are there streaming problems that require asymptotically fewer qubits than bits?

- [Le Gall '06] Can require exponentially less space than classical algorithms (but for an “unnatural” problem).
Quantum-Classical Separations

Are there streaming problems that require asymptotically fewer qubits than bits?

- [Le Gall '06] Can require exponentially less space than classical algorithms (but for an “unnatural” problem).
- [Montanaro ‘16] Quantum advantage for moment estimation with many ($\omega(1)$) passes over the input.
Are there streaming problems that require asymptotically fewer qubits than bits?

- [Le Gall ‘06] Can require exponentially less space than classical algorithms (but for an “unnatural” problem).
- [Montanaro ‘16] Quantum advantage for moment estimation with many ($\omega(1)$) passes over the input.

With this sketch we obtain quantum advantage for single-pass streaming problems of independent interest.
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

\[ T = 0 \]
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

\[ T = 0 \]
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

\[ T = 0 \]
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

\[ T = 0 \]
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

\[ T = 1 \]
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

Applications in social science, spam detection, etc.

The simplest graph counting problem that requires non-local information.

For our purposes: assume $\Omega(m)$ edge-disjoint triangles in a $\Theta(m)$-edge graph.
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

\[ T = 2 \]
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

![Diagram of a graph with 3 nodes and 3 edges, forming a triangle.]

$T = 2$

- Applications in social science, spam detection, etc.
Triangle Counting

Given a graph one edge at a time, count the number of three-cliques.

- Applications in social science, spam detection, etc.
- The simplest graph counting problem that requires *non-local* information.
Given a graph one edge at a time, count the number of three-cliques.

- Applications in social science, spam detection, etc.
- The simplest graph counting problem that requires *non-local* information.
- For our purposes: assume $\Omega(m)$ edge-disjoint triangles in a $\Theta(m)$-edge graph.
One hard case is an $m$-star with triangles (possibly) completed by a matching.
One hard case is an $m$-star with triangles (possibly) completed by a matching.
One hard case is an $m$-star with triangles (possibly) completed by a matching.
One hard case is an $m$-star with triangles (possibly) completed by a matching.
One hard case is an $m$-star with triangles (possibly) completed by a matching.

Intuitively: if we keep $k$ star edges, we have a $\sim T \times \left( \frac{k}{m} \right)^2 = \Theta \left( \frac{k^2}{m} \right)$ chance of getting both edges of at least one of $T = \Theta(m)$ triangles, so need $k = \Omega(\sqrt{m})$. 
One hard case is an $m$-star with triangles (possibly) completed by a matching.

Intuitively: if we keep $k$ star edges, we have a $\sim T \times \left( \frac{k}{m} \right)^2 = \Theta \left( \frac{k^2}{m} \right)$ chance of getting both edges of at least one of $T = \Theta(m)$ triangles, so need $k = \Omega(\sqrt{m})$.

Optimal up to log factors for general (classical) algorithms.
Sketching a Star Graph

In the two-player setting, our sketch solves this instance immediately in $O(\log n)$ space: make $S$ the set of neighbors of the star vertex, and $P$ the set of edges in the matching.
Sketching a Star Graph

In the two-player setting, our sketch solves this instance immediately in $O(\log n)$ space: make $S$ the set of neighbors of the star vertex, and $P$ the set of edges in the matching.

- Querying the sketch for the number of pairs contained in $S$ is equivalent to asking the number of triangles in the graph.

Implies exponential quantum algorithm for the two-player version of this specific instance of triangle counting.
Sketching a Star Graph

In the two-player setting, our sketch solves this instance immediately in $O(\log n)$ space: make $S$ the set of neighbors of the star vertex, and $P$ the set of edges in the matching.

