Instance-optimal Database Joins

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Relational AI

Logic and Algorithms in Database Theory and AI Boot Camp
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Aug 21, 2023

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Beyond Worst-case Analysis in DB: Why?

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    - Build indices, DS, etc
  - Query Evaluation
    - Reuse Prebuilt DSs for many queries (amortization)
    - Sublinear time is possible

Beyond Worst-case Analysis: Some Models

- Parameterized Complexity
- Adaptive Analysis
- Instance Optimality
- Average-case
- ...

Instance Optimality: Goal

- Given an input instance I to some problem P

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- Design an algorithm A^* whose runtime is $O(m \cdot f(I))$ for every I
  - m is the optimality ratio

Instance Optimality: General Approach

- Every algorithm must produce a proof $\mathcal{C}$ of output correctness (certificate)

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- Every algorithm must produce a proof $\mathcal{C}$ of output correctness (certificate)
- The minimum certificate size $|\mathcal{C}|$ is a lower bound on the runtime

Instance Optimality: A Meta-Algorithm

- Fagin et al, JCSS'03: Database aggregation problem

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- Analysis
  - $|\mathcal{C}| \leqslant m \cdot |\mathcal{C}'|,$ for any certificate $\mathcal{C}'$

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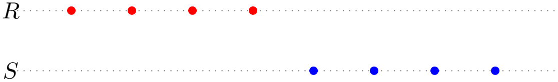
- Some instances are easier than others

Instance 1



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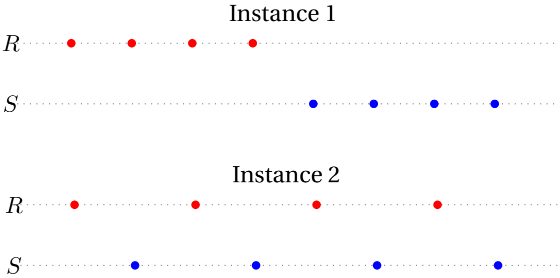


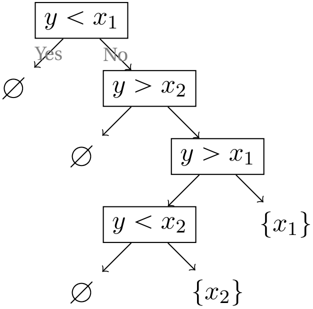
Instance-optimal Set Intersection

- Hwang and Lin, SIAM'72: "Leap-frogging" intersection
- Demaine et al., SODA'00: A form of comparison certificates
- Barbay and Kenyon, SODA'02: "Partition" certificates
- Ngo et al., PODS'14: "Stronger" comparison certificates

Instance-optimal Set Intersection

- Algorithm ⇒ Decision Tree

$$\{x_1 < x_2\} \cap \{y\}$$



Instance-optimal Set Intersection

- Algorithm $\Rightarrow$ Decision Tree
- Worst-case runtime $\Rightarrow$ Tree depth

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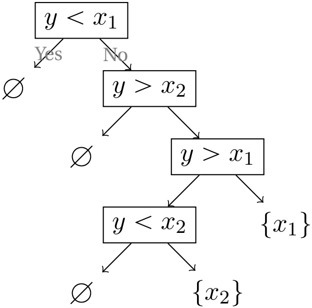

Instance-optimal Set Intersection

- Algorithm $\Rightarrow$ Decision Tree
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- Instance-specific runtime $\Rightarrow$
  Leaf depth

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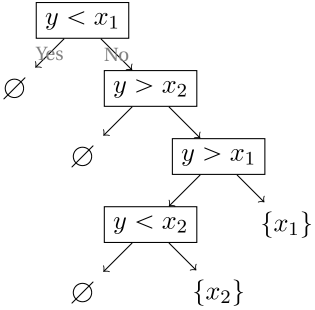

Instance-optimal Set Intersection

- Algorithm ⇒ Decision Tree
- Worst-case runtime ⇒
  Tree depth
- Instance-specific runtime ⇒
  Leaf depth
- Instance Certificate ⇒
  Leaf-to-root path

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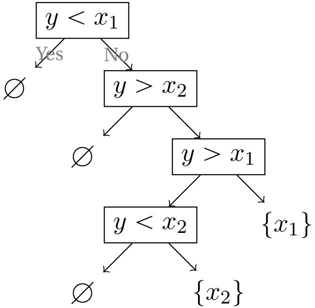

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- Consider the class of algorithms that access the input only through comparisons
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    - $R[i]$ is the i-th smallest element in R
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    - $\theta \in \{<, =, >\}$

Comparison-based Certificates

- Input
  - $R = \{1, 5, 7\}$
  - $S = \{2, 3, 4, 7, 9, 10\}$
- Output
  - $Q = \{7\}$

Comparison-based Certificates

- Input
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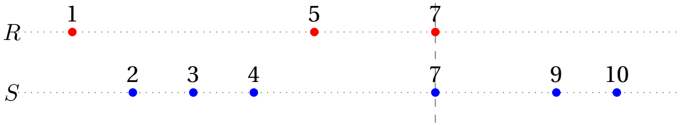
- Comparison-based certificate
  - $R[1] < S[1]$
  - $R[2] > S[3]$
  - $R[3] = S[4]$
  - $R[4] = \infty$



Gap-based Certificates

- $\mathcal{C}_{\Box}$ is a collection of gap intervals from R and S that cover every point *not* in $R \cap S$

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- $\mathcal{C}_\square$ is a collection of gap intervals from R and S that cover every point *not* in $R \cap S$
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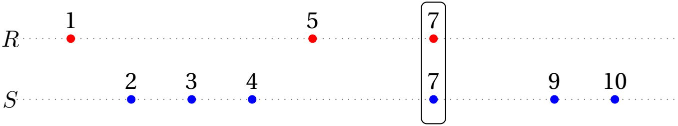

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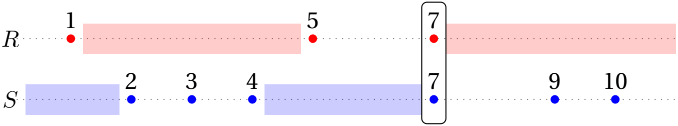

From $\mathcal{C}_<$ to $\mathcal{C}_\square$

$$|\mathcal{C}_\square| + Z = O(|\mathcal{C}_<|)$$

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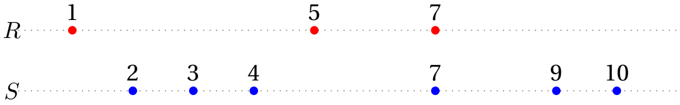
Proof idea:

- Take (R, S) and $\mathcal{C}_<$
- $\mathcal{C}_\square \leftarrow \emptyset, \quad \mathcal{Z} \leftarrow \emptyset$



$$\mathcal{C}_< = \{R[1] < S[1], \quad R[2] > S[3], \quad R[3] = S[4], \quad R[4] = \infty\}$$

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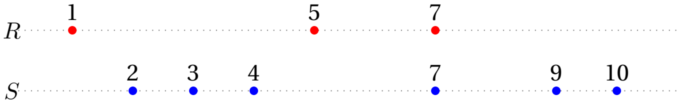

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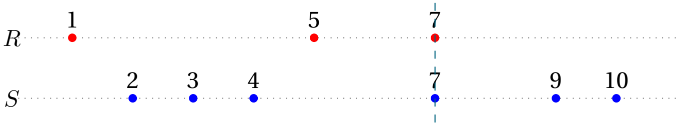

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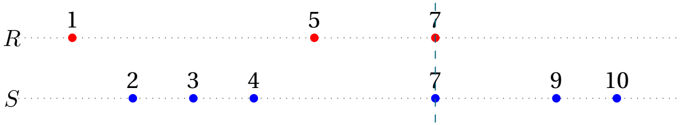

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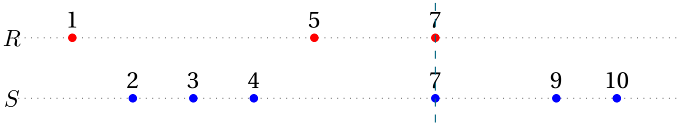

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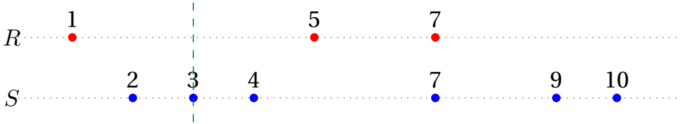

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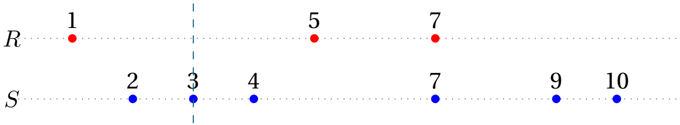

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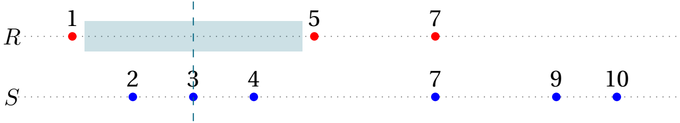

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An Instance-Optimal Algorithm for $\cap$

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- Repeat: Find the smallest t outside $\mathcal{C}_{\square} \cup \mathcal{Z}$
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    - Add t to $\mathcal{Z}$
  - Otherwise
    - Find $R[i] < t < R[i+1]$
    - Find $S[j] < t < S[j+1]$
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Lemma: $|\mathcal{C}_\square| \leqslant 2 \cdot |\mathcal{C}'_\square|, \quad$ for any $\mathcal{C}'_\square$

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Lemma: $|\mathcal{C}_\square| \leqslant 2 \cdot |\mathcal{C}'_\square|, \quad$ for any $\mathcal{C}'_\square$

Runtime: $O(|\mathcal{C}_\square| + Z) = O(|\mathcal{C}_<|)$

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Instance-optimal Database Joins

- Database Join Query

$$Q(\mathbf{X}) = \bigwedge_F R_F(\mathbf{X}_F)$$

- Examples
  - $Q(A, B) = R(A, B) \wedge S(A) \wedge T(B)$
  - $Q(A, B, C) = R(A, B) \wedge S(B, C) \wedge T(C, A)$
  - $Q(A) = R(A) \wedge S(A)$

Relation Indices $\Rightarrow$ Comparison Certificates

- $R = \{(2,1),(2,2),(2,3),(4,2)\}$
- Suppose $R(A,B)$ is indexed first on A and then on B

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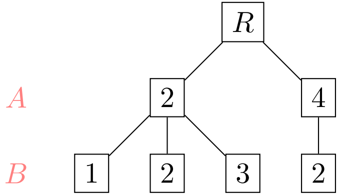

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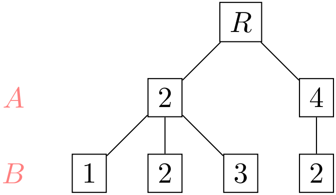

$$
\begin{aligned}
R[1] &= 2 \\
R[2] &= 4 \\
R[1,1] &= 1 \\
R[2,1] &= 2 \\
R[2,2] &= \infty
\end{aligned}
$$

Relation Indices ⇒ Comparison Certificates

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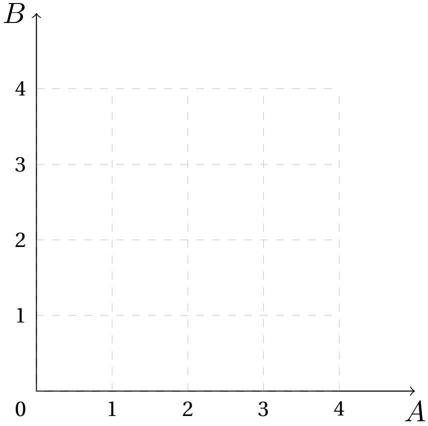

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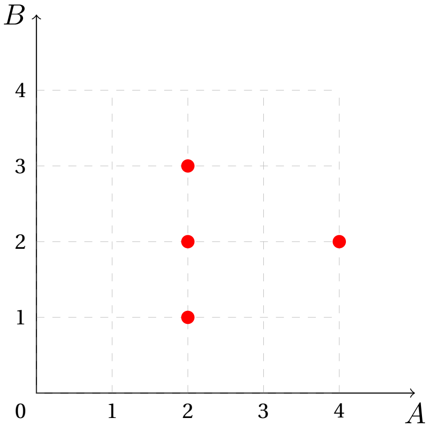

