Tight Bounds for Volumetric Spanners in All Norms

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Problem Definition

Input: A set of vectors $X = \{v_1, \cdots, v_n\} \subset R^d$

Goal: Find a “bases” $B \subseteq X$ (specified by a subset $S \subset [n]$) s.t.

- Each $v \in X$ can be written as a linear combination of bases: $v = \sum_{i \in S} \alpha_i v_i$
- Coefficients are small (e.g., $|\alpha_i| \le C$ for every i),
- Size of the bases $|S|$ is small

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- Each $v \in X$ can be written as a linear combination of bases: $v = \sum_{i \in S} \alpha_i v_i$
- **Coefficients vector has small norm, $\|\boldsymbol{\alpha}\|$**
- Size of the bases $|S|$ is small

Background (Part I)

- **Auerbach Bases**
  - Any compact set $X \subseteq R^d$ admits a basis of size d with $||\alpha||_\infty \leq 1$ [Auerbach'29]
  - Rediscovered in ML community [Awerbuch and Kleinberg, COLT05]
  - Algorithms for online optimization and multi-armed bandit problems
  - Also referred to as **Barycentric Spanners**

For the online linear optimization problem, a novel idea in our work is to compute a special basis for the vector space spanned by the strategy set. This basis, which is called a *barycentric spanner*, has the property that all other strategies can be expressed as linear combinations *with bounded coefficients* of the basis elements. Against an oblivious adversary, we can sample basis ele-

Background (Part II)

- **Auerbach Bases/Barycentric Spanners** [Auerbach'29] [Awerbuch and Kleinberg'05]
- **Volumetric Spanners** [Hazan, Karnin and Meka'14][Hazan and Karnin'14]
  - Introduced for *multi-armed bandit* and *experimental design*
  - Defined w.r.t ℓ_2-norm *(bounding variance)*

A subset of $\mathcal{K} = \{v_1, \cdots, v_n\} \subset \mathbb{R}^d$ specified by $S \subseteq [n]$ is a *volumetric spanner* for $\mathcal{K}$, if $\mathcal{K} \subseteq \left\{\sum_{i\in S} \alpha_i v_i \,\middle|\, \sum_{i\in S} \alpha_i^2 \leq 1\right\}$

Theorem. A volumetric spanner of $\mathcal{K}$ of size at most $12d$ can be constructed in poly(n, d) time.

Open Question [Hazan, Karnin'16]: Improve the $12d$ upper bound. They showed lower bound is $d + 1$

Background (Part III)

- **Auerbach Bases/Barycentric Spanners** [Auerbach'29] [Awerbuch and Kleinberg'05]
- **Volumetric Spanners** [Hazan, Karnin and Meka'14][Hazan and Karnin'14]
- **Well-Conditioned Basis, Spanning Subsets**
  - Applications in matrix sketching and LRA [Dasgupta et al., SODA'08]
  - Recent improvements for LRA and related problems [Woodruff and Yasuda, STOC'23]

Given a set of vectors $A = \{v_1, \cdots, v_n\} \subset \mathbb{R}^d$, a set $S \subseteq [n]$ is a *$\boldsymbol{c}$-spanning subset* for A, if for every $v \in A$, $\exists \alpha \in \mathbb{R}^{|S|}$, s.t. $\sum_{i \in S} \alpha_i \cdot v_i = v$ and $\|\alpha\|_2 \leq \boldsymbol{c}$.

Background (Part III)

- **Auerbach Bases/Barycentric Spanners** [Auerbach’29] [Awerbuch and Kleinberg’05]
- **Volumetric Spanners** [Hazan, Karnin and Meka’14][Hazan and Karnin’14]
- **Well-Conditioned Basis, Spanning Subsets**
  - Applications in matrix sketching and LRA [Dasgupta et al., SODA’08]
  - Recent improvements for LRA and related problems [Woodruff and Yasuda, STOC’23]
    - By coreset for John Ellipsoids [Todd’16]
    - Superseded by [Hazan and Karnin’14] & [Hazan, Karnin and Meka’14]
    - Applications to a host of matrix approximation problems

Theorem. A $(1+\epsilon)$-spanning subset for A of size $O(d \log\log d + d/\epsilon)$ can be constructed in poly$(d, \epsilon, nnz(A))$ time.