- Querying the sketch for $\#$ of pairs contained in $S$ is equivalent to asking $\#$ of triangles in the graph.
- Implies exponential quantum algorithm for the two-player version of this specific instance of triangle counting.
Generalizing to a general triangle-free graph \((V, G)\) followed by a matching \(M\) is simple: make \(S\) the set of all edges in the first graph, and set \(P = \{(uv, uw) : u \in V, vw \in M\}\).
Generalizing to a general triangle-free graph \((V, G)\) followed by a matching \(M\) is simple: make \(S\) the set of all edges in the first graph, and set 
\[
P = \{(uv, uw) : u \in V, vw \in M\}.
\]

Problems arise when the triangle-completing edges don’t form a matching.
Our sketch doesn’t work for non-matching edges because they don’t give us disjoint pairs.

$\Rightarrow S$

Requires copying sketch max-degree times, which is in general infeasible. Worst-case instances turn out to be “classically easy.” Interpolation between classical and quantum estimators then allows a $O\left(\frac{n^2}{5}\right)$ vs. $\Omega\left(\sqrt{n}\right)$ quantum space advantage.
Sketching General Graphs

Our sketch doesn’t work for non-matching edges because they don’t give us disjoint pairs.

Requires copying sketch max-degree times, which is in general infeasible.
Sketching General Graphs

Our sketch doesn’t work for non-matching edges because they don’t give us disjoint pairs.

Requires copying sketch max-degree times, which is in general infeasible.

- Worst-case instances turn out to be “classically easy”. 
Sketching General Graphs

Our sketch doesn’t work for non-matching edges because they don’t give us disjoint pairs.

Requires copying sketch max-degree times, which is in general infeasible.

- Worst-case instances turn out to be “classically easy”.
- Interpolation between classical and quantum estimators then allows a $O\left(n^{2/5}\right)$ v. $\Omega(\sqrt{n})$ quantum space advantage.
Sketching in the Stream

To make use of this in the stream, we need a couple more properties of the sketch:
To make use of this in the stream, we need a couple more properties of the sketch:

**Swapping** Given a pair \((x, y)\), if \(x \in S\) and \(y \not\in S\): remove \(x\) and add \(y\). If \(y \in S\) and \(x \not\in S\), do the opposite. Otherwise do nothing.
To make use of this in the stream, we need a couple more properties of the sketch:

**Swapping** Given a pair \((x, y)\), if \(x \in S\) and \(y \not\in S\): remove \(x\) and add \(y\). If \(y \in S\) and \(x \not\in S\), do the opposite. Otherwise do nothing.

**Partial Query** Given a pair \(p\): return \(p\) and destroy \(S\) with probability \(2/|S|\) if \(p \subseteq S\). If \(p\) is not returned, set \(S = S \setminus p\).
Sketching in the Stream

To make use of this in the stream, we need a couple more properties of the sketch:

**Swapping** Given a pair \((x, y)\), if \(x \in S\) and \(y \notin S\): remove \(x\) and add \(y\). If \(y \in S\) and \(x \notin S\), do the opposite. Otherwise do nothing.

**Partial Query** Given a pair \(p\): return \(p\) and destroy \(S\) with probability \(2/|S|\) if \(p \subseteq S\). If \(p\) is not returned, set \(S = S \setminus p\). (as with the full query setting, \(p\) can also be returned when \(|p \cap S| = 1\), but will come with a random \(\pm 1\) sign)
Sketching in the Stream

To make use of this in the stream, we need a couple more properties of the sketch:

**Swapping** Given a pair \((x, y)\), if \(x \in S\) and \(y \notin S\): remove \(x\) and add \(y\). If \(y \in S\) and \(x \notin S\), do the opposite. Otherwise do nothing.

**Partial Query** Given a pair \(p\): return \(p\) and destroy \(S\) with probability \(2/|S|\) if \(p \subseteq S\). If \(p\) is not returned, set \(S = S \setminus p\). (as with the full query setting, \(p\) can also be returned when \(|p \cap S| = 1\), but will come with a random \(\pm 1\) sign)

Executing a full query is then equivalent to doing a partial query for each \(p \in P\) (in any order).
Quantum Advantage for Triangle Counting

By starting with $S = [m]$ a set of $m$, “dummy variables”, we can construct the sketch in the stream, swapping out one dummy variable for an edge whenever we see one.