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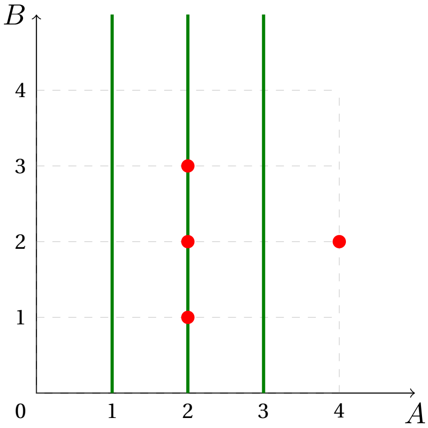
  - $S = \{1, 2, 3\}$



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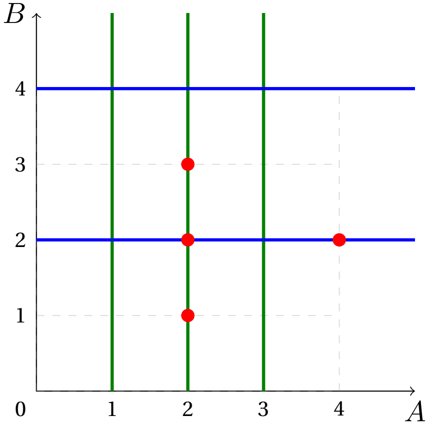
  - $T = \{2, 4\}$



Relation Indices ⇒ Comparison Certificates

- $Q(A, B) = R(A, B) \wedge S(A) \wedge T(B)$
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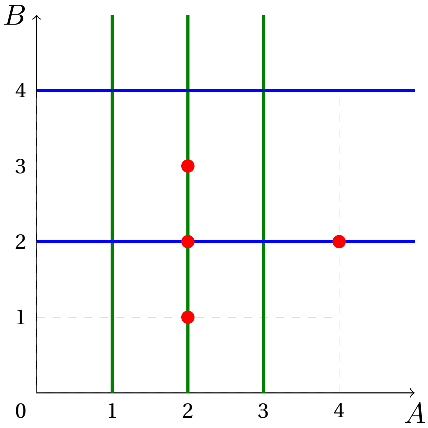

$[A, B]$-Comparison Certificate:

$$\begin{aligned} R[1] &= S[2] \\ R[2] &> S[3] \\ S[4] &= \infty \end{aligned}$$

$$\begin{aligned} T[1] &= R[1,2] \\ T[2] &> R[1,3] \\ R[1,4] &= \infty \end{aligned}$$

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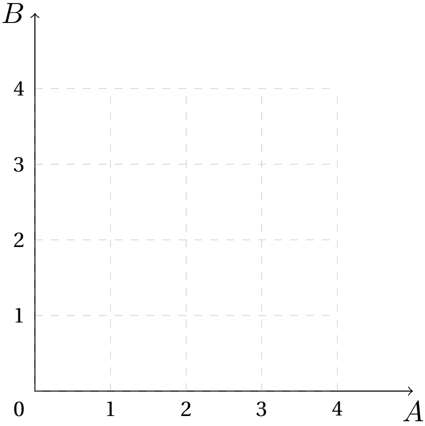

Relation Indices ⇒ Gap Certificates

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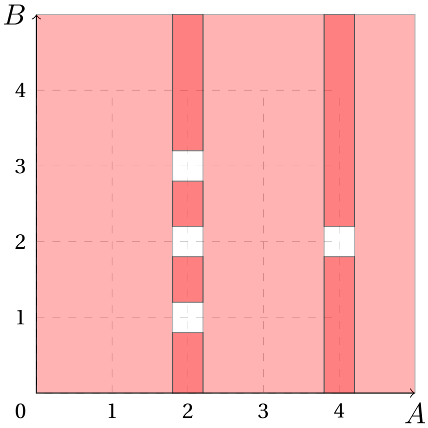

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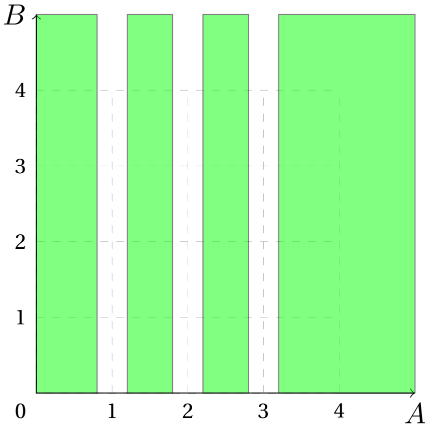

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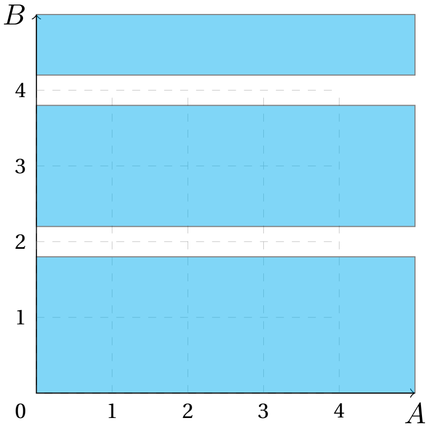

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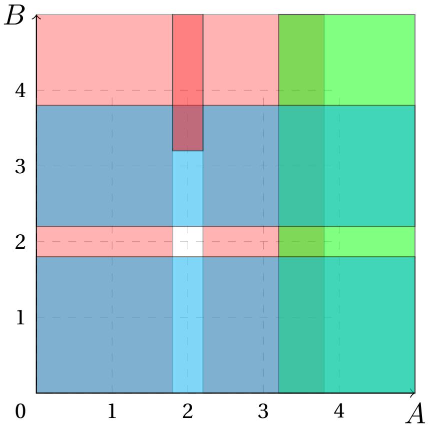


$[A, B]$-Gap Certificate

Background

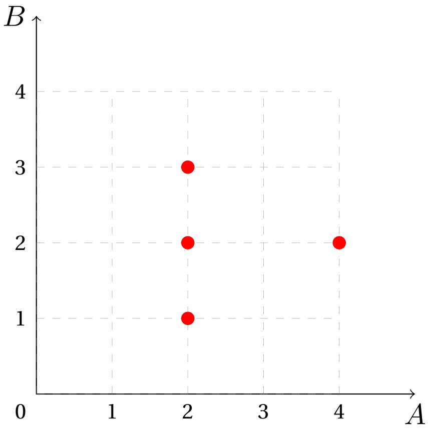
- Ngo et al, PODS'14:
  - $|\mathcal{C}_{\Box}^{\text{gao}}| + Z = O(|\mathcal{C}_{<}^{\text{gao}}|)$
  - Minesweeper algorithm
    - First Instance-optimal Join Algorithm
    - $O(|\mathcal{C}_{<}^{\text{gao}}| + Z)$ for β-acyclic queries
    - $O(|\mathcal{C}_{<}^{\text{gao}}|^{w+1} + Z)$ for treewidth w-queries

Background

- Abo Khamis et al, PODS'15:
  - A tighter notion of certificate $|\mathcal{C}_{\Box}| \leqslant |\mathcal{C}_{\Box}^{\text{gao}}|$
  - Tetris algorithm
    - works over different kinds of indexes.
    - achieves the fractional hypertree-width bound.
    - achieves a series of instance-optimality results.
  - A **proof system** for joins where
    - proof complexity lower bounds/upper bounds are developed.
    - proof sizes precisely capture the runtime of Tetris.

Multiple Indexes $\Rightarrow \mathcal{C}_\square$

$$R = \{(2,1),(2,2),(2,3),(4,2)\}$$



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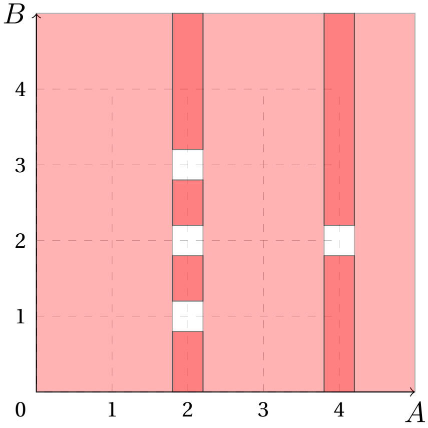

[A,B]-Index

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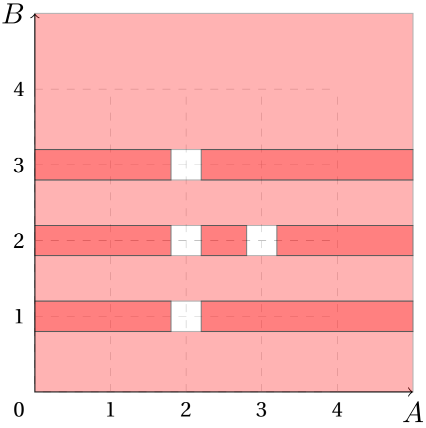

[B,A]-Index

Multiple Indexes $\Rightarrow \mathcal{C}_\Box$

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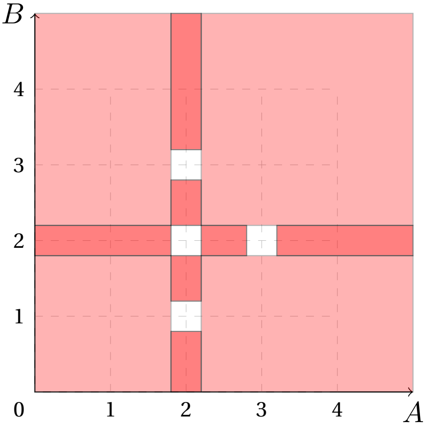

Quad tree

Box Cover Problem

Problem (BCP)

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Relational Join can be reduced to BCP

BCP Certificates

Definition (Box Certificate)

Given a set of boxes $\mathcal{A}$, a *box certificate* $\mathcal{C}_\square$ for $\mathcal{A}$ is a *minimum-sized* subset of $\mathcal{A}$ such that

$$\bigcup_{\mathbf{c} \in \mathcal{C}_\square} \mathbf{c} = \bigcup_{\mathbf{a} \in \mathcal{A}} \mathbf{a}.$$

Table of Contents

Dyadic Boxes

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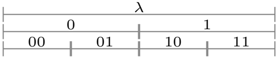

Dyadic Boxes

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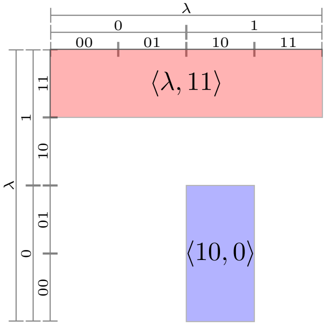
- A dyadic box is an n-tuple of binary strings of length $\leqslant d$.



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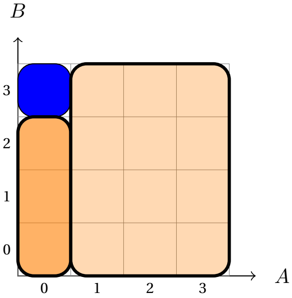

Gap boxes for $R(A, B)$

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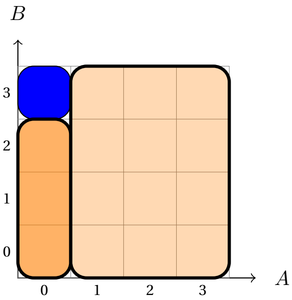

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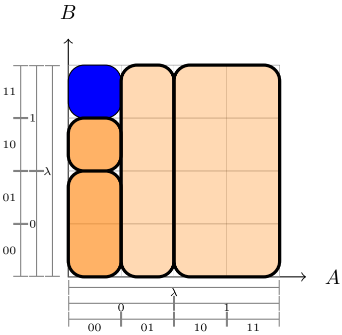

Corresponding dyadic boxes

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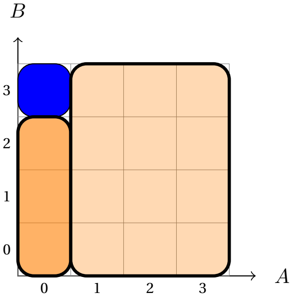

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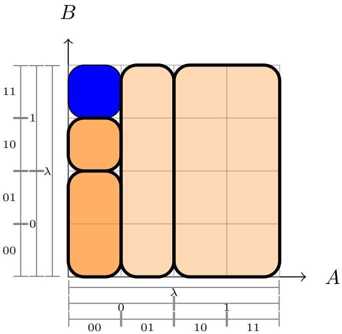

Corresponding dyadic boxes

- Every n-tuple is contained in $\leqslant d^n = \tilde{O}(1)$ dyadic boxes.

