Sublinear Time Eigenvalue Approximation via Random Sampling

Cameron Musco

University of Massachusetts Amherst

Joint with: Rajarshi Bhattacharjee (UMass), Gregory Dexter (Purdue), Petros Drineas (Purdue), Archan Ray (UMass)

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Eigenvalue Approximation

Basic linear algebraic primitive: Given symmetric $A \in \mathbb{R}^{n \times n}$, compute approximations to all of A's eigenvalues.

- Nearly exact computation of all eigenvalues in $O(n^\omega)$ time via full eigendecomposition — but this is prohibitive for large n.
- Accurate approximation to k largest magnitude eigenvalues using $\tilde{O}(k)$ matrix vector multiplications with A (power method, Krylov subspace methods, `eigs`). $\tilde{O}(n^2 \cdot k)$ time for dense matrices.

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How well can we approximate the spectrum in sublinear time, i.e., $o(n^2)$ time for dense matrices?

Summary

Today:

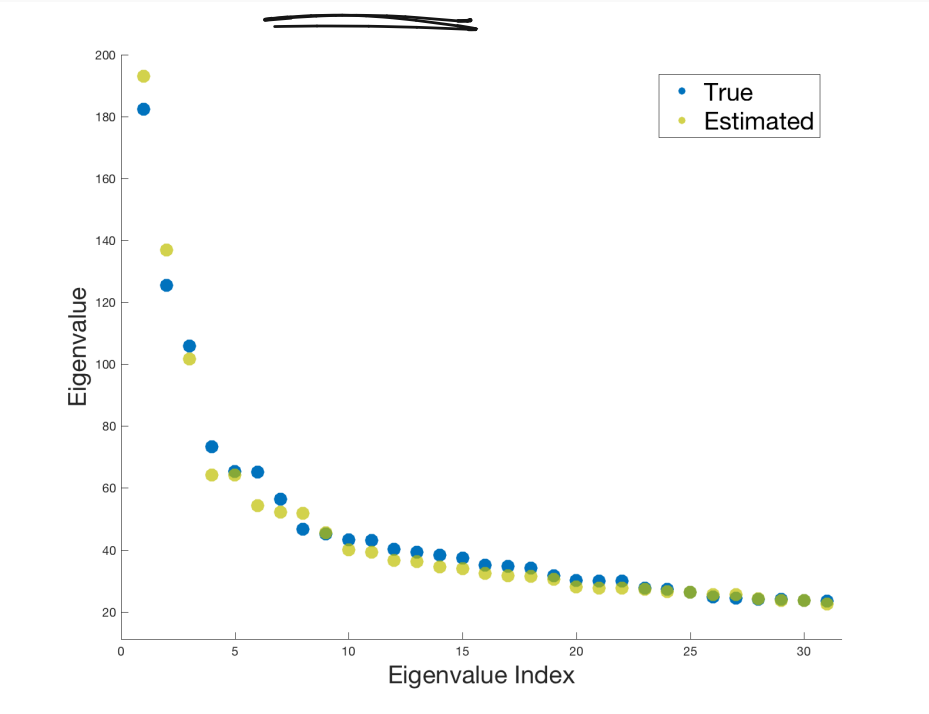
- Very simple sublinear time algorithm for approximating all eigenvalues of any symmetric matrix $A \in \mathbb{R}^{n \times n}$ up to additive error $\pm \epsilon n \cdot \|A\|_\infty$ using $\mathbf{poly}(\log n, 1/\epsilon)$ samples.
- Just sample a uniform random principal submatrix and computes its eigenvalues.
- Improved results when you can sample rows/columns with probabilities proportional to their sparsity or squared Euclidean norms. Give error $\epsilon \sqrt{\mathbf{nnz}(A)} \cdot \|A\|_\infty$ and $\epsilon \cdot \|A\|_F$ respectively.
- Lots of open questions on sublinear time algorithms for eigenspectrum estimation.

Our Main Result

Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $\|A\|_\infty \leq 1$, and eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Main Result: There is an algorithm that reads $O\left(\frac{\log^3 n}{\epsilon^5}\right)$ entries of A and outputs $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \tilde{\lambda}_n$ such that, for all $i \in [n]$,

$$|\lambda_i - \tilde{\lambda}_i| \leq \epsilon \cdot n.$$



Some Remarks

How good are $\pm\epsilon n$ additive error approximations to each of A's eigenvalues?

- $|\lambda_i| \leq \|A\|_F \leq n$ for all i.
- $\sum \lambda_i^2 = \|A\|_F^2 \leq n^2$. So there are at most $1/\epsilon^2$ outlying eigenvalues with $|\lambda_i| \geq \epsilon \cdot n$.
- These are the only eigenvalues for which we give a non-trivial approximation.