Theorem (Informal, K. ’21)
There is a $\tilde{O}(m^2/\sqrt{5})$-qubit streaming algorithm for counting triangles in the stream. ($\Omega(\sqrt{m})$ classically)
Quantum Advantage for Triangle Counting

- By starting with $S = [m]$ a set of $m$, “dummy variables”, we can construct the sketch in the stream, swapping out one dummy variable for an edge whenever we see one.
- By using partial queries, we can check whether an edge completes triangles with a pair of previously-arrived edges, without having to store that edge for later.
Quantum Advantage for Triangle Counting

- By starting with \( S = [m] \) a set of \( m \), “dummy variables”, we can construct the sketch in the stream, swapping out one dummy variable for an edge whenever we see one.
- By using partial queries, we can check whether an edge completes triangles with a pair of previously-arrived edges, without having to store that edge for later.

This gives us quantum space advantage for counting triangles in the stream.

**Theorem (Informal, K. ‘21)**

There is a \( \widetilde{O}\left(m^{2/5}\right) \)-qubit streaming algorithm for counting triangles in the stream. (\( \Omega(\sqrt{m}) \) classically)
Interlude: Implementing the Streaming Properties of the Sketch
To implement our swap operation, we need one more quantum primitive.

### Unitary Evolution

A superposition $\sum_{x \in S} \alpha_x |x\rangle$ can be converted into a new one by any unitary (length-preserving) linear transformation.
Swapping

To implement our swap operation, we need one more quantum primitive.

Unitary Evolution

A superposition $\sum_{x \in S} \alpha_x |x\rangle$ can be converted into a new one by any unitary (length-preserving) linear transformation.

- One such transformation is swapping the basis elements $|x\rangle$ and $|y\rangle$. 
Swapping

To implement our swap operation, we need one more quantum primitive.

Unitary Evolution

A superposition $\sum_{x \in S} \alpha_x |x\rangle$ can be converted into a new one by any unitary (length-preserving) linear transformation.

- One such transformation is swapping the basis elements $|x\rangle$ and $|y\rangle$.
- As our sketch is $\frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$, this will do nothing if neither or both of $x$ and $y$ are in $S$, and swap them if exactly one is.
Partial Queries

We can implement our partial query operation using the measurement postulate we already have.

**Projective Measurement**

Using labeled projectors \((\Lambda_i)\) s.t. \(\sum_i \Lambda_i = I\), measure state \(\psi\). With probability \(\|\Lambda_i \psi\|^2\), get result \(i\) and transform \(\psi\) to \(\Lambda_i \psi / \|\Lambda_i \psi\|_2\).
Partial Queries

We can implement our partial query operation using the measurement postulate we already have.

Projective Measurement

Using labeled projectors \((\Lambda_i)\) s.t. \(\sum_i \Lambda_i = I\), measure state \(\psi\). With probability \(\|\Lambda_i \psi\|^2\), get result \(i\) and transform \(\psi\) to \(\Lambda_i \psi / \|\Lambda_i \psi\|_2\).

- Rather than measure with a large collection of projectors, we use \(\frac{|x\rangle \pm |y\rangle}{\sqrt{2}}\) and the projector onto the rest of the space.
Partial Queries

We can implement our partial query operation using the measurement postulate we already have.

**Projective Measurement**

Using labeled projectors \((\Lambda_i)\) s.t. \(\sum_i \Lambda_i = I\), measure state \(\psi\). With probability \(\|\Lambda_i\psi\|^2\), get result \(i\) and transform \(\psi\) to \(\Lambda_i\psi/\|\Lambda_i\psi\|_2\).

- Rather than measure with a large collection of projectors, we use \(\frac{|x\rangle \pm |y\rangle}{\sqrt{2}}\) and the projector onto the rest of the space.
- Our probabilities for returning \(\frac{|x\rangle \pm |y\rangle}{\sqrt{2}}\) are the same as they would have been when measuring it as a one of a large collection of pairs.
Partial Queries

We can implement our partial query operation using the measurement postulate we already have.