Geometric Resolution ...

... is an inference system for BCP.

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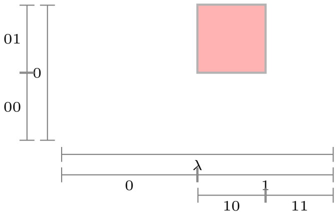

$\langle 10, 01 \rangle$

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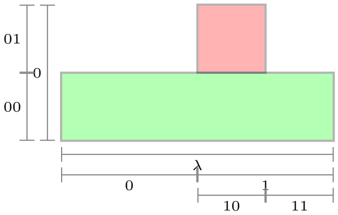

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$\langle \lambda, 00 \rangle$

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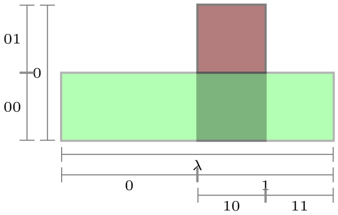

$$\frac{\color{red}{\langle 10, 01\rangle} \quad \color{green}{\langle \lambda, 00\rangle}}{\langle 10, 0\rangle}$$

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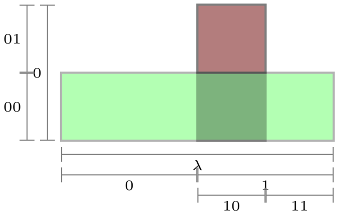

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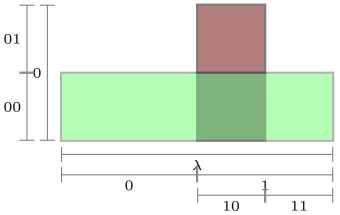

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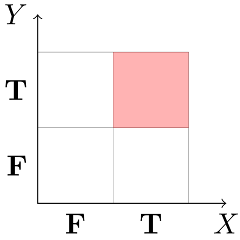

$\color{red}{\bar{X} \vee \bar{Y}}$

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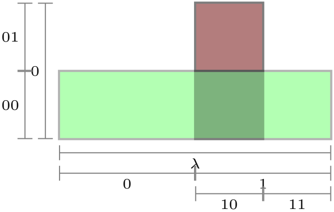

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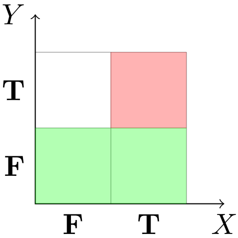

$\bar{X} \vee \bar{Y}$

Y

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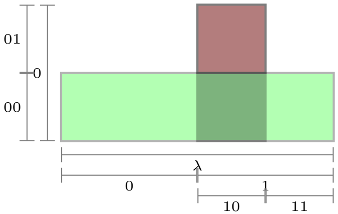

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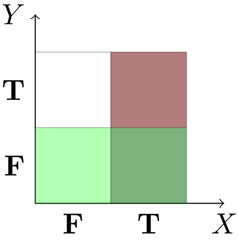

$$\frac{\begin{array}{c}\bar{X} \vee \bar{Y} \\ Y\end{array}}{\bar{X}}$$

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- Three main variations:
  - Geometric Resolution
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  - Tree Ordered Geometric Resolution

(General) Geometric Resolution

$$\mathbf{w} = \mathsf{Resolve}(\mathbf{w}_1, \mathbf{w}_2)$$

$$\begin{array}{rcl}
\mathbf{w}_1 & = & \langle y_1 \;,\; \ldots \;,\; y_{\ell-1} \;,\; x_\ell 0 \;,\; y_{\ell+1} \;,\; \ldots \;,\; y_n \rangle \\
\mathbf{w}_2 & = & \langle z_1 \;,\; \ldots \;,\; z_{\ell-1} \;,\; x_\ell 1 \;,\; z_{\ell+1} \;,\; \ldots \;,\; z_n \rangle \\
\hline
\mathbf{w} & = & \langle \ldots \;,\; y_{\ell-1} \cap z_{\ell-1} \;,\; x_\ell \;,\; y_{\ell+1} \cap z_{\ell+1} \;,\; \ldots \rangle
\end{array}$$

Ordered Geometric Resolution

$$\mathbf{w} = \mathsf{Resolve}(\mathbf{w}_1, \mathbf{w}_2)$$

$$\begin{array}{rcl}
\mathbf{w}_1 & = & \langle y_1 \ , \ \ldots \ , \ y_{\ell-1} \ , \ x_\ell 0 \ , \ \lambda \ , \ \ldots \ , \ \lambda\rangle \\
\mathbf{w}_2 & = & \langle z_1 \ , \ \ldots \ , \ z_{\ell-1} \ , \ x_\ell 1 \ , \ \lambda \ , \ \ldots \ , \ \lambda\rangle \\
\hline
\mathbf{w} & = & \langle \ldots \ , \ y_{\ell-1} \cap z_{\ell-1} \ , \ x_\ell \ , \ \lambda \ , \ \ldots \ , \ \lambda\rangle
\end{array}$$

Tree-Ordered Geometric Resolution

- Proof is a Tree (as opposed to DAG)
  - No caching

Table of Contents

Tetris: a recursive algorithm for BCP

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is b covered by the union of boxes in $\mathcal{A}$?

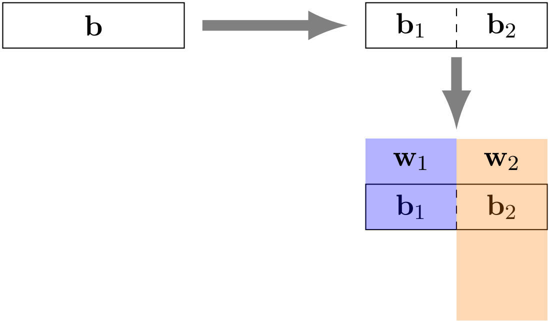
$\boxed{\mathrm{b}}$

Tetris: a recursive algorithm for BCP

split $\mathbf{b}$ into two halves $\mathbf{b}_1, \mathbf{b}_2$,

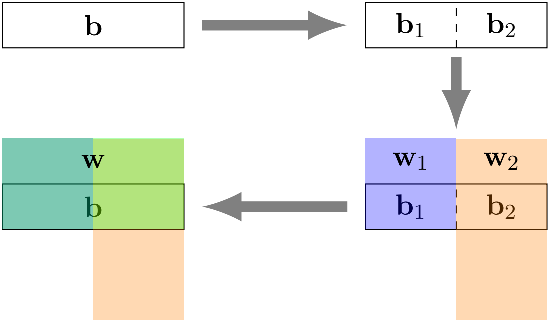
Tetris: a recursive algorithm for BCP

recursively verify that $\mathbf{b}_1$ and $\mathbf{b}_2$ are covered
through finding two witnesses $\mathbf{w}_1, \mathbf{w}_2$ that cover $\mathbf{b}_1, \mathbf{b}_2$,



Tetris: a recursive algorithm for BCP

$\mathbf{w} = \mathsf{Resolve}(\mathbf{w}_1, \mathbf{w}_2)$, then $\mathbf{w}$ covers $\mathbf{b}$,
add $\mathbf{w}$ to $\mathcal{A}$, $\mathbf{w}$ is a witness for $\mathbf{b}$.



Two main analytical components

- runtime $= \Theta(\#\text{resolutions})$

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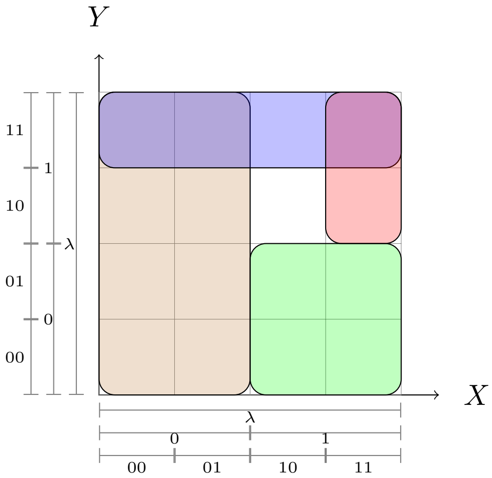
- Different initializations lead to different results
  - Tetris-Preloaded (load all input boxes)
  - Tetris-Reloaded (load as needed)
  - Tetris-Balanced (work under a transformed space)

Tetris-Preloaded: Example



Input Boxes

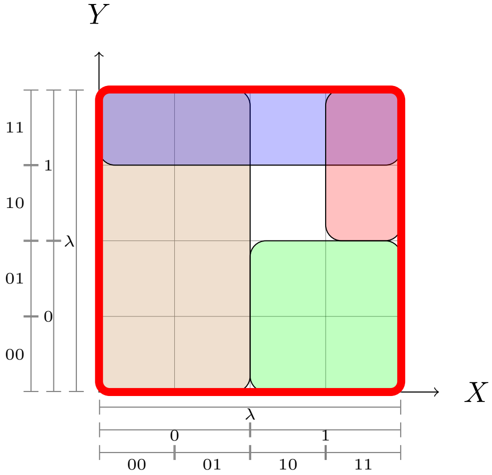
$\langle 0, \lambda \rangle$

$\langle 1, 0 \rangle$

$\langle \lambda, 11 \rangle$

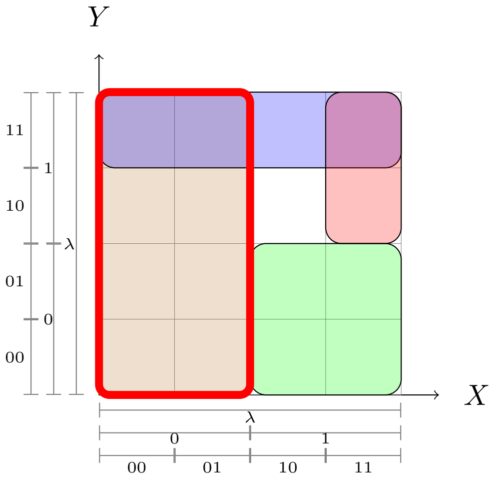
$\langle 11, 1 \rangle$

Tetris-Preloaded: Example



Is $\langle \lambda, \lambda \rangle$ covered?
No
Split into $\langle 0, \lambda \rangle$ and $\langle 1, \lambda \rangle$

Tetris-Preloaded: Example



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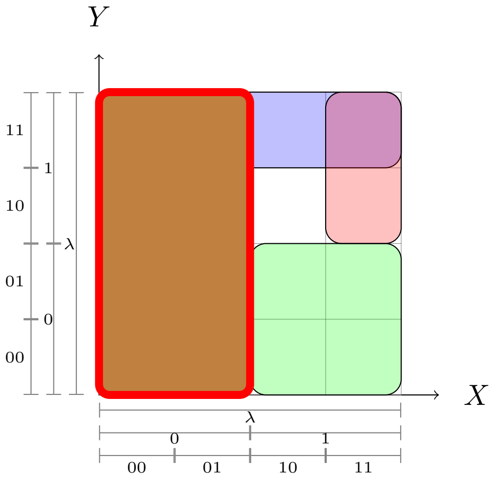
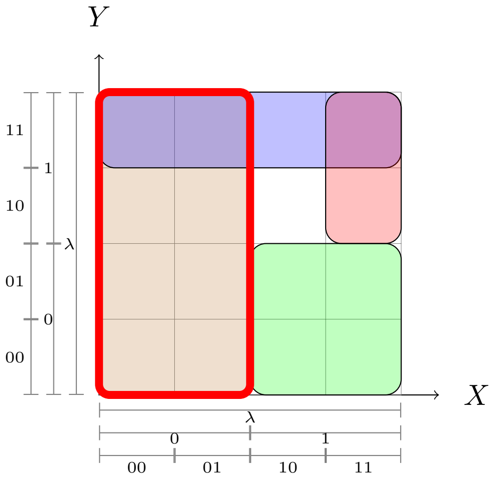