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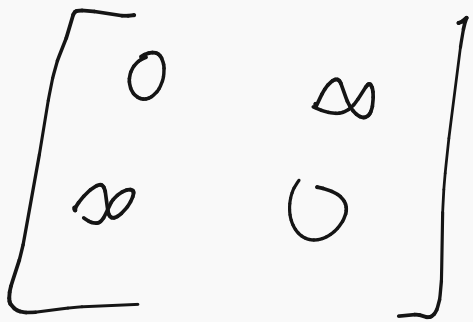
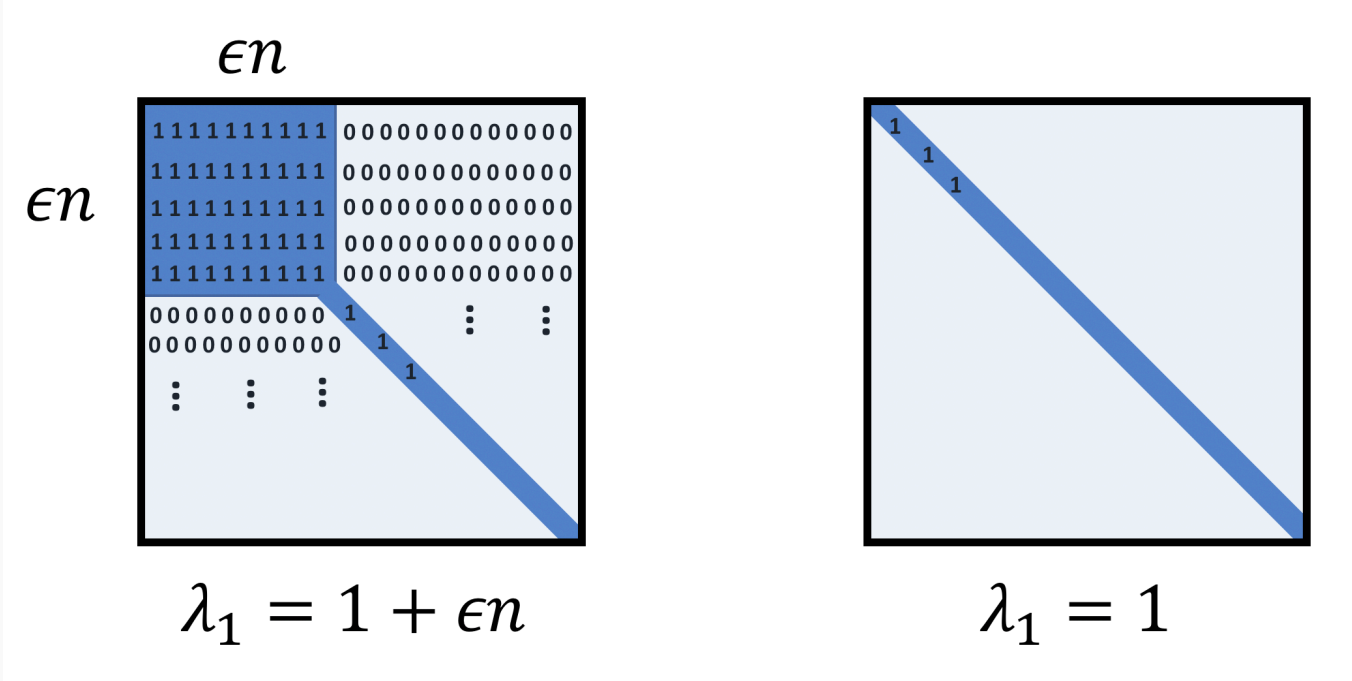
- These are the only eigenvalues for which we give a non-trivial approximation.
- It is easy to see that additive error scaling linearly in $n \cdot \|A\|_\infty$ is necessary.



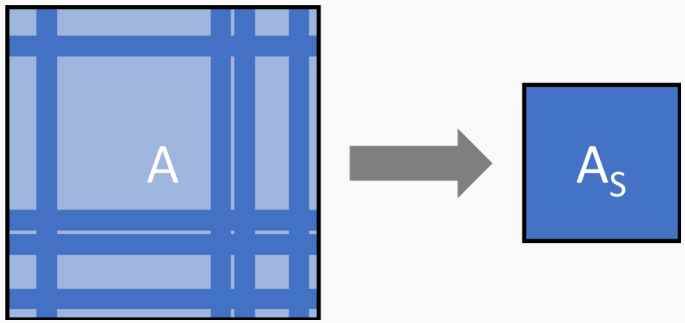
Lower Bound Instance



Only $\approx \epsilon^2 n^2$ entries differ across these matrices. Need to read at least $\Omega(1/\epsilon^2)$ entries before you can distinguish them with good probability.

The Algorithm

Our algorithm just computes the eigenvalues of a small random principal submatrix of A.