Projective Measurement

Using labeled projectors \((\Lambda_i)\) s.t. \(\sum_i \Lambda_i = I\), measure state \(\psi\). With probability \(\|\Lambda_i\psi\|^2\), get result \(i\) and transform \(\psi\) to \(\Lambda_i\psi/\|\Lambda_i\psi\|_2\).

- Rather than measure with a large collection of projectors, we use \(|x\rangle \pm |y\rangle / \sqrt{2}\) and the projector onto the rest of the space.
- Our probabilities for returning \(|x\rangle \pm |y\rangle / \sqrt{2}\) are the same as they would have been when measuring it as a one of a large collection of pairs.
- If we do not return one of them, the superposition is projected onto the space orthogonal to \(\text{span}(|x\rangle, |y\rangle)\), i.e. \(x\) and \(y\) are deleted from \(S\).
Application: Exponential Advantage for Maximum Directed Cut
Maximum Directed Cut

Given a directed graph, what is the maximum, over partitions \( V_0 \cap V_1 = V \), number of edges \( u \to v \) such that \( u \in V_0 \) and \( v \in V_1 \)?
Maximum Directed Cut

Given a directed graph, what is the maximum, over partitions $V_0 \cap V_1 = V$, number of edges $\overrightarrow{uv}$ such that $u \in V_0$ and $v \in V_1$?

$K = \text{Max-DiCut}(G) = 5$
Maximum Directed Cut

Given a directed graph, what is the maximum, over partitions $V_0 \cap V_1 = V$, number of edges $\overrightarrow{uv}$ such that $u \in V_0$ and $v \in V_1$?

$K = \text{Max-DiCut}(G) = 5$

We want to output an $\alpha$-approximation in the minimum amount of space, i.e. $K' \in [\alpha K, K]$, using as little space as possible.
Complexity of Approximating Max-DiCut

- [Chou, Golovnev, Velusamy ’20] Beating a $4/9$-approximation requires $\Omega(\sqrt{n})$ space classically.

- [Feige, Jozeph ’15] A $0.4835 > \frac{4}{9}$-approximation is possible given a histogram of the number of edges going between vertices with biases in various ranges.

- [Saxena, Singer, Sudan, Velusamy ’23] This can be achieved in $\tilde{O}(\sqrt{n})$ space classically.

- [K., Parekh, Voronova] This can be achieved in polylog$(n)$ space with our quantum sketch.
Complexity of Approximating Max-DiCut

- [Chou, Golovnev, Velusamy ‘20] Beating a $4/9$-approximation requires $\Omega(\sqrt{n})$ space classically.
- [Feige, Jozeph ‘15] A $0.4835 > 4/9$-approximation is possible given a histogram of the number of edges going between vertices with *biases* in various ranges.
Complexity of Approximating Max-DiCut

- [Chou, Golovnev, Velusamy ‘20] Beating a 4/9-approximation requires $\Omega(\sqrt{n})$ space classically.
- [Feige, Jozeph ‘15] A $0.4835 > 4/9$-approximation is possible given a histogram of the number of edges going between vertices with biases in various ranges.
- [Saxena, Singer, Sudan, Velusamy ‘23] This can be achieved in $\tilde{O}(\sqrt{n})$ space classically.
Complexity of Approximating Max-DiCut

- [Chou, Golovnev, Velusamy ‘20] Beating a $4/9$-approximation requires $\Omega(\sqrt{n})$ space classically.
- [Feige, Jozeph ‘15] A $0.4835 > 4/9$-approximation is possible given a histogram of the number of edges going between vertices with biases in various ranges.
- [Saxena, Singer, Sudan, Velusamy ‘23] This can be achieved in $\tilde{O}(\sqrt{n})$ space classically.
- [K., Parekh, Voronova] This can be achieved in $\text{polylog}(n)$ space with our quantum sketch.
(First Order) Bias Histogram

The bias of a vertex $v$ is $\frac{d_{v}^{\text{out}} - d_{v}^{\text{in}}}{d_{v}}$. 