Is $\langle 0, \lambda \rangle$ covered?
Yes by $\langle 0, \lambda \rangle$

Tetris-Preloaded: Example



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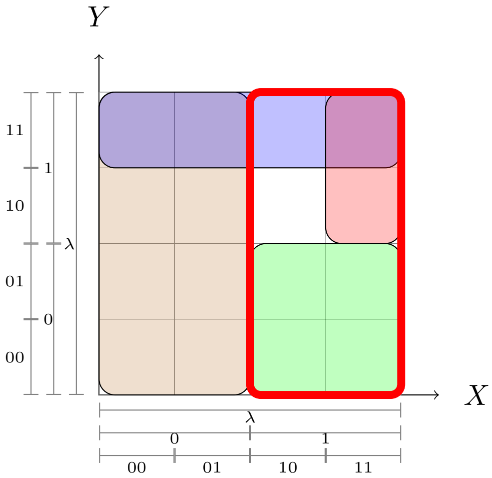
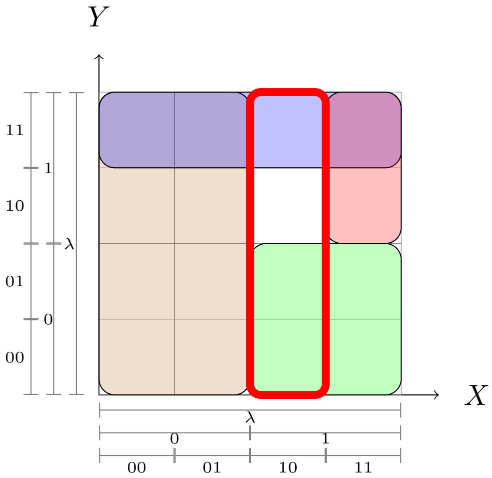


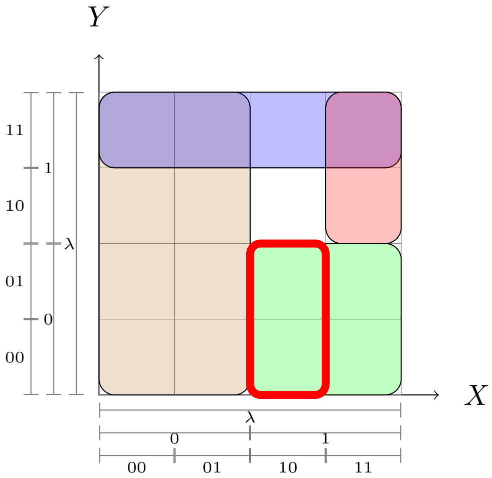
Is $\langle 1, \lambda \rangle$ covered?
No
Split into $\langle 10, \lambda \rangle$ and $\langle 11, \lambda \rangle$

Tetris-Preloaded: Example



Is $\langle 10, \lambda \rangle$ covered?
No
Split into $\langle 10, 0 \rangle$ and $\langle 10, 1 \rangle$

Tetris-Preloaded: Example



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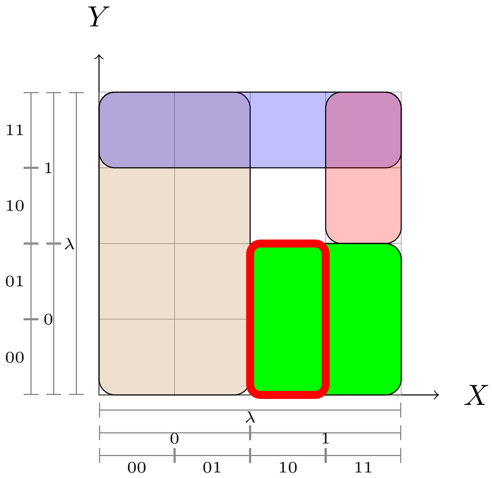
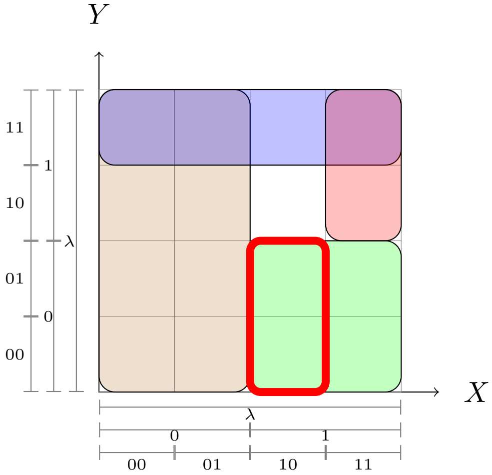


Is $\langle 10, 0\rangle$ covered?

Yes by $\langle 1, 0\rangle$

Tetris-Preloaded: Example



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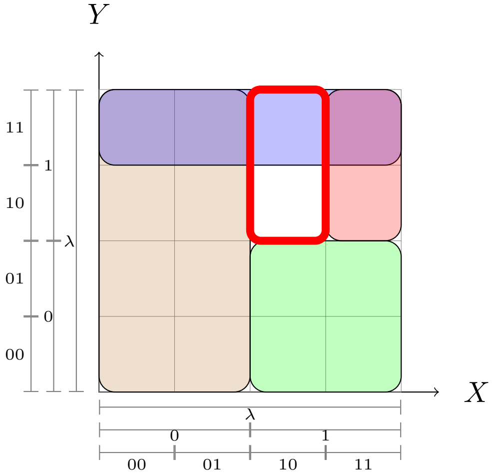
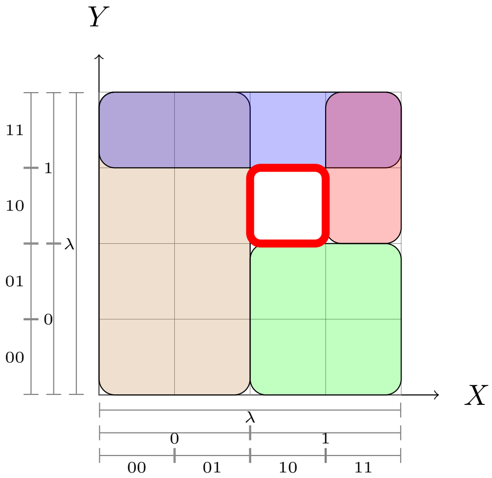
Yes by $\langle 1, 0\rangle$

Tetris-Preloaded: Example



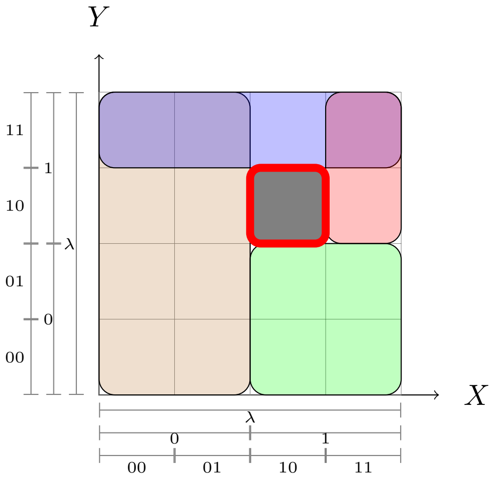
Is $\langle 10, 1\rangle$ covered?
No
Split into $\langle 10, 10\rangle$ and $\langle 10, 11\rangle$

Tetris-Preloaded: Example



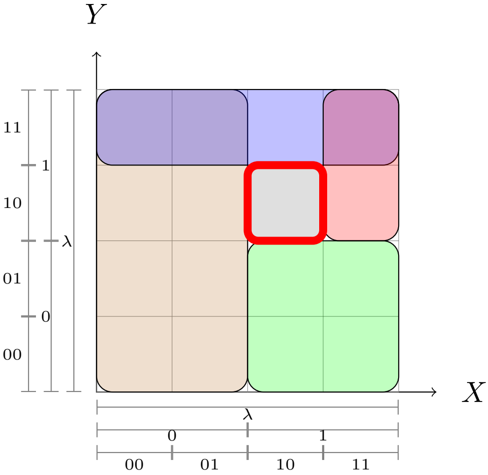
Is $\langle 10, 10 \rangle$ covered?
No
It cannot be split

Tetris-Preloaded: Example



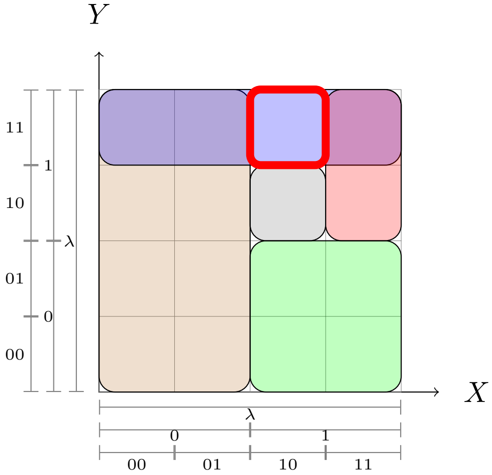
Is $\langle 10, 10 \rangle$ covered?
No
It cannot be split
Output $\langle 10, 10 \rangle$
Add a box $\langle 10, 10 \rangle$

Tetris-Preloaded: Example



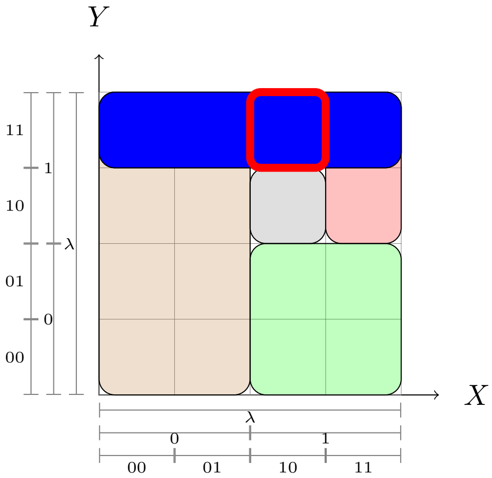
Is $\langle 10, 10\rangle$ covered?
No
It cannot be split
Output $\langle 10, 10\rangle$
Add a box $\langle 10, 10\rangle$

Tetris-Preloaded: Example



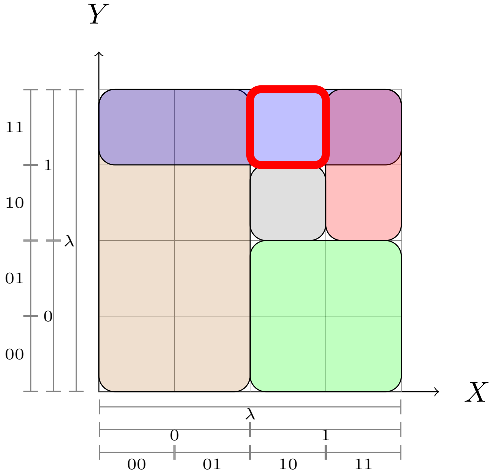
Is $\langle 10, 11 \rangle$ covered?