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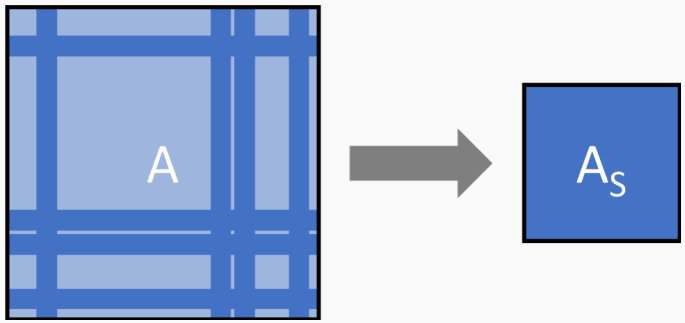

1. Let $s = \frac{c \log^3 n}{\epsilon^3}$, and let A_S be the random principal submatrix of A where each row/column is included independently with probability $\frac{s}{n}$.
2. Compute all eigenvalues of $\underbrace{\frac{n}{s}}$ $\cdot A_S$.
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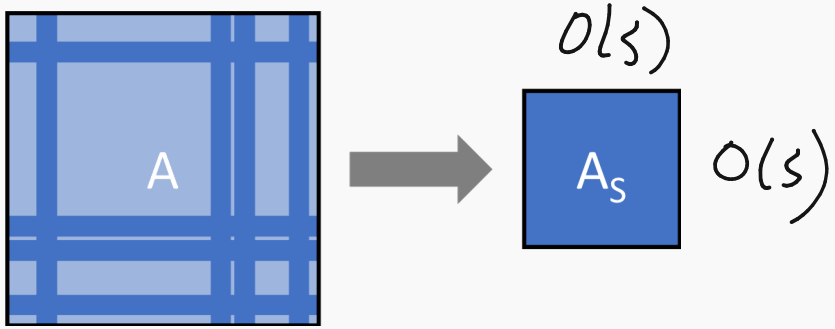

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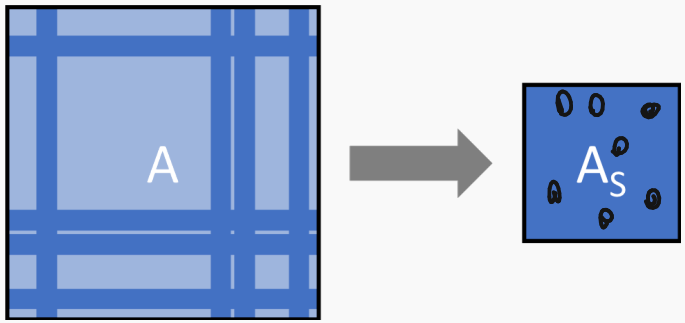

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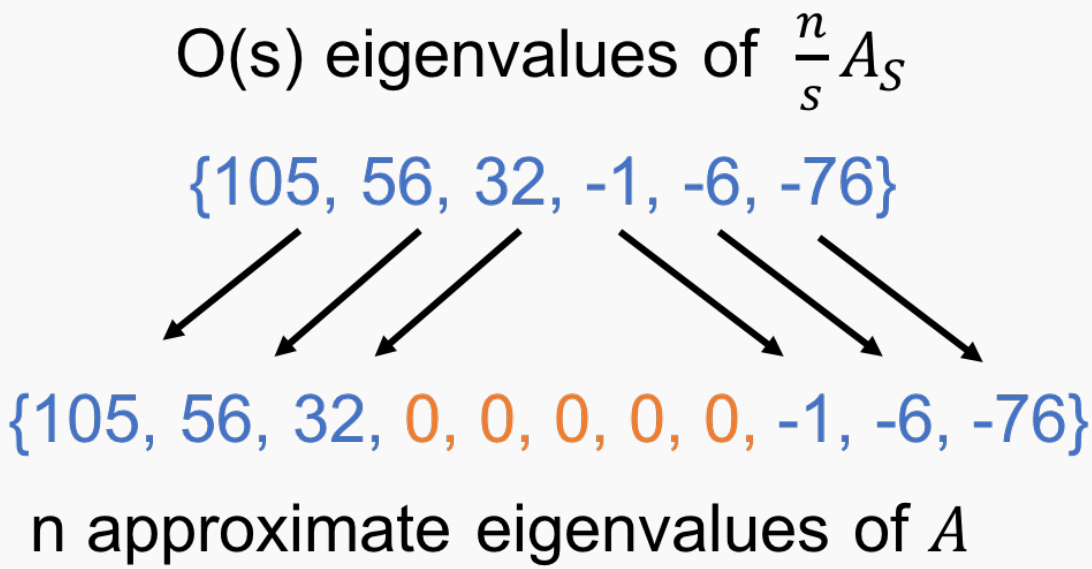
Note: To obtain improved sample complexity, we further sparsify A_S. I will ignore this optimization in this talk.

Eigenvalue Alignment

Approximate the large positive eigenvalues using the positive eigenvalues of A_S, the large negative ones using the negative eigenvalues of A_S, and the rest by 0.