Applications to Matrix Approx. Problems

[Woodruff and Yasuda'23]

- Oblivious ℓ_p Subspace Embedding
- Entrywise Huber Low Rank Approximation
- Average Top k Subspace Embedding
- Cascaded Norm Subspace Embedding
- Entrywise ℓ_p Low Rank Approximation
- ℓ_p Column Subset Selection
- Online …

Our Contributions

- Simple Local Search algorithm for **volumetric spanners**

Open Question [Hazan, Karnin'16]: Improve the $12d$ upper bound. They showed lower bound is $d + 1$

Theorem. A volumetric spanner of vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^d$ of size $\leq 3d$ can be constructed in poly(n, d) time using a single-swap Local Search.

Our Contributions

- Simple Local Search algorithm for **volumetric spanners**
- Improve both prior works

I. [Hazan et al.'14]

- Our algorithm is faster (roughly by a factor of d^2)
- Our algorithm is much simpler
  - They use spectral sparsification [Batson, Spielman, Srivastava'12] (by keeping track of barrier potential functions) + rounding step
  - Simpler volumetric spanners construction of size $O(d \log d \log n)$ in time $O(nd^2)$

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I. [Hazan et al.'14]

II. [Todd'16] & [Woodruff and Yasuda'23]

  - A greedy approach, but incurs additional $\log \log d$ factor in the size

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- Simple Local Search algorithm for **volumetric spanners**
- Improve both prior works
- For all ℓ_p norms
  - Local search still finds near optimal guarantee

i. $p = 1$: $n = \exp(d)$ vectors s.t. any spanner of size $< n$ has $\|\alpha\|_1 = \tilde{\Omega}(\sqrt{n})$

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- Simple Local Search algorithm for **volumetric spanners**
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  - Local search still finds near optimal guarantee

$1 \leq p \leq q$ and $x \in \mathbb{R}^n$, $\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \cdot \|x\|_q$

i. $p = 1$: $n = \exp(d)$ vectors s.t. any spanner of size $< n$ has $\|\alpha\|_1 = \widetilde{\Omega}(\sqrt{n})$

ii. $1 < p \leq 2$: any ℓ_p spanner with $\|\alpha\|_p \leq 1$ has size $\Omega\left(\left(\frac{d}{\log n}\right)^{\frac{p}{2p-2}}\right)$, ℓ_2 spanner's guarantee gives $O\left(d^{\frac{p}{2p-2}}\right)$ size

iii. $p > 2$: by ℓ_2 spanner construction, ℓ_p spanner of size $< 3d$ exists

Our Contributions

- Simple Local Search algorithm for **volumetric spanners**
- Improve both prior works
- For all ℓ_p norms
- Coresets for Minimum Volume Enclosing Ellipsoid (MVEE)

Given a symmetric convex body K, MVEE(K) denotes a minimum volume ellipsoid $\mathcal{E}$ s.t. $\mathcal{E} \supset K$.

Theorem. For any $K \subset \mathbb{R}^d$, there exists $S \subset K$ of size $O(\frac{d}{\epsilon})$ such that,

$$\mathrm{vol}\big(\mathrm{MVEE}(K)\big) \leq (1+\epsilon)^d \cdot \mathrm{vol}(\mathrm{MVEE}(S))$$

Our Contributions

- Simple Local Search algorithm for **volumetric spanners**
- Improve both prior works
- For all ℓ_p norms
- Coresets for Minimum Volume Enclosing Ellipsoid (MVEE)
    - ❑ Contact points of convex bodies [Rudelson'97][Srivastava'12]:
        - $H \subset K \subset (1+\epsilon)H$ and $|H| = O(d/\epsilon^2)$
    - ❑ Coreset of size $(d \log\log d + d/\epsilon)$ [Todd'16] and $O(d^2/\epsilon)$ [Kumar, Yildirim'05]