Tetris-Preloaded: Example



Is $\langle 10, 11 \rangle$ covered?
Yes by $\langle \lambda, 11 \rangle$

Tetris-Preloaded: Example



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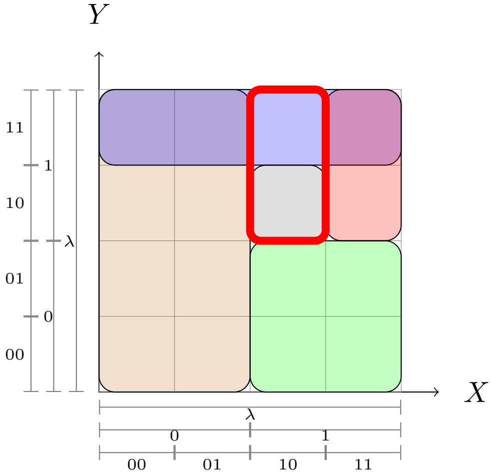
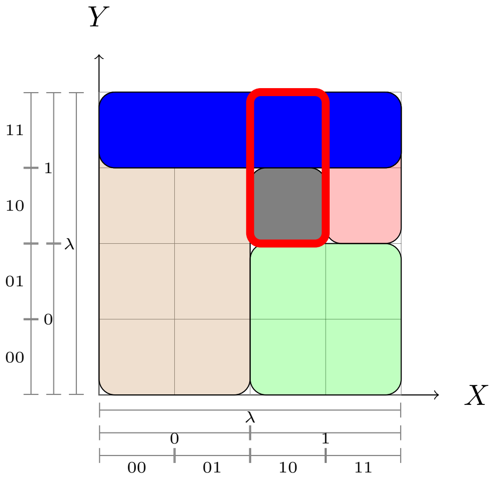
Tetris-Preloaded: Example



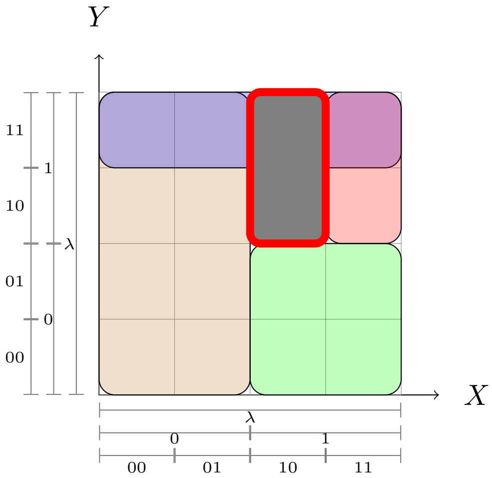
Backtrack to $\langle 10, 1 \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 10, 1 \rangle$
Resolve
$\langle \lambda, 11 \rangle$
$\langle 10, 10 \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 10, 1\rangle$

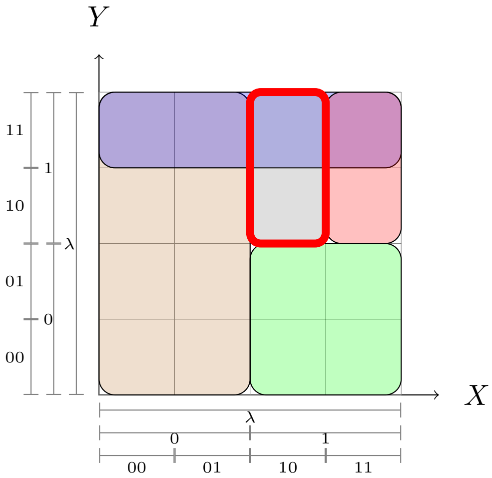
Resolve

$\langle \lambda, 11\rangle$

$\langle 10, 10\rangle$

$\langle 10, 1\rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 10, 1 \rangle$

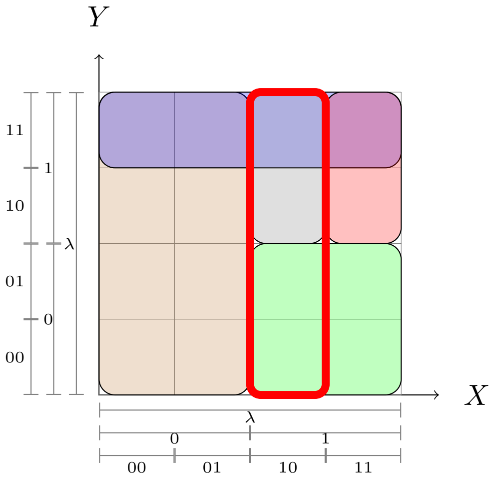
Resolve

$\langle \lambda, 11 \rangle$

$\langle 10, 10 \rangle$

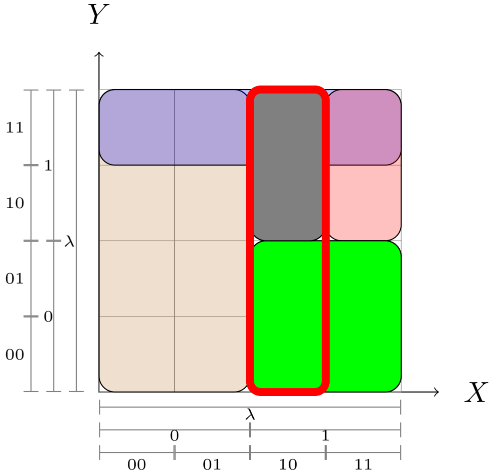
$\langle 10, 1 \rangle$

Tetris-Preloaded: Example



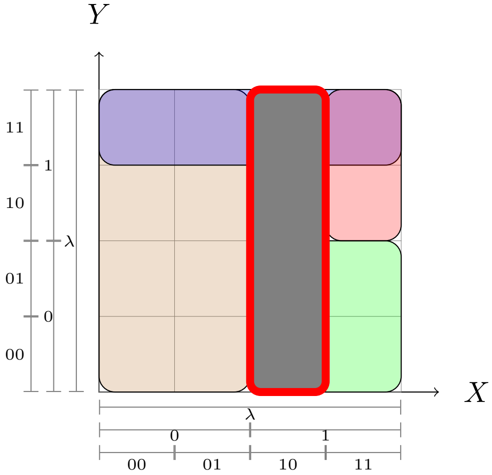
Backtrack to $\langle 10, \lambda \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 10, \lambda \rangle$
Resolve
$\langle 10, 1 \rangle$
$\underline{\langle 1, 0 \rangle}$

Tetris-Preloaded: Example



Backtrack to $\langle 10, \lambda \rangle$

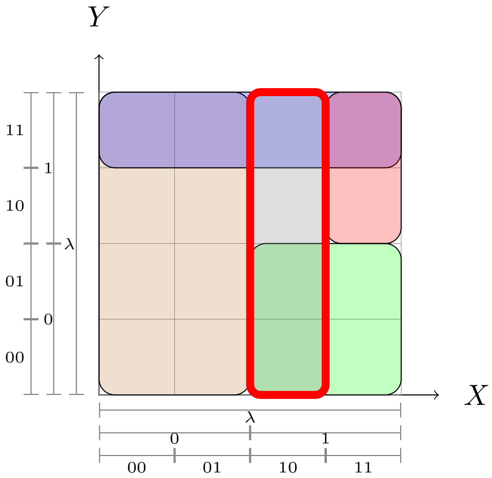
Resolve

$\langle 10, 1 \rangle$

$\langle 1, 0 \rangle$

$\overline{\langle 10, \lambda \rangle}$

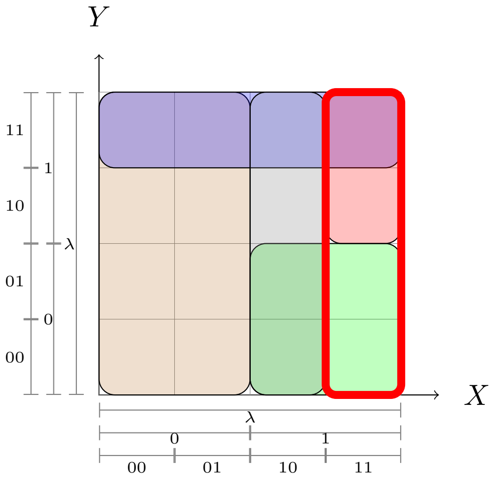
Tetris-Preloaded: Example



Backtrack to $\langle 10, \lambda \rangle$
Resolve

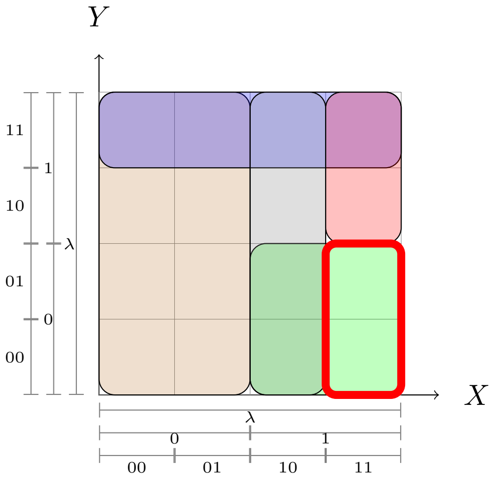
$$\frac{\begin{array}{l}\langle 10, 1 \rangle \\ \langle 1, 0 \rangle\end{array}}{\langle 10, \lambda \rangle}$$

Tetris-Preloaded: Example



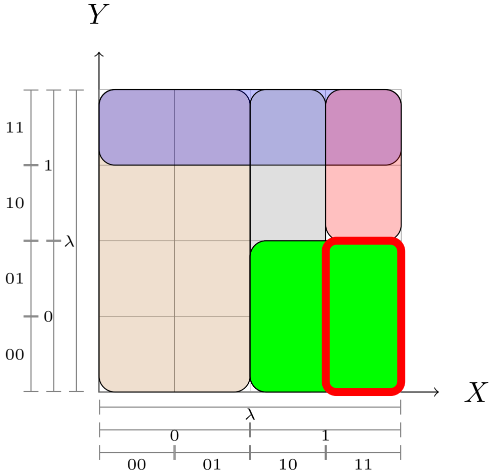
Is $\langle 11, \lambda \rangle$ covered?
No
Split into $\langle 11, 0 \rangle$ and $\langle 11, 1 \rangle$

Tetris-Preloaded: Example



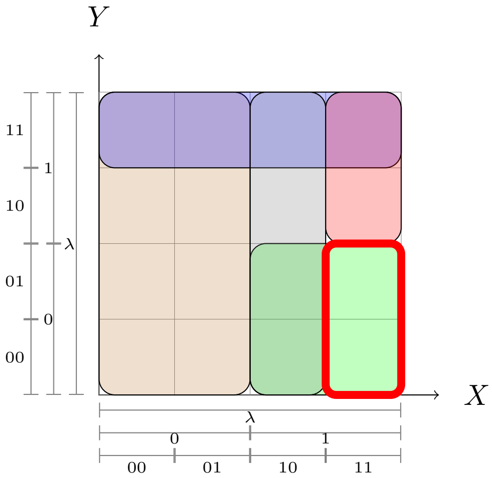
Is $\langle 11, 0\rangle$ covered?

Tetris-Preloaded: Example



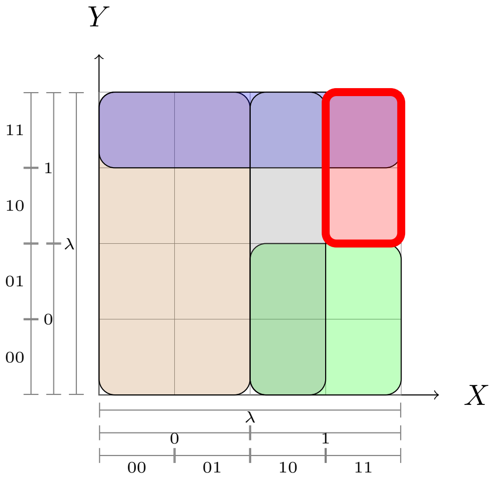
Is $\langle 11, 0\rangle$ covered?
Yes by $\langle 1, 0\rangle$

Tetris-Preloaded: Example



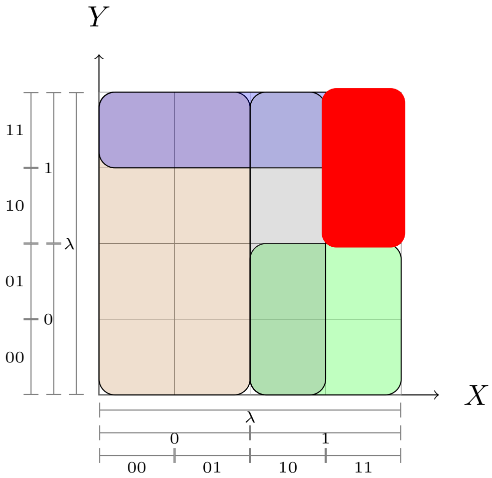
Is $\langle 11, 0\rangle$ covered?
Yes by $\langle 1, 0\rangle$

Tetris-Preloaded: Example



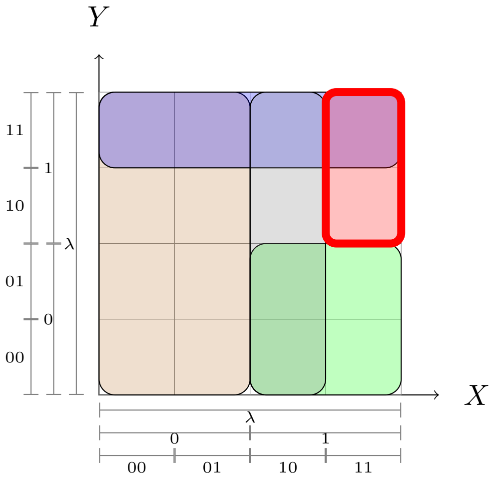
Is $\langle 11, 1 \rangle$ covered?