Improved Bounds with Non-Uniform Sampling

Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $\|A\|_\infty \leq 1$, $\mathbf{nnz}(A)$ non-zero entries, and $\mathbf{nnz}(A_i)$ entries in row i.

Sparse Matrix Result: Given the ability to sample $i \in [n]$ with probability $\propto \frac{\mathbf{nnz}(A_i)}{\mathbf{nnz}(A)}$, there is an algorithm that reads $O\left(\frac{\log^{16} n}{\epsilon^{16}}\right)$ entries of A and outputs $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \tilde{\lambda}_n$ such that, for all $i \in [n]$,

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- Sparsity sampling requires sublinear queries per sample in the standard graph query model, where A is the adjacency matrix.
- Also possible via sampling a random non-zero entry when A is stored in sparse matrix format.
- Surprisingly, simply computing the eigenvalues of a random submatrix does not suffice here. Need to carefully zero out some entries of the sampled matrix.

Improved Bounds with Non-Uniform Sampling

Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $\|A_i\|_2^2$ equal to the squared Euclidean norm of row i.

Norm-Based Sampling Result: Given the ability to sample $i \in [n]$ with probability $\propto \frac{\|A_i\|_2^2}{\|A\|_F^2}$, there is an algorithm that reads $O\left(\frac{\log^{20} n}{\epsilon^{16}}\right)$ entries of A and outputs $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \tilde{\lambda}_n$ such that, for all $i \in [n]$,

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- Observe that $|\lambda_i| \leq \|A\|_F \leq \sqrt{\mathsf{nnz}(A)} \leq n$ for all i.
- Norm based sampling has been considered since early work on randomized linear algebra (e.g., [Freize, Kannan Vempala '04].
- Recently it has received significant attention in work on 'quantum-inspired' classical algorithms [Tang '19].
- Our result matches known sublinear time quantum algorithms for singular value approximation up to $\mathbf{poly}(\log n, 1/\epsilon)$ factors [Kerenidis, Prakash '16].

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- Our point-wise approximation guarantee immediately implies such a testing result, but can be stronger. However, our ϵ and $\log n$ dependence is worse.
- Subsequent to our work, [Needell, Swartworth, and Woodruff '22, '23] give optimal eigenvalue approximation algorithms in the matrix vector query model. Also see [Andoni, Nguyen '13]

Related Work

Several other works look at sublinear time spectral density estimation for normalized graph adjacency matrices, which have $\|A\|_\infty \leq 1$.

- The goal is to approximate the spectral density: the distribution placing mass $1/n$ at each eigenvalue.
- [Cohen-Steiner, Kong, Sohler, and Valiant '18] give a $2^{O(1/\epsilon)}$ time algorithm for ϵ error approximation in the Wasserstein-1 distance.
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- [Braverman, Krishnan, and Musco '22] give a $\tilde{O}(n/\epsilon^c)$ time algorithm for the same task.
- Our result gives ϵn error approximation in the Wasserstein-1 distance.
- Note that the eigenvalues of a general matrix with $\|A\|_\infty \leq 1$ lie in $[-n, n]$. Those of a normalized adjacency matrix lie in $[-1, 1]$.

Broader Context

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- $\tilde{O}(nk/\epsilon^c)$ time algorithms for near optimal rank-k approximation of positive semidefinite and distance matrices [Musco Musco '17, Musco Woodruff '17, Bakshi Woodruff '18, Indyk et al. '19]
- $\tilde{O}(d \cdot n^{1.173})$ time algorithm for estimating the top eigenvalue of a Gaussian kernel matrix [Backurs Indyk Musco Wagner '21]
- Sublinear time algorithms for structured matrices via sublinear time matrix vector multiplication [Shi Woodruff '19]
- 'Quantum-inspired' algorithms for linear algebra [Tang '18, Chepurko Clarkson Horesh Lin Woodruff '21]
- Classic additive error randomized SVD [Frieze Kannan Vempala '04, Drineas Kannan Mahoney '06].

Proof Approach

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Recall: For a uniformly random principal submatrix A_S, need to show that the eigenvalues of $\frac{n}{s} \cdot A_S$, appropriately padded with zeros, approximate all eigenvalues of A to error $\pm\epsilon n$.

- A_S will be $O(s) \times O(s)$ for $s = \mathbf{poly}(\log n, 1/\epsilon)$.