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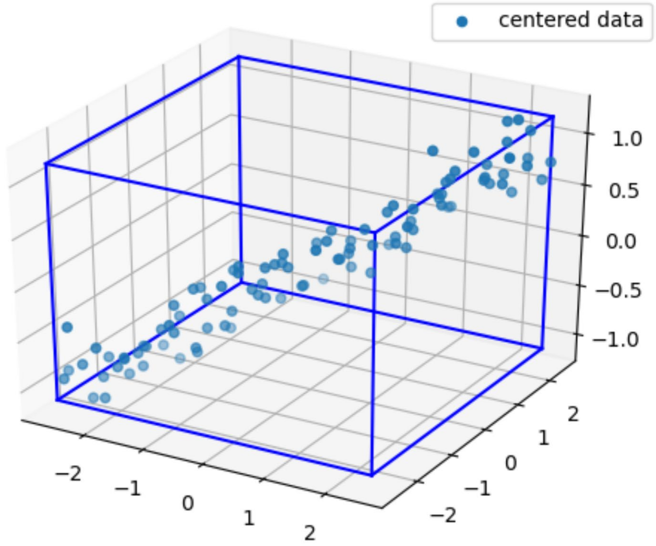
Our Contributions



credit: https://logicatcore.github.io/scratchpad/

- Simple Local Search algorithm for **volumetric**
- Improve both prior works
- For all ℓ_p norms
- Coresets for Minimum Volume Enclosing Ellipsoid (MVEE)
  - Weaker notion than standard coreset guarantees
  - Similar to the coreset definition of [Todd'16] & [Kumar and Yildirim'05]
  - Matches the coreset size for a much simpler object:
    - Axis-parallel bounding box of a set of points (*two extreme points along each axis*)

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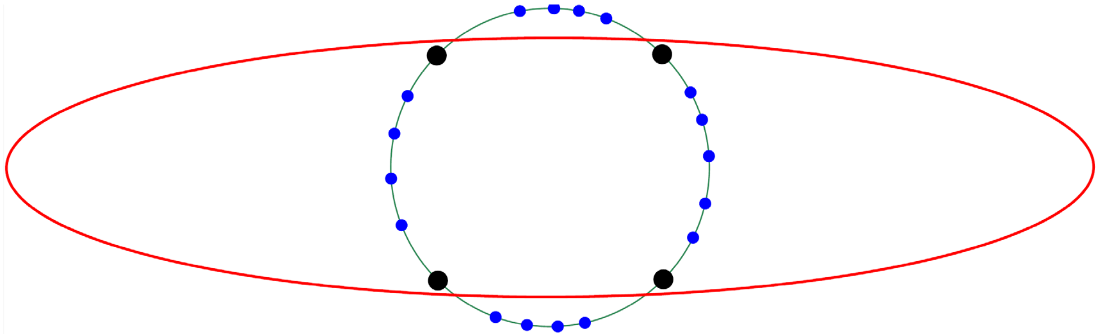

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- Improve both prior works
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  - Matches the coreset size for a much simpler object:
    - Axis-parallel bounding box of a set of points (*two extreme points along each axis*)
  - Ours is not a strong coreset
    - E.g., for any $\mathcal{E} \supset S$, $(1+\epsilon)\mathcal{E} \supset X$

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Local Search

The promised algorithm

Local Search Algorithm

Input. Set of vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^d$, δ and size parameter $r \geq d$

Output. Volumetric spanner of size r

w.l.o.g. vectors are of rank d

- **initialize** S of size r
  - i. d vectors in S by the greedy algorithm of volume maximization [Civril & Magdon-Ismail'09], and
  - ii. $r - d$ arbitrary vectors from $\{v_1, \cdots, v_n\}$
- **set** $M = \sum_{i \in S} v_i v_i^\top$
- **while** $\exists i \in S, j \in [n] \setminus S$ s.t. $\det(M - v_i v_i^\top + v_j v_j^\top) > (1 + \delta) \det(M)$ **do**
  - i. $S \leftarrow S \setminus \{i\} \cup \{j\}$
  - ii. $M \leftarrow M - v_i v_i^\top + v_j v_j^\top$
- **returns** S