<table>
<thead>
<tr>
<th>Head Bias</th>
<th>$-1 \leq b &lt; -0.5$</th>
<th>$-0.5 \leq b &lt; 0$</th>
<th>$0 \leq b &lt; 0.5$</th>
<th>$0.5 \leq b &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tail Bias</td>
<td>$-0.5 \leq b &lt; 0$</td>
<td>$0 \leq b &lt; 0.5$</td>
<td>$0.5 \leq b &lt; 1$</td>
<td>$1 \leq b &lt; 1.5$</td>
</tr>
</tbody>
</table>

Given a partition of $(B_i)$ of $[-1, 1]$ by thresholds, we want to know how many edges there are from $B_i$ to $B_j$ for each $i$, $j$. 
(First Order) Bias Histogram

The bias of a vertex \( v \) is \( \frac{d_{v}^{\text{out}} - d_{v}^{\text{in}}}{d_{v}} \).

<table>
<thead>
<tr>
<th>Head Bias</th>
<th>(-1 \leq b &lt; -0.5)</th>
<th>(-0.5 \leq b &lt; 0)</th>
<th>(0 \leq b &lt; 0.5)</th>
<th>(0.5 \leq b \leq 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tail Bias</td>
<td>(-0.5 \leq b &lt; 0)</td>
<td>1245</td>
<td>2333</td>
<td>2974</td>
</tr>
<tr>
<td></td>
<td>(0 \leq b &lt; 0.5)</td>
<td>9361</td>
<td>8421</td>
<td>82</td>
</tr>
<tr>
<td></td>
<td>(0 \leq b &lt; 0.5)</td>
<td>955</td>
<td>2133</td>
<td>5369</td>
</tr>
<tr>
<td></td>
<td>(0.5 \leq b \leq 1)</td>
<td>3530</td>
<td>5312</td>
<td>4789</td>
</tr>
</tbody>
</table>

Given a partition of \((B_i)\) of \([-1, 1]\) by thresholds, we want to know how many edges there are from \(B_i\) to \(B_j\) for each \(i, j\).
Estimating the Bias Histogram

Return to the two-player setting: Alice and Bob both have graphs, and Alice wants to send Bob a message that will let him estimate their joint bias histogram.
Estimating the Bias Histogram

Return to the two-player setting: Alice and Bob both have graphs, and Alice wants to send Bob a message that will let him estimate their joint bias histogram.

- Alice’s edges are easy: she can sample edges and send them to Bob with their endpoint out- and in- degrees.
Return to the two-player setting: Alice and Bob both have graphs, and Alice wants to send Bob a message that will let him estimate their joint bias histogram.

- Alice’s edges are easy: she can sample edges and send them to Bob with their endpoint out- and in- degrees.
- So the challenge is sampling head-tail degree pairs from Alice’s graph that correspond to edges in Bob’s graph.
Sketching for the Bias Histogram

We can use our sketch to sample labeled pairs of Alice’s vertices corresponding to Bob’s edges, allowing him to calculate the biases of the endpoints of a sample of his edges with $O(\log(n))$ qubits of communication from Alice.

\[
\begin{align*}
S
\end{align*}
\]
Sketching for the Bias Histogram

We can use our sketch to sample labeled pairs of Alice’s vertices corresponding to Bob’s edges, allowing him to calculate the biases of the endpoints of a sample of his edges with $O(\log(n))$ qubits of communication from Alice.

- Actually implementing this requires “coarsening” Alice’s possible biases and having Bob query every possible coarsened bias pair for an edge.
Sketching for the Bias Histogram

We can use our sketch to sample labeled pairs of Alice’s vertices corresponding to Bob’s edges, allowing him to calculate the biases of the endpoints of a sample of his edges with $O(\log(n))$ qubits of communication from Alice.

Actually implementing this requires “coarsening” Alice’s possible biases and having Bob query every possible coarsened bias pair for an edge. Works if he has a matching. But what if he doesn’t?
Suppose Alice knew how many edges Bob had incident to each of her vertices.
Suppose Alice knew how many edges Bob had incident to each of her vertices. Now she can copy each vertex with corresponding multiplicity before sketching, and Bob can then measure with a matching on the copied vertices.
Suppose Alice knew how many edges Bob had incident to each of her vertices.