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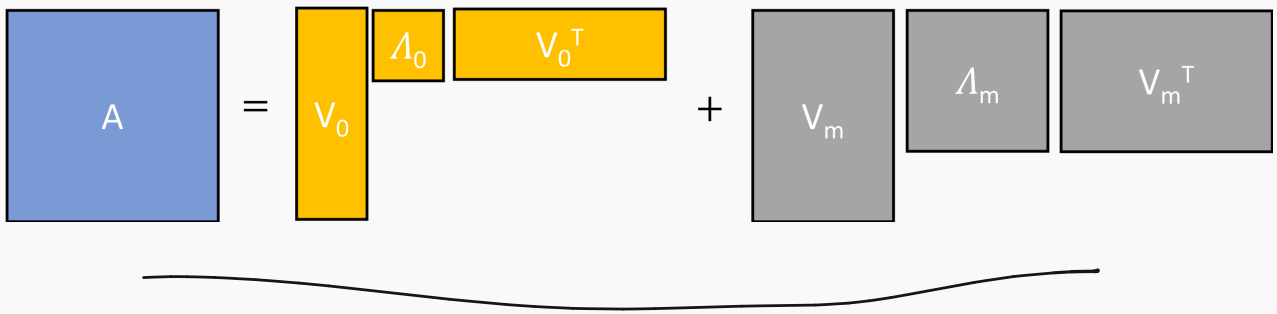
- Let $V_o \in \mathbb{R}^{n \times n_o}$ have columns equal to all eigenvectors with corresponding eigenvalues satisfying $|\lambda_i| \geq \epsilon n$. Let $V_m \in \mathbb{R}^{n \times n_m}$ have columns equal to the remaining eigenvectors.
- Let $\Lambda_o \in \mathbb{R}^{n_o \times n_o}$ and $\Lambda_m \in \mathbb{R}^{n_m \times n_m}$ be the corresponding diagonal eigenvalue matrices.
- Write $A = A_o + A_m$ where $A_o = V_o \Lambda_o V_o^T$ and $A_m = V_m \Lambda_m V_m^T$.



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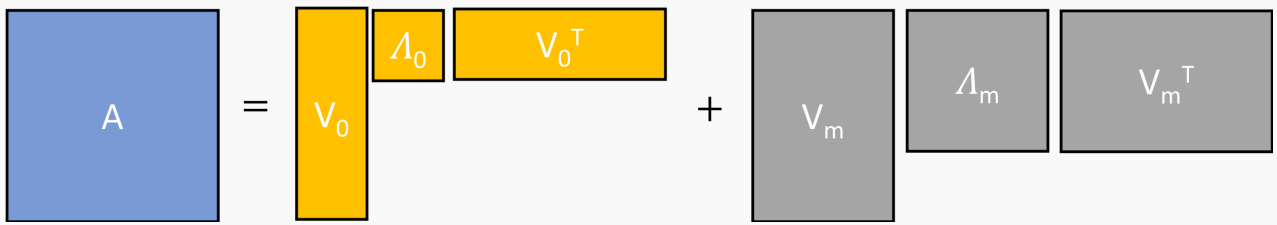

- Can similarly write $\frac{n}{s} \cdot A_S = S^T A S = S^T A_o S + S^T A_m S$.

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Step 2: Show that the eigenvalues of $S^T A_m S$ are all small in magnitude — i.e. $\leq \epsilon n$.

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Step 2: Show that the eigenvalues of $S^T A_m S$ are all small in magnitude — i.e. $\leq \epsilon n$.

Step 3: By Weyl's inequality, this gives that the eigenvalues of $S^T A S$, appropriately padded with zeros, approximate those of A to error $\pm O(\epsilon n)$.

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The above bound was an important part of [Bakshi, Chepurko, and Jayaram '20]. We show a related bound, that $\|[V_o]_{i,:}\|_2^2 \leq \frac{1}{\epsilon^2 n}$. I.e., we show that the leverage scores of V_o are uniformly bounded.

Sampling Outlying Eigendirections

So far: Can show that the outlying eigenspace of A is incoherent, with i^{th} leverage score bounded by $\|[V_o]_{i,:}\|_2^2 \leq \frac{1}{\epsilon^2 n}$.

Sampling Outlying Eigendirections

So far: Can show that the outlying eigenspace of A is incoherent, with i^{th} leverage score bounded by $\|[V_o]_{i,:}\|_2^2 \leq \frac{1}{\epsilon^2 n}$.

- Via a standard matrix Bernstein bound, can show that if we take $s = \tilde{O}(1/\epsilon^4)$ samples, with high probability $V_o^T S S^T V_o \approx V_o^T V_o \approx I$.
- Can use this to argue that the nonzero eigenvalues of $S^T A_o S = S^T V_o \Lambda_o V_o S$ are close to those of Λ_o — i.e., close to the outlying eigenvalues in A_o.
- This completes Step 1 of the proof.