Tetris-Preloaded: Example



Is $\langle 11, 1 \rangle$ covered?
Yes by $\langle 11, 1 \rangle$

Tetris-Preloaded: Example



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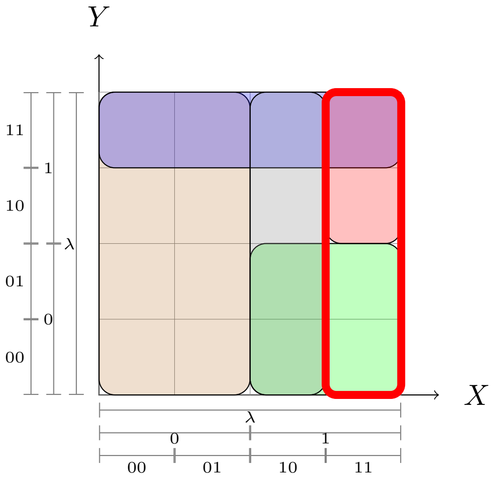
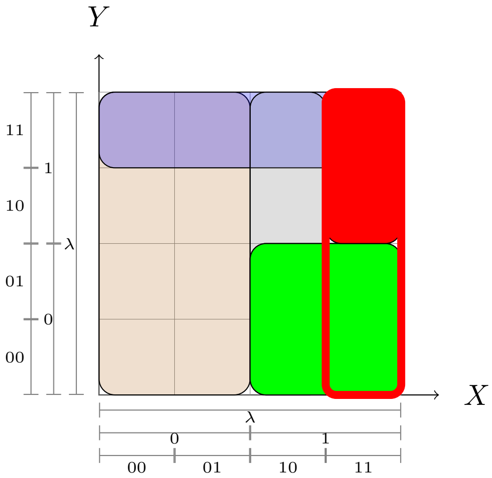
Yes by $\langle 11, 1 \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 11, \lambda \rangle$

Tetris-Preloaded: Example



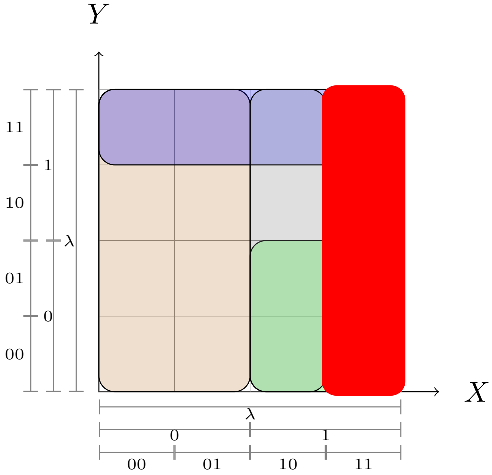
Backtrack to $\langle 11, \lambda \rangle$

Resolve

$\langle 11, 1 \rangle$

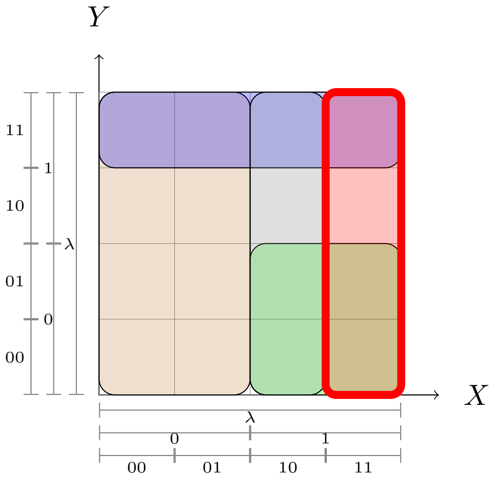
$\langle 1, 0 \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 11, \lambda \rangle$
Resolve
$\langle 11, 1 \rangle$
$\langle 1, 0 \rangle$
$\overline{\langle 11, \lambda \rangle}$

Tetris-Preloaded: Example

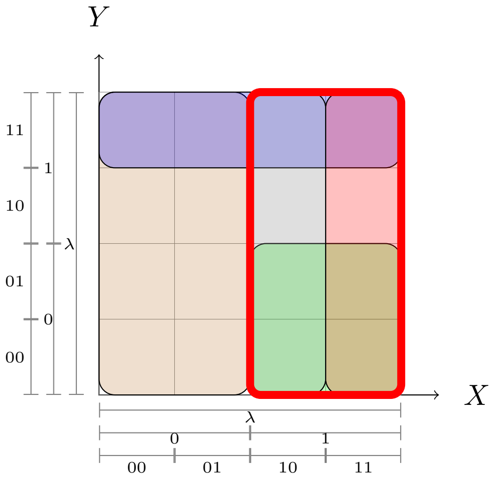


Backtrack to $\langle 11, \lambda \rangle$

Resolve

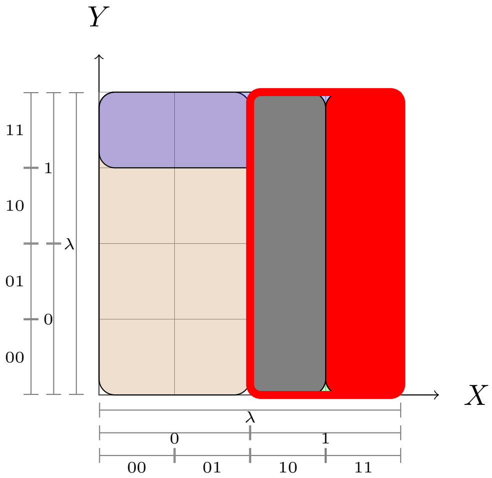
$$\frac{\langle 11, 1 \rangle \quad \langle 1, 0 \rangle}{\langle 11, \lambda \rangle}$$

Tetris-Preloaded: Example



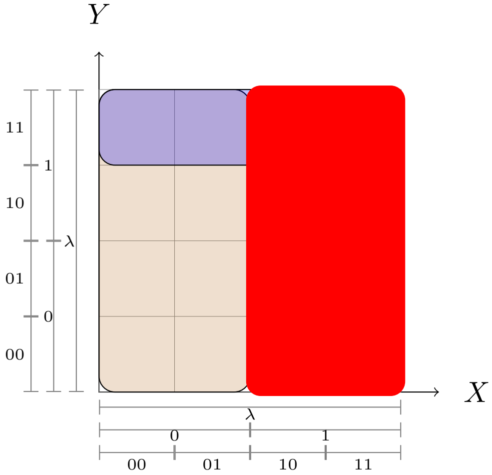
Backtrack to $\langle 1, \lambda \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle 1, \lambda \rangle$
Resolve
$\langle 10, \lambda \rangle$
$\langle 11, \lambda \rangle$

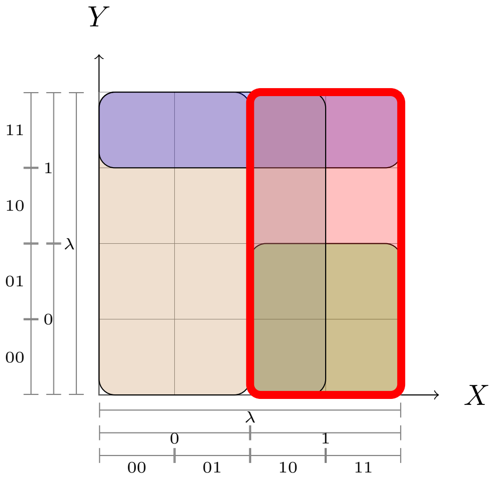
Tetris-Preloaded: Example



Backtrack to $\langle 1, \lambda \rangle$
Resolve
$\langle 10, \lambda \rangle$
$\langle 11, \lambda \rangle$

$\langle 1, \lambda \rangle$

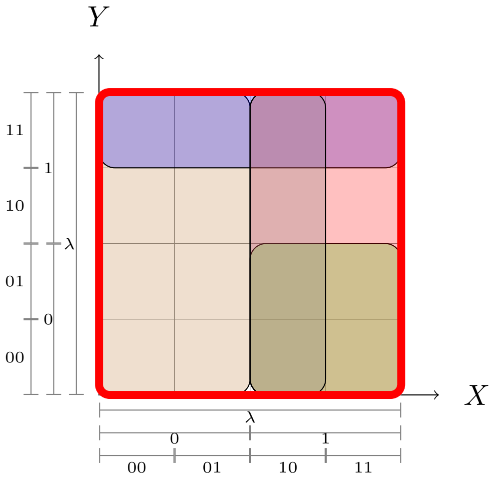
Tetris-Preloaded: Example



Backtrack to $\langle 1, \lambda \rangle$
Resolve

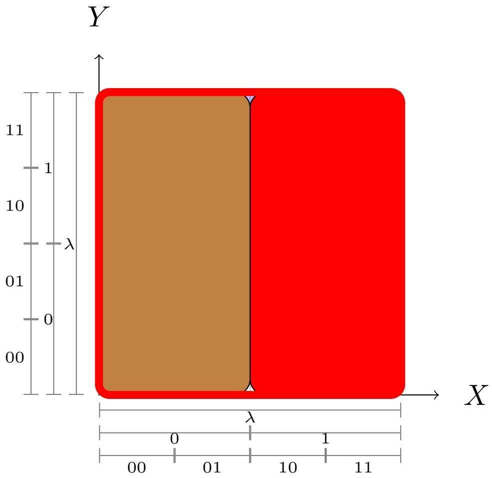
$$\frac{\langle 10, \lambda \rangle \quad \langle 11, \lambda \rangle}{\langle 1, \lambda \rangle}$$

Tetris-Preloaded: Example



Backtrack to $\langle \lambda, \lambda \rangle$

Tetris-Preloaded: Example



Backtrack to $\langle \lambda, \lambda \rangle$
Resolve
$\langle 0, \lambda \rangle$
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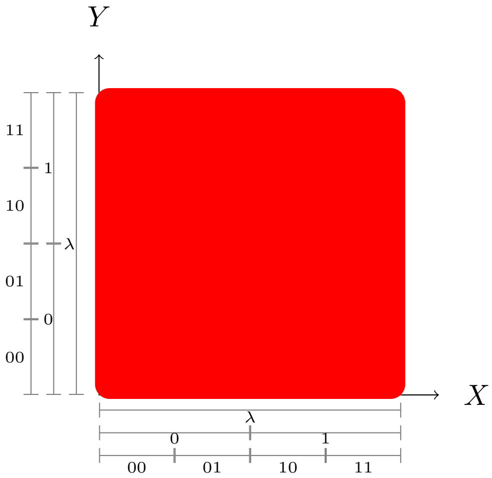


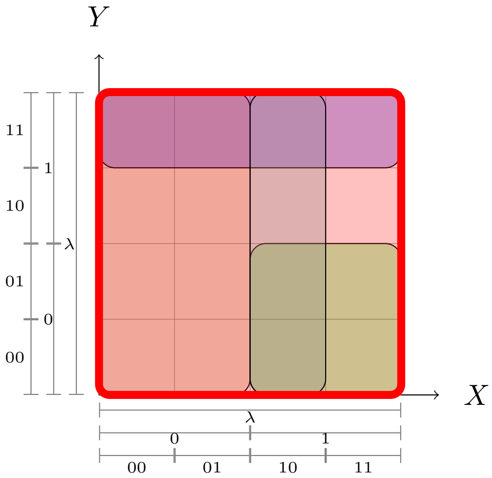
Backtrack to $\langle \lambda, \lambda \rangle$
Resolve

$\langle 0, \lambda \rangle$

$\langle 1, \lambda \rangle$

$\langle \lambda, \lambda \rangle$

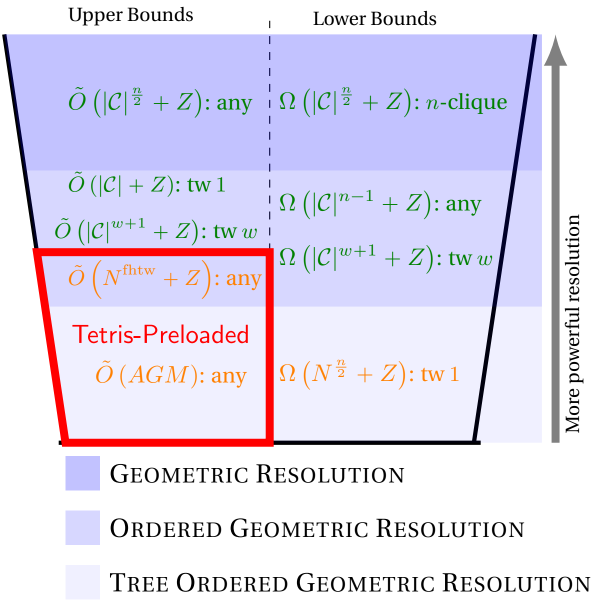
Tetris-Preloaded: Example



Backtrack to $\langle \lambda, \lambda \rangle$
Resolve

$$\frac{\begin{array}{c}\langle 0, \lambda \rangle \\ \langle 1, \lambda \rangle\end{array}}{\langle \lambda, \lambda \rangle}$$

Done!