Now she can copy each vertex with corresponding multiplicity before sketching, and Bob can then measure with a matching on the copied vertices.

- She will still put at most $m = |E_{\text{Bob}}|$ elements total in her sketch set $S$, and so the needed property of $|p \in P : p \subseteq S| = \Omega(|S|)$ is preserved.
Guessing the Future

How can Alice copy correctly without knowing what Bob’s degrees will be?

She can use her own degrees instead (multiplied by some large constant). This works because when her degree for a vertex is much smaller than Bob’s degree, that vertex’s bias is almost exactly determined by Bob’s input. We can therefore combine classical and quantum sampling to estimate the bias histogram in polylog($n$) space.
Guessing the Future

How can Alice copy correctly without knowing what Bob’s degrees will be?

- She can use her own degrees instead (multiplied by some large constant).
How can Alice copy correctly without knowing what Bob’s degrees will be?

- She can use her *own* degrees instead (multiplied by some large constant).
- This works because when her degree for a vertex is much smaller than Bob’s degree, that vertex’s bias is almost exactly determined by Bob’s input.
Guessing the Future

How can Alice copy correctly without knowing what Bob’s degrees will be?

- She can use her *own* degrees instead (multiplied by some large constant).
- This works because when her degree for a vertex is much smaller than Bob’s degree, that vertex’s bias is almost exactly determined by Bob’s input.
- We can therefore combine classical and quantum sampling to estimate the bias histogram in polylog($n$) space.
Sketching in the Stream

Idealized algorithm:

Maintain a sketch of $S = \{(v, b'v) : v \in V\}$, where $b'v$ is some appropriate coarsening of the biases, and where $(v, b'v)$ is copied with multiplicity $d_v$.

Query $((u, b'u), (v, b'v))$ for the possible biases $(b'u, b'v)$ on seeing an edge $\rightarrow uv$, then calculate biases for sampled vertices using the rest of the stream.

Use classical sampling to estimate edge counts for edges $\rightarrow uv$ such that $b_u$ or $b_v$ are dominated by edges that arrive after $\rightarrow uv$. 
Sketching in the Stream

Idealized algorithm:

- Maintain a sketch of $S = \{(v, b'_v) : v \in V\}$, where $b'_v$ is some appropriate coarsening of the biases, and where $(v, b'_v)$ is copied with multiplicity $d_v$. 

Query $(u, b'_u)$, $(v, b'_v)$ for the possible biases $(b'_u, b'_v)$ on seeing an edge $\vec{uv}$, then calculate biases for sampled vertices using the rest of the stream. Use classical sampling to estimate edge counts for edges $\vec{uv}$ such that $b_u$ or $b_v$ are dominated by edges that arrive after $\vec{uv}$. 
Sketching in the Stream

Idealized algorithm:

- Maintain a sketch of $S = \{(v, b'_v) : v \in V\}$, where $b'_v$ is some appropriate coarsening of the biases, and where $(v, b'_v)$ is copied with multiplicity $d_v$.

- Query $((u, b'_u), (v, b'_v))$ for the possible biases $(b'_u, b'_v)$ on seeing an edge $\vec{uv}$, then calculate biases for sampled vertices using the rest of the stream.
Sketching in the Stream

Idealized algorithm:

- Maintain a sketch of $S = \{(v, b'_v) : v \in V\}$, where $b'_v$ is some appropriate coarsening of the biases, and where $(v, b'_v)$ is copied with multiplicity $d_v$.
- Query $((u, b'_u), (v, b'_v))$ for the possible biases $(b'_u, b'_v)$ on seeing an edge $\overrightarrow{uv}$, then calculate biases for sampled vertices using the rest of the stream.
- Use classical sampling to estimate edge counts for edges $\overrightarrow{uv}$ such that $b_u$ or $b_v$ are dominated by edges that arrive after $\overrightarrow{uv}$. 
Sketching in the Stream

Problem: maintaining the $S$ described would require knowing $d_u$, $d_v$ when we see an edge $\overrightarrow{uv}$.
Sketching in the Stream

Problem: maintaining the $S$ described would require knowing $d_u$, $d_v$ when we see an edge $\overrightarrow{uv}$.