Similar Local Search in the context of Det. Maximization [Madan et al.'19]

Local Search Algorithm: Runtime Analysis

Input. Set of vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^d$, δ and size parameter $r \geq d$

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  i. d vectors by the greedy volume maximization, and
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1) In every iteration, $\det(M)$ increases by a factor $(1+\delta)$

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1) In every iteration, $\det(M)$ increases by a factor $(1+\delta)$

2) Bounding the initial det, $\det(M^{\text{init}})$

Local Search Algorithm: Runtime Analysis

Goal: Bound $\det(M)$ **initialization step**

$T^* \subset [n]$: of size d, maximizing $\det\left(\sum_{i \in T^*} v_i v_i^\top\right)$

$G \subset [n]$: returned by the greedy algorithm **[CM'09]**

arameter $r \geq d$

1) In every iteration, $\det(M)$ increases by a factor $(1+\delta)$

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$> (1+\delta)\det(M)$ **do**

Local Search Algorithm: RUNTIME ANALYSIS

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$G \subset [n]$: returned by the greedy algorithm **[CM'09]**

$$\det(M^{\text{init}}) = \sum_{T \subset S, |T|=d} \det\left(\sum_{i \in T} v_i v_i^\top\right)$$

Cauchy-Binet Formula

arameter $r \geq d$

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$$\geq \det\left(\sum_{i\in G} v_i v_i^\top\right)$$

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Cauchy-Binet Formula

$$\det(M^{\text{init}}) = \sum_{T \subset S, |T| = d} \det\left(\sum_{i \in T} v_i v_i^\top\right)$$

$$\geq \det\left(\sum_{i \in G} v_i v_i^\top\right)$$

$$\geq \frac{1}{d!} \cdot \det\left(\sum_{i \in T^*} v_i v_i^\top\right)$$

By [CM'09]

arameter $r \geq d$

1) In every iteration, $\det(M)$ increases by a factor $(1 + \delta)$

2) Bounding the initial det, $\det(M^{\text{init}})$

$> (1 + \delta) \det(M)$ do

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Input. Set of vectors $\{v_1, \cdots, v_n\} \subset \mathbb{R}^d$, δ and size parameter $r \geq d$

Output. Volumetric spanner of size r

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- **while** $\exists i \in S, j \in [n] \setminus S$ s.t. $\det(M - v_i v_i^\top + v_j v_j^\top) >$
  i. $S \leftarrow S \setminus \{i\} \cup \{j\}$
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- **returns** S

1) In every iteration, $\det(M)$ increases by a factor $(1 + \delta)$

2) Bounding the initial det, $\det(M^{\text{init}})$

3) Bounding $\det(M^*)$**,** maximum possible det?

Local Search Algorithm: Runtime Analysis

Goal: Bound $\det(M^*)$

$T^* \subset [n]$: of **size d**, maximizing $\det\left(\sum_{i \in T^*} v_i v_i^\top\right)$

$$\det(M^*) = \sum_{T \subset S^*, |T| = d} \det\left(\sum_{i \in T} v_i v_i^\top\right)$$

Cauchy-Binet Formula

$$\leq \binom{r}{d} \cdot \det\left(\sum_{i \in T^*} v_i v_i^\top\right)$$

arameter $r \geq d$

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Local Search Algorithm: Runtime Analysis

Bounding Number of Iterations:

$$\det(M^*) \le \binom{r}{d} \cdot \det\left(\sum_{i\in T^*} v_i v_i^\top\right)$$

$$\det(M^{\text{init}}) \ge \frac{1}{(d!)^2} \cdot \det\left(\sum_{i\in T^*} v_i v_i^\top\right)$$