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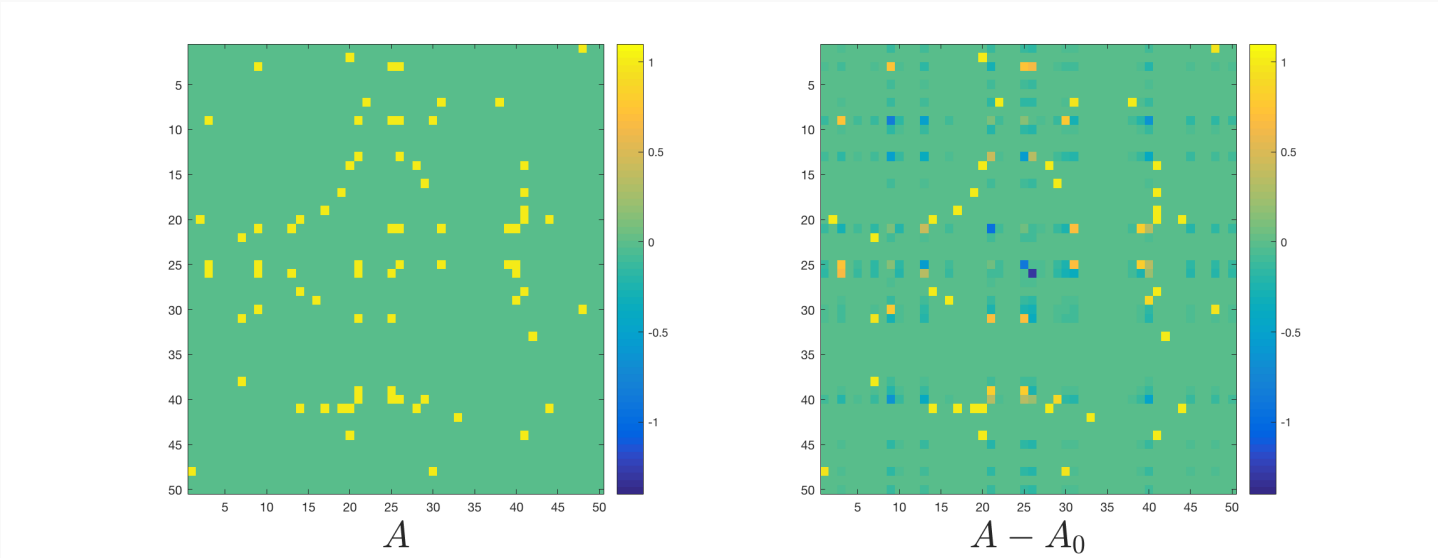

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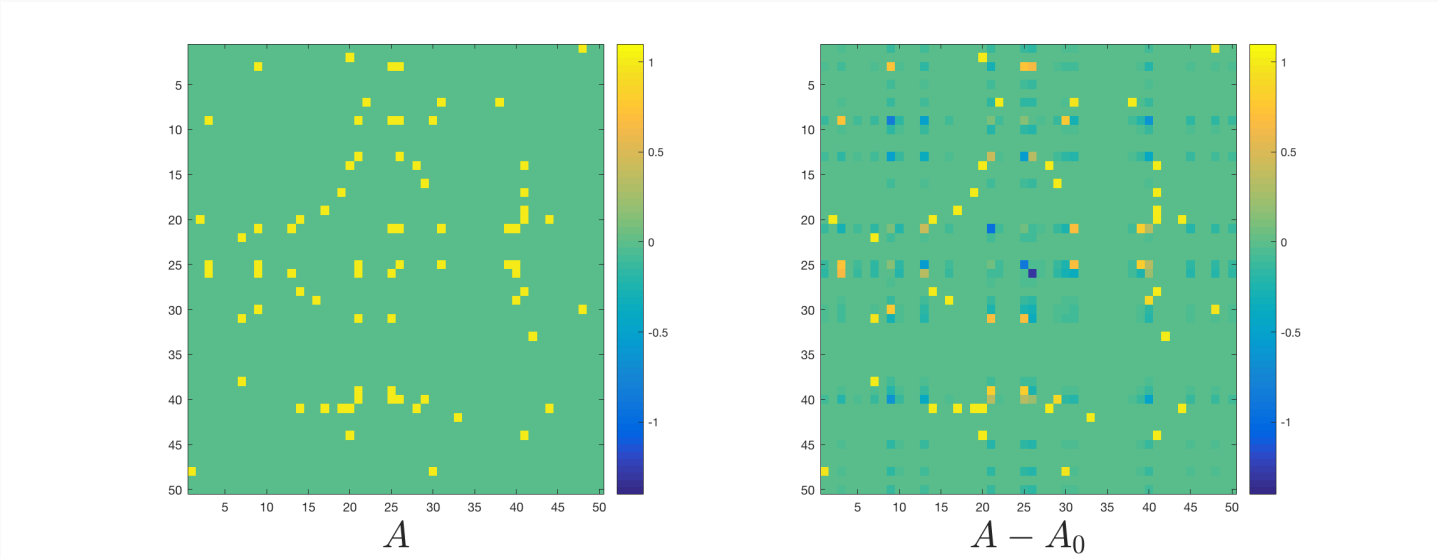

- Can then apply spectral norm bounds for random principal submatrices of bounded entry matrices [Rudelson Vershynin '07, Tropp '08], to show that $\|S^T A_m S\|_2 \leq \epsilon n$ when $s = \tilde{O}(1/\epsilon^2)$.