- Basic idea: we can copy a vertex every time an edge is seen incident to it, and use the number of times it has been copied to encode its degree.
Problem: maintaining the $S$ described would require knowing $d_u, d_v$ when we see an edge $\overrightarrow{uv}$.

- Basic idea: we can copy a vertex every time an edge is seen incident to it, and use the number of times it has been copied to encode its degree.
- On seeing an edge incident to $v$, add $(v, 0)$ to $S$ after performing the swap $(v, i), (v, i + 1)$ for every $i$. 
Problem: maintaining the $S$ described would require knowing $d_u, d_v$ when we see an edge $uv$.

- Basic idea: we can copy a vertex every time an edge is seen incident to it, and use the number of times it has been copied to encode its degree.
- On seeing an edge incident to $v$, add $(v, 0)$ to $S$ after performing the swap $(v, i), (v, i + 1)$ for every $i$.
- Now $S$ contains $(v, 0) \ldots (v, d_v)$, and we can then query $((u, d_1), (v, d_2))$ if we want to sample restricted to edges with endpoint degrees at least $(d_1, d_2)$. 
Problem: maintaining the $S$ described would require knowing $d_u, d_v$ when we see an edge $\overrightarrow{uv}$.

- Basic idea: we can copy a vertex every time an edge is seen incident to it, and use the number of times it has been copied to encode its degree.
- On seeing an edge incident to $v$, add $(v, 0)$ to $S$ after performing the swap $(v, i), (v, i + 1)$ for every $i$.
- Now $S$ contains $(v, 0) \ldots (v, d_v)$, and we can then query $((u, d_1), (v, d_2))$ if we want to sample restricted to edges with endpoint degrees at least $(d_1, d_2)$.
- Actual algorithm is complicated by need to track in- and out-degrees separately.
Exponential Quantum Advantage for Maximum Directed Cut

Theorem (Informal, K., Parekh, Voronova)

There is a polylog($n$) space quantum streaming algorithm that 0.4835-approximates the Max-DiCut value of a graph.
Theorem (Informal, K., Parekh, Voronova)

There is a polylog($n$) space quantum streaming algorithm that 0.4835-approximates the Max-DiCut value of a graph.

Contrasts with the undirected problem, where no quantum advantage is possible for any approximation ratio [Kapralov, Krachun ‘19], [K., Parekh ‘22].
Conclusion

There is a simple, $O(\log n)$ space quantum sketch that allows sampling from a set based on set of disjoint pairs unknown at the time of sketching.
Conclusion

There is a simple, $O(\log n)$ space quantum sketch that allows sampling from a set based on set of disjoint pairs unknown at the time of sketching.

- Can be used to obtain quantum streaming space advantages for triangle counting.

Open Questions

What is the correct complexity for triangle counting?

Can we characterize which CSPs admit quantum space advantage in the stream?
Conclusion

There is a simple, $O(\log n)$ space quantum sketch that allows sampling from a set based on set of disjoint pairs unknown at the time of sketching.

- Can be used to obtain quantum streaming space advantages for triangle counting.
- And *exponential* advantages for $\text{MAX-DICUT}$.

Open Questions

- What is the correct complexity for triangle counting?
- Can we characterize which CSPs admit quantum space advantage in the stream?
Conclusion

There is a simple, $O(\log n)$ space quantum sketch that allows sampling from a set based on set of disjoint pairs unknown at the time of sketching.

- Can be used to obtain quantum streaming space advantages for triangle counting.
- And exponential advantages for $\text{Max-DiCut}$.

Open Questions

- What is the correct complexity for triangle counting?
There is a simple, $O(\log n)$ space quantum sketch that allows sampling from a set based on set of disjoint pairs unknown at the time of sketching.

- Can be used to obtain quantum streaming space advantages for triangle counting.
- And exponential advantages for \text{Max-DiCut}.

**Open Questions**

- What is the correct complexity for triangle counting?
- Can we characterize which CSPs admit quantum space advantage in the stream?