$\Longrightarrow$ **# of iterations**: $O\left(\frac{\log\left((d!)^2\cdot\binom{r}{d}\right)}{\delta}\right) = O\left(\frac{d\log r}{\delta}\right)$

$\Longrightarrow$ **runtime**: $O\left(\frac{d\log r}{\delta}\cdot nrd^2\right)$

e parameter $r \ge d$

1) In every iteration, $\det(M)$ increases by a factor $(1+\delta)$

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Local Search Algorithm: APPROXIMATION

Input. Set of vectors $\{v_1, \cdots, v_n\}$

Output. Volumetric Spanner of

For every $j \notin S$, $\forall i \in S$, *"when local search terminates"*

$$\det\left(M - v_i v_i^\top + v_j v_j^\top\right) < (1+\delta)\det(M)$$

Goal:

- U is a matrix whose columns are $\{v_i | i \in S\}$

Show there exists $\alpha_j \in \mathbb{R}^r$ s.t. $v_j = U\alpha_j$ & $\|\alpha_j\|_2 \leq 1$

- **initialize** S of size r
  1. d vectors by the greedy volume r
  2. $r - d$ arbitrary vectors from $\{v_1, \cdots, v_n\}$
- **set** $M = \sum_{i \in S} v_i v_i^\top$
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Bound on $\|\alpha_j\|$:

take $\alpha_j = U^+ v_j$

$$\Longrightarrow \|\alpha_j\|_2^2 = \alpha_j^\top \alpha_j = v_j^\top (U^+)^\top U^+ v_j = v_j^\top (UU^\top)^{-1} v_j$$

$$\Longrightarrow \|\alpha_j\|_2^2 = v_j^\top (M)^{-1} v_j \stackrel{\text{def}}{=} \tau_j$$

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Output. Volumetric Spanner of size r

leverage score

$\tau_i := v_i^\top M^{-1} v_i$, $\tau_{ij} := v_i^\top M^{-1} v_j$

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$\tau_i := v_i^\top M^{-1} v_i$, $\tau_{ij} := v_i^\top M^{-1} v_j$

i. $\sum_{i \in S} \tau_i = d$

ii. For any $i, j \in [d]$, $\tau_{ij} = \tau_{ji}$

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- **returns** S

$$\det(M - v_i v_i^\top + v_j v_j^\top) = \det(M)\left((1 - \tau_i)(1 + \tau_j) + \tau_{ij}^2\right)$$

Local Search Algorithm: Approximation

Matrix Determinant Lemma (Invertible A)

$$\det(A + uv^\top) = (1 + v^\top A^{-1} u) \det(A)$$

Sherman-Morrison Formula (Invertible A)

$$(A + uv^\top)^{-1} = A^{-1} + \frac{A^{-1} u v^\top A^{-1}}{1 + v^\top A^{-1} u}$$

$\tau_i := v_i^\top M^{-1} v_i$, $\tau_{ij} := v_i^\top M^{-1} v_j$

i. $\sum_{i \in S} \tau_i = d$

ii. For any $i, j \in [d]$, $\tau_{ij} = \tau_{ji}$

d size parameter $r \geq d$

➢ initialize S of size r

and

➢ while $\exists i \in S, j \in [n] \setminus S$ s.t. $\det(M - v_i v_i^\top + v_j v_j^\top) > (1+\delta)\det(M)$ do

i. $S \leftarrow S \setminus \{i\} \cup \{j\}$

ii. $M \leftarrow M - v_i v_i^\top + v_j v_j^\top$

Matrix Det. Lemma + Sherman-Morrison Formula

$$\det(M - v_i v_i^\top + v_j v_j^\top) = \det(M)\left((1 - \tau_i)(1 + \tau_j) + \tau_{ij}^2\right)$$

Local Search Algorithm: APPROXIMATION

For every $j \notin S$, $\forall i \in S$, *“when local search terminates”*

$$\det\left(M - v_i v_i^\top + v_j v_j^\top\right) < (1+\delta)\det(M)$$

Goal:

- U is a matrix whose columns are $\{v_i | i \in S\}$

Show there exists $\alpha_j \in \mathbb{R}^r$ s.t. $v_j = U\alpha_j$ & $\|\alpha_j\|_2 \leq 1$

$\tau_i := v_i^\top M^{-1} v_i$, $\tau_{ij} := v_i^\top M^{-1} v_j$

i. $\sum_{i \in S} \tau_i = d$

ii. For any $i, j \in [d]$, $\tau_{ij} = \tau_{ji}$

$$\det\left(M - v_i v_i^\top + v_j v_j^\top\right) = \det(M)\left((1-\tau_i)(1+\tau_j) + \tau_{ij}^2\right)$$

parameter $r \geq d$

- set $M = \sum_{i \in S} v_i v_i^\top$
- while $\exists i \in S, j \in [n] \setminus S$ s.t. $\det(M - v_i v_i^\top + v_j v_j^\top) > (1+\delta)\det(M)$ do
  - i. $S \leftarrow S \setminus \{i\} \cup \{j\}$
  - ii. $M \leftarrow M - v_i v_i^\top + v_j v_j^\top$
- returns S

Local Search Algorithm: APPROXIMATION

For every $j \notin S$, $\forall i \in S$, *"when local search terminates"*

$$\left((1-\tau_i)(1+\tau_j)+\tau_{ij}^2\right)\det(M) < (1+\delta)\ \det(M)$$

For a fixed j, sum over all $i \in [S]$:

$$\sum_{i\in[S]}\left((1-\tau_i)(1+\tau_j)+\tau_{ij}^2\right) < r(1+\delta)$$

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Local Search Algorithm: APPROXIMATION

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Hence, $\tau_j \leq \frac{d+r\delta}{r-d+1}$

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Local Search Algorithm: APPROXIMATION

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Hence, $\tau_j \le \frac{d+r\delta}{r-d+1} \xrightarrow{r=3d,\,\delta=1/3} \|\alpha_j\|_2 = \tau_j < 1$

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Volumetric Spanner Bounds (ℓ_p norm)

$p = 2$

- Local Search finds a ℓ_2 spanner of size at most $3d$

Volumetric Spanner Bounds (ℓ_p norm)

$p = 2$

$p = 1$

- Almost orthogonal vectors $v_1 \cdots v_m \in \mathbb{R}^d$
- $m = \exp(\Omega(d))$ and $\forall i, j \in [m]$, $|\langle v_i, v_j \rangle| = O(\sqrt{(\log m)/d})$
- **Any ℓ_1 spanner of size $< m$ has $\|\alpha\|_1 \geq \Omega(\sqrt{d/\log m})$**
- Matching upper bound by ℓ_2 spanner and Cauchy-Schwarz

Volumetric Spanner Bounds (ℓ_p norm)

$p = 2$

$p = 1$

$1 < p < 2$

- Lower bound: $\|\alpha\|_p = \Omega\left(\left(\frac{d}{\log n}\right)^{\frac{p}{2p-2}}\right)$
- Upper bound: $\|\alpha\|_p = O(d^{\frac{p}{2p-2}})$

Volumetric Spanner Bounds (ℓ_p norm)

$p = 2$

$p = 1$

$1 < p < 2$

$p > 2$

- Following from the ℓ_2 spanner, ℓ_p spanner of size $\leq 3d$ exists

Summary and Open Questions

- Analysis of simple local search for volumetric spanner
- Applications to Matrix Approximation Problems

Open Question [Hazan, Karnin'16]: Improve the $12d$ upper bound. They showed lower bound is $d + 1$

Summary and Open Questions

- Analysis of simple local search for volumetric spanner
- Applications to Matrix Approximation Problems

Open Question [Hazan, Karnin'16]: Improve the $\mathbf{3d-1}$ upper bound. They showed lower bound is $d+1$

- Gaps for other norms
- Further applications of volumetric spanners

Thank You!