Upper Bounds
Lower Bounds
$\tilde{O}\left(|\mathcal{C}|^{\frac{n}{2}} + Z\right)$: any
$\Omega\left(|\mathcal{C}|^{\frac{n}{2}} + Z\right)$: n-clique
$\tilde{O}\left(|\mathcal{C}| + Z\right)$: tw 1
$\Omega\left(|\mathcal{C}|^{n-1} + Z\right)$: any
$\tilde{O}\left(|\mathcal{C}|^{w+1} + Z\right)$: tw w
$\Omega\left(|\mathcal{C}|^{w+1} + Z\right)$: tw w
$\tilde{O}\left(N^{\text{fhtw}} + Z\right)$: any
Tetris-Preloaded
$\tilde{O}\left(AGM\right)$: any
$\Omega\left(N^{\frac{n}{2}} + Z\right)$: tw 1
More powerful resolution
Geometric Resolution
Ordered Geometric Resolution
Tree Ordered Geometric Resolution

Tetris-Reloaded

- Algorithm
  1. $\mathcal{C}_{\square} \leftarrow \emptyset$
  2. Fix a dimension ordering
  3. Run Tetris. If an uncovered point $\mathbf{b}$ is found
     - Query for boxes covering $\mathbf{b}$ $\quad (\tilde{O}(1))$
     - Load them into $\mathcal{C}_{\square}$
     - Repeat

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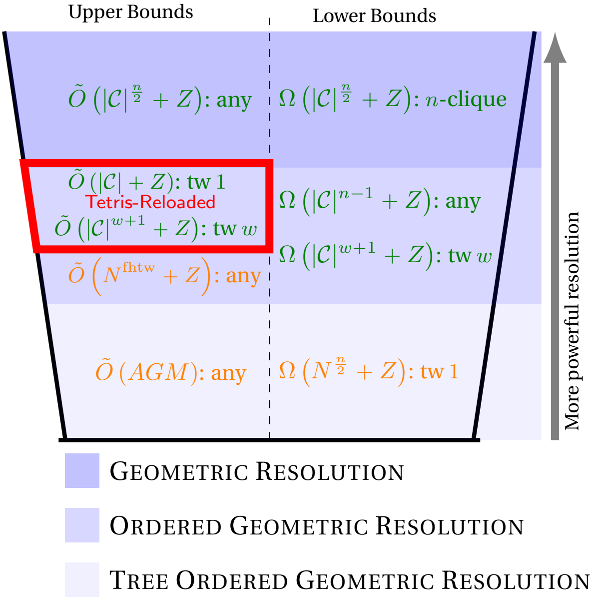
- Analysis
  - $|\mathcal{C}_\square| = \tilde{O}(|\mathcal{C}'_\square|), \quad$ for any $\mathcal{C}'_\square$



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Tetris-Reloaded
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$\Omega\left(|\mathcal{C}|^{n-1} + Z\right)$: any
$\Omega\left(|\mathcal{C}|^{w+1} + Z\right)$: tw w
$\tilde{O}\left(N^{\text{fhtw}} + Z\right)$: any
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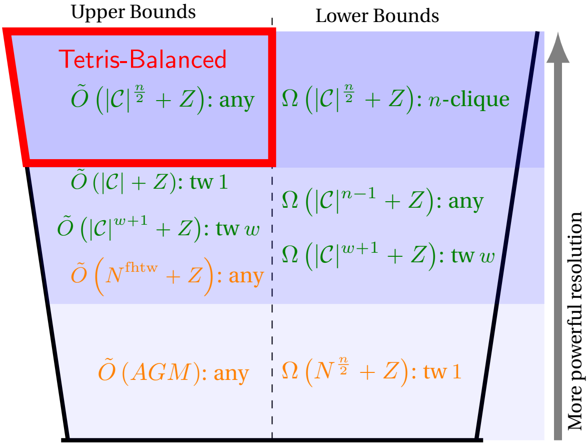

Geometric Resolution

Ordered Geometric Resolution

Tree Ordered Geometric Resolution

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Open Problems

- A $\tilde{O}(P)$-algorithm?

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- Other models for instance optimality?
- A notion of certificates for algebraic algorithms?
  - Algebraic algorithms can break the $\Omega(|\mathcal{C}_\Box|^{n/2})$-lower bound
    - e.g. listing triangles in $O(N^{1.408} + N^{1.222} Z^{0.186})$ [Björklund et al, ICALP'14]

Open Problems (cont.)

- Count/Aggregate queries?

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- Instance Optimality under Updates (IVM)?

Many Thanks!
Any Questions/Comments?

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  3. Run Tetris

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Tetris-Preloaded

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- Underlying Proof System
  - Ordered Geometric Resolution
- Runtime Bounds
  - $\tilde{O}(N + N^{\text{fhtw}} + Z)$
    - $\tilde{O}(N + Z)$ for acyclic queries
    - $\tilde{O}(\mathsf{AGM})$ even without caching (Tree Ordered Geometric Resolution)

Tetris-Reloaded: More details

- Underlying Proof System
  - ORDERED GEOMETRIC RESOLUTION

Tetris-Reloaded: More details

- Underlying Proof System
  - ORDERED GEOMETRIC RESOLUTION
- Runtime Bounds
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  - $\tilde{O}(|\mathcal{C}_\square|^{w+1} + Z)$

Tetris-Reloaded: More details

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  - ORDERED GEOMETRIC RESOLUTION
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- Lower Bounds for ORDERED GEOMETRIC RESOLUTION
  - $\Omega(|\mathcal{C}_\Box| + Z)$ for treewidth $w = 1$
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  - $\Omega(|\mathcal{C}_\Box|^{n-1} + Z)$ for n-clique

Tetris-Reloaded: More details

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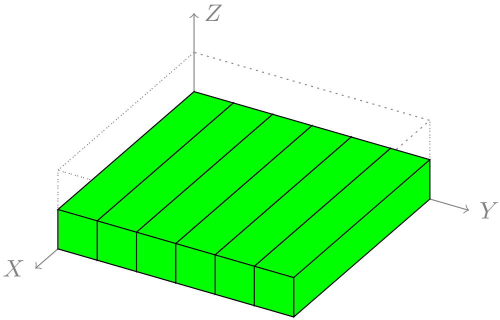
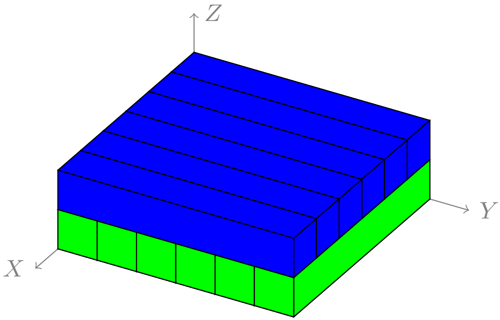
  - $\Omega(|\mathcal{C}_\square| + Z)$ for treewidth $w = 1$
  - $\Omega(|\mathcal{C}_\square|^{w+1} + Z)$
  - $\Omega(|\mathcal{C}_\square|^{n-1} + Z)$ for n-clique
    - *But AGM bound for an n-clique is $\tilde{O}(N^{n/2})$*

$\Omega(|\mathcal{C}_\square|^{n-1})$ for Ordered Geometric Resolution

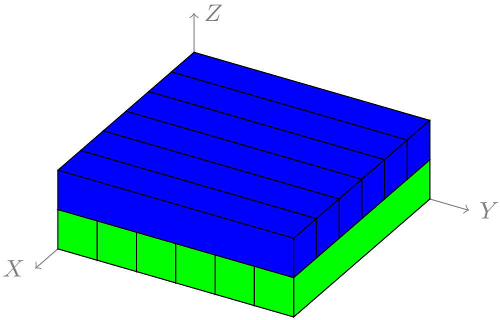


$\Omega(|\mathcal{C}_\square|^{n-1})$ for Ordered Geometric Resolution



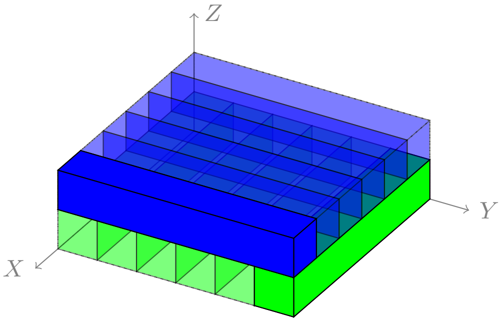
Consider the above certificate $\mathcal{C}_\square$

$\Omega(|\mathcal{C}_\square|^{n-1})$ for Ordered Geometric Resolution



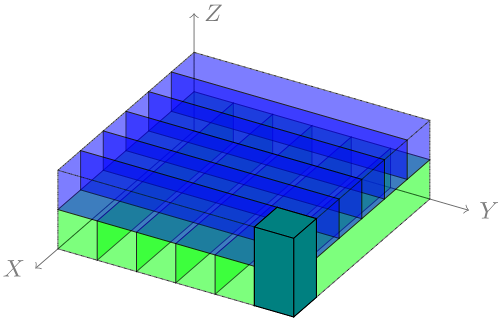
Ordered resolution under any order starting with Z results in $|\mathcal{C}_\square|^2$

$\Omega(|\mathcal{C}_\Box|^{n-1})$ for Ordered Geometric Resolution



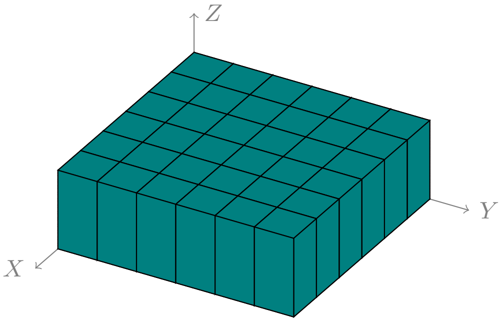
Ordered resolution under any order starting with Z results in $|\mathcal{C}_\Box|^2$

$\Omega(|\mathcal{C}_\square|^{n-1})$ for Ordered Geometric Resolution



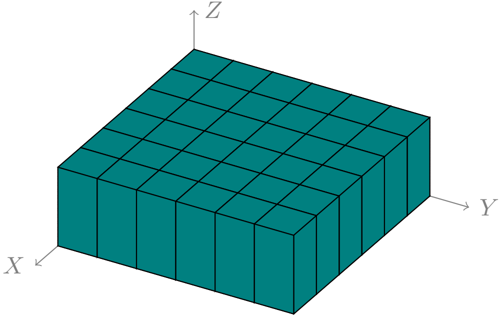
Ordered resolution under any order starting with Z results in $|\mathcal{C}_\square|^2$

$\Omega(|\mathcal{C}_\square|^{n-1})$ for Ordered Geometric Resolution



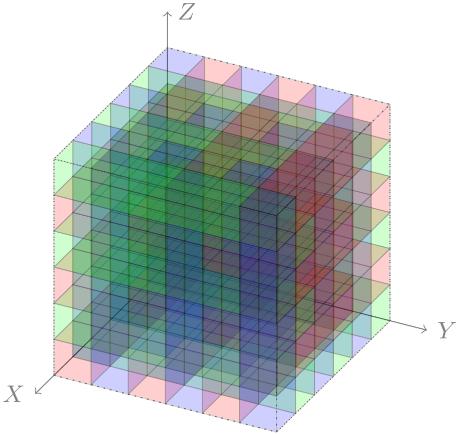
Ordered resolution under any order starting with Z results in $|\mathcal{C}_\square|^2$

$\Omega(|\mathcal{C}_\square|^{n-1})$ for Ordered Geometric Resolution



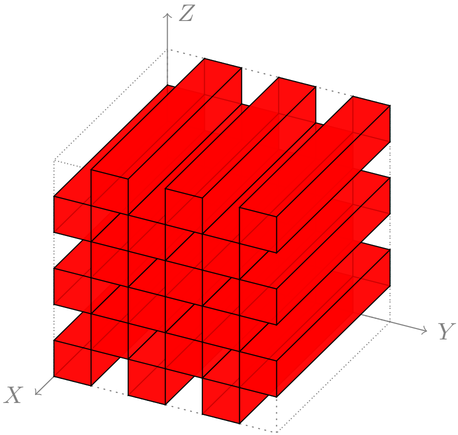
By concatenating together 3 rotated instances of the above, we get a lower bound of $|\mathcal{C}_\square|^2$ for any fixed order