Proof Recap

Step 0: Split $A = A_o + A_m$ into its outlying and middle eigendirections.

Step 1: Prove that the outlying eigendirections of A are incoherent, and thus, uniform sampling approximately preserves the eigenvalues of A_o. I.e., the non-zero eigenvalues of $S^T A_o S$ approximate all the eigenvalues of A_o to $\pm\epsilon n$ error.

Step 2: Use the incoherence of A_o to argue that $A_m = A - A_o$ is entrywise bounded, and thus $\|S^T A_m S\|_2 \leq \epsilon n$.

Step 3: Combine the above to show that, after padding by zeros, the eigenvalues of $\frac{n}{s} \cdot A_S = S^T A S = S^T A_o S + S^T A_m S$ approximate those of A up to $\pm\epsilon n$ error.

Improved Bounds with Non-Uniform Sampling

Non-Uniform Sampling

Natural extension of random submatrix algorithm to sparsity-based sampling:

1. Let $s = \textbf{poly}(\log n, 1/\epsilon)$, and let A_S be the random principal submatrix of A where each row/column is included independently with probability $p_i = s \cdot \frac{\textbf{nnz}(A_i)}{\textbf{nnz}(A)}$.
2. Let D be the diagonal matrix with $D_{i,i} = \frac{1}{\sqrt{p_j}}$ if the i^{th} sampled row/column is row j.
3. Compute all eigenvalues of DA_SD.
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Observe that if the rows have uniform sparsity, $DA_SD = \frac{n}{s} \cdot A_S$, and we have exactly the uniform sampling algorithm.

Challenge 1: The Identity

Say that $A = I$, so sparsity-based sampling is just uniform sampling, so $DA_SD = \frac{n}{s} \cdot A_S$. Also $\mathbf{nnz}(A) = n$.

- Want to approximate all eigenvalues up to $\pm\epsilon\sqrt{n}$

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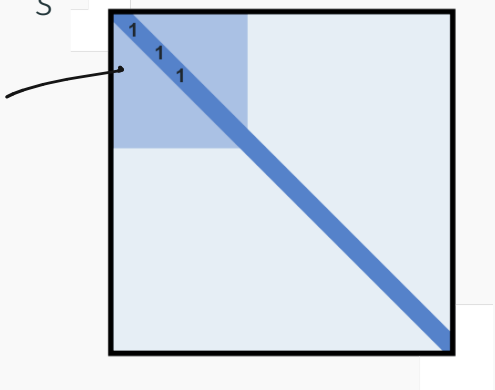

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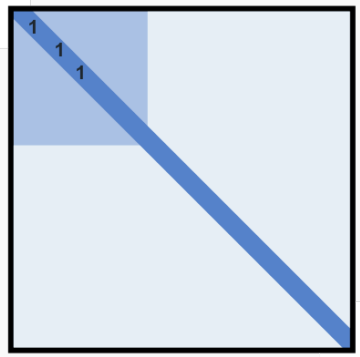

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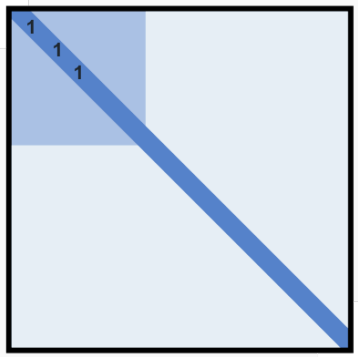

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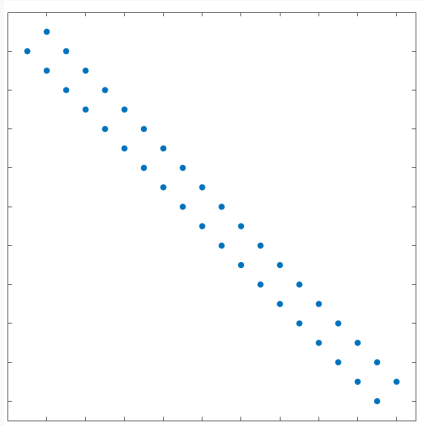
Simple Fix: Set the diagonal of A_S to 0. Introduces at most ± 1 error into the eigenvalue estimates and resolves this issue. When $A = I$, $A_S = 0$. So our eigenvalue estimates all have error 1.

Challenge 2: Sparse Rows and Columns

Consider A with $A_{i,i+1} = A_{i+1,i} = 1$ for all $i = 1, \ldots, n-1$.