$\Omega(|\mathcal{C}_{\Box}|^{n/2})$ for (Unordered) Geometric Resolution



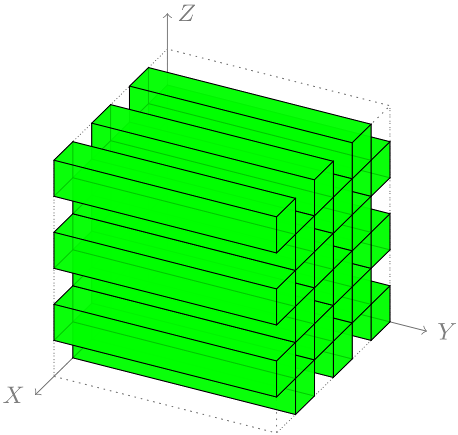
- Let $m := \sqrt{|\mathcal{C}_{\Box}|/3}$

$\Omega(|\mathcal{C}_{\Box}|^{n/2})$ for (Unordered) Geometric Resolution



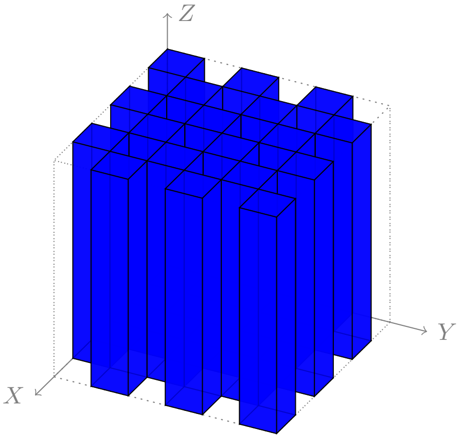
- Let $m := \sqrt{|\mathcal{C}_{\Box}|/3}$
- $m \times m$ red boxes

$\Omega(|\mathcal{C}_\square|^{n/2})$ for (Unordered) GEOMETRIC RESOLUTION



- Let $m := \sqrt{|\mathcal{C}_\square|/3}$
- $m \times m$ red boxes
- $m \times m$ green boxes

$\Omega(|\mathcal{C}_\square|^{n/2})$ for (Unordered) GEOMETRIC RESOLUTION



- Let $m := \sqrt{|\mathcal{C}_\square|/3}$
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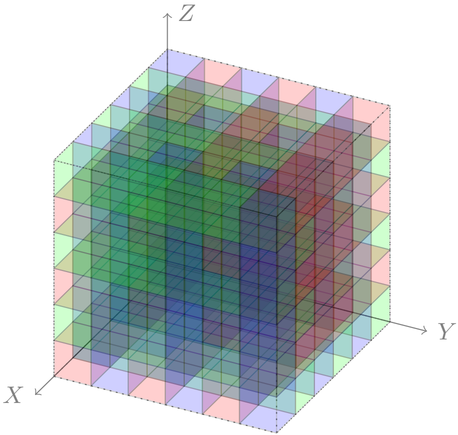
- $m \times m$ green boxes
- $m \times m$ blue boxes

$\Omega(|\mathcal{C}_\square|^{n/2})$ for (Unordered) Geometric Resolution



- Let $m := \sqrt{|\mathcal{C}_\square|/3}$
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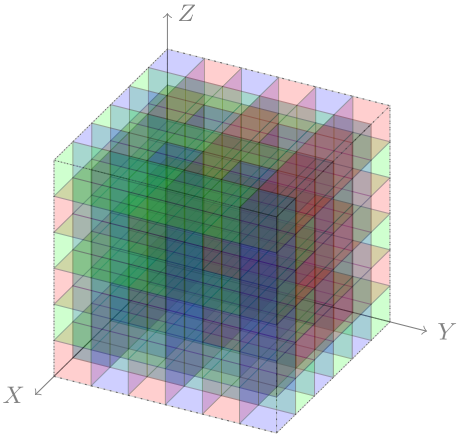
$\Omega(|\mathcal{C}_\square|^{n/2})$ for (Unordered) Geometric Resolution



- Let $m := \sqrt{|\mathcal{C}_\square|/3}$
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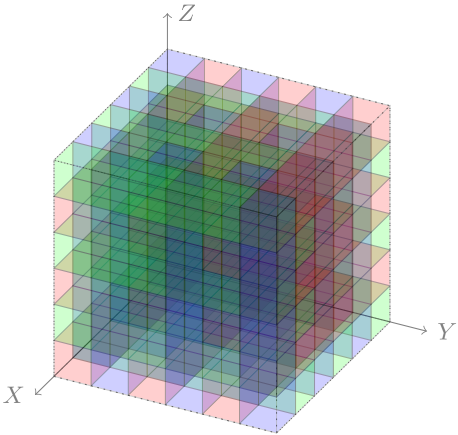
- $m \times m$ blue boxes

Resolving any two boxes results in a box of size 2

$\Omega(|\mathcal{C}_\square|^{n/2})$ for (Unordered) GEOMETRIC RESOLUTION



- Let $m := \sqrt{|\mathcal{C}_\square|/3}$
- $m \times m$ red boxes
- $m \times m$ green boxes
- $m \times m$ blue boxes

Resolving any two boxes results in a box of size 2
(This does NOT prove the lower bound! Just for intuition..)

Tetris-Balanced

- Algorithm

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- Algorithm
  1. Suppose the input space has n-dimensions $A_1, ..., A_n$.

Tetris-Balanced

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       "Split" dimension A_i into two smaller dimensions (A'_i, A''_i) in a "balanced" way.

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     - For each $i \in \{1, ..., n-2\}$
       "Split" dimension A_i into two smaller dimensions (A_i', A_i'') in a "balanced" way.
  2. Fix the dimension order:

$$(A_1', A_2', \ldots, A_{n-2}', A_{n-1}, A_n, A_{n-2}'', A_{n-3}'', \ldots, A_1'')$$

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  3. Run Tetris-Reloaded (in the new space of dimension $2n-2$)

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- Underlying Proof System

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  3. Run Tetris-Reloaded (in the new space of dimension $2n-2$)
- Underlying Proof System
  - GEOMETRIC RESOLUTION

Tetris-Balanced

- Algorithm
  1. Suppose the input space has n-dimensions $A_1, ..., A_n$.
     - For each $i \in \{1, ..., n-2\}$
       "Split" dimension A_i into two smaller dimensions (A'_i, A''_i) in a "balanced" way.
  2. Fix the dimension order:

$$(A'_1, A'_2, \ldots, A'_{n-2}, A_{n-1}, A_n, A''_{n-2}, A''_{n-3}, \ldots, A''_1)$$

  3. Run Tetris-Reloaded (in the new space of dimension $2n-2$)
- Underlying Proof System
  - Geometric Resolution
- Runtime Bound

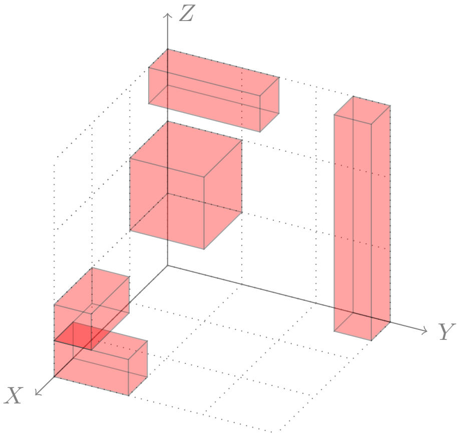
Tetris-Balanced

- Algorithm
  1. Suppose the input space has n-dimensions $A_1, ..., A_n$.
     - For each $i \in \{1, ..., n-2\}$
       "Split" dimension A_i into two smaller dimensions (A_i', A_i'') in a "balanced" way.
  2. Fix the dimension order:

$$(A_1', A_2', \ldots, A_{n-2}', A_{n-1}, A_n, A_{n-2}'', A_{n-3}'', \ldots, A_1'')$$

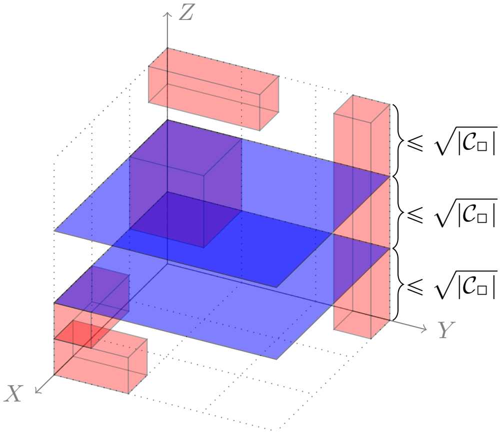
  3. Run Tetris-Reloaded (in the new space of dimension $2n-2$)
- Underlying Proof System
  - GEOMETRIC RESOLUTION
- Runtime Bound
  - $\tilde{O}(|\mathcal{C}_\square|^{n/2} + Z)$

Tetris-Balanced: Splitting Explained



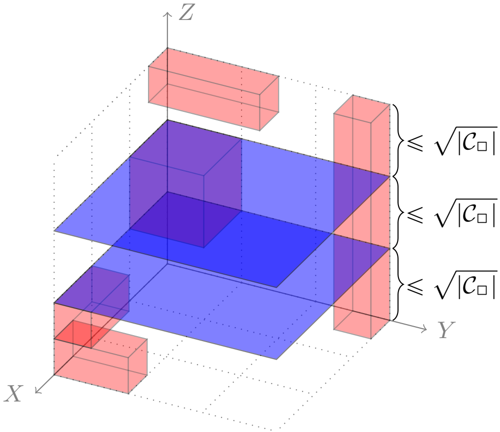
Consider the above box set $\mathcal{C}_{\square}$

Tetris-Balanced: Splitting Explained



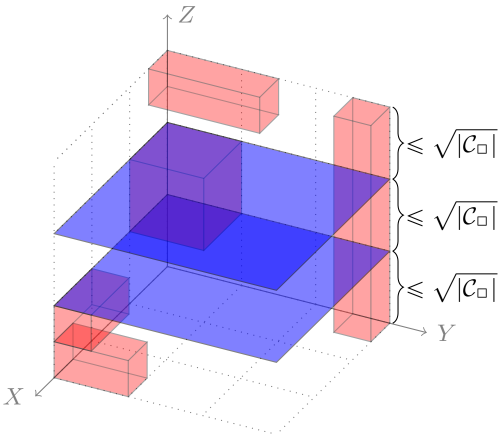
Split Z into $\sqrt{|\mathcal{C}_\square|}$ slices where each slice has $\sqrt{|\mathcal{C}_\square|}$ boxes fully contained in the slice

Tetris-Balanced: Splitting Explained



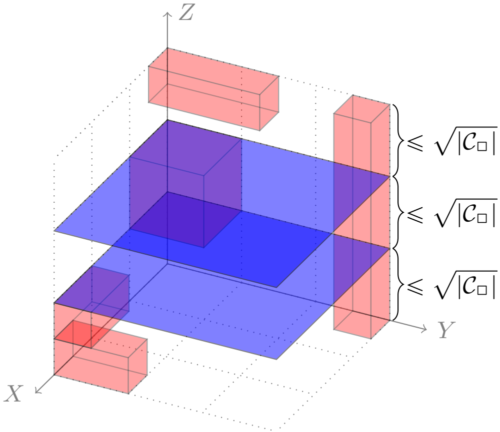
Do resolution over Z only within each slice

Tetris-Balanced: Splitting Explained



Then resolve over X and Y

Tetris-Balanced: Splitting Explained



Then resolve the slices together over Z

Some Followup Works

- K. Alway, "Domain Ordering and Box Cover Problems for Beyond Worst-Case Join Processing", Master Thesis, Waterloo, 2019.
  - Given a relation R with N tuples, generate all maximal dyadic gap boxes of R in time $\tilde{O}(N)$.
  - Strengthens the notion of $\mathcal{C}_{\square}$.
- J. Dobler, and A. Rudra, "Implementation of Tetris as a Model Counter", ArXiV 2017.