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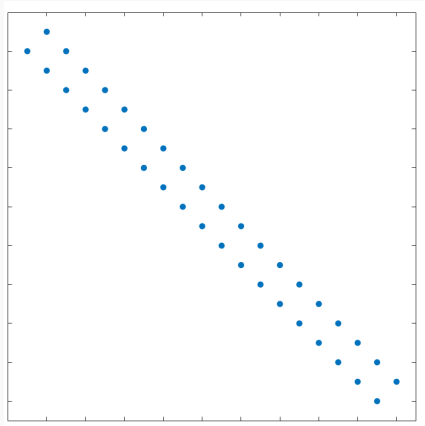

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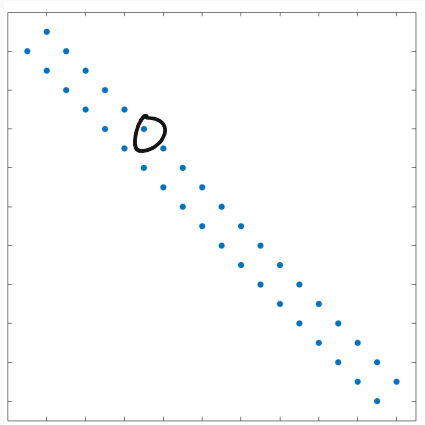

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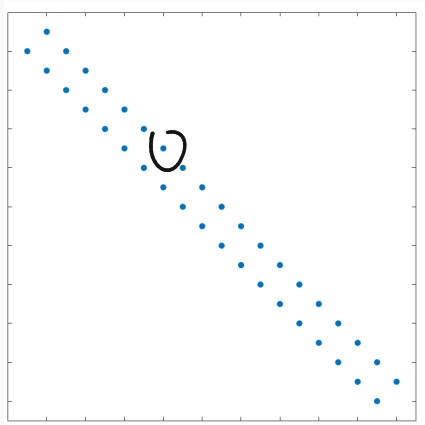

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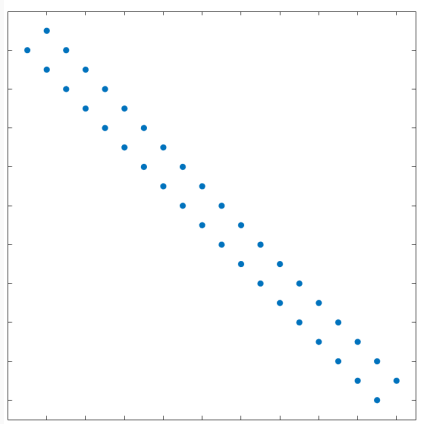

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- Can find many related examples: entries at the intersection of sparse rows/columns get scaled up too much in DA_SD, leading to large estimation errors.

Zeroing Out Entries

- To handle these cases, we argue that zeroing out the entries of A lying at the intersection of sparse rows and columns does not significantly alter the eigenvalues.
- Ensures that after sampling, no entries are scaled up too much in DA_SD, and lets us extend our uniform sampling proof to give $\pm\epsilon\sqrt{\mathbf{nnz}(A)}$ error with sparsity-based sampling.
- See paper for the full argument. Challenging to obtain bounds on S^TA_mS when S is sampled non-uniformly
- Our approach loses many $\mathbf{poly}(\log n, 1/\epsilon)$ factors. What is the right algorithm/analysis here?

Open Questions

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- Our uniform sampling bound samples a $\tilde{O}(1/\epsilon^3) \times \tilde{O}(1/\epsilon^3)$ random principal submatrix to give $\pm\epsilon n$ approximations.
- We believe this can be tightened to $O(1/\epsilon^2) \times O(1/\epsilon^2)$, matching a lower bound for algorithms based on principal submatrix sampling.
- When A is PSD, a trivial proof based on approximate matrix multiplication obtains the above bound.
- A simple matrix Bernstein bound shows that if we independently sample $O(\log n/\epsilon^2)$ rows and columns then we can approximate all singular values to $\pm\epsilon n$.
- Perhaps techniques from [Woodruff, Swartworth '23], who give optimal bounds in the matvec query model can be useful.

Open Questions

- Can we obtain tight $\tilde{O}(1/\epsilon^2)$ query complexity for computing $\pm\epsilon n$ approximations to all eigenvalues? Requires going beyond principal submatrix sampling, for which a simple $\Omega(1/\epsilon^4)$ lower bound holds. What is even a plausible algorithm here?
- Can we approximate $\|A\|_1 = \sum_{i=1}^{n} |\lambda_i|$ to error $1/2 \cdot n^{3/2}$ using $o(n^2)$ queries to A? [Balcan, Li, Woodruff, Zhang '18] show that $\tilde{\Omega}(n)$ is required. Key challenge problem in understanding how to approximate bulk spectral properties.
- For what classes of structured matrices can we give stronger approximation bounds? E.g., interesting bounds are known for normalized graph adjacency matrices. What else?

Follow Up Work

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- Can be improved to $O(n/\epsilon^2)$ entries when A is PSD.
- What else is possible with deterministic queries?