Parallel Discrete Sampling via Continuous Walks

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joint work with

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Tianyu Liu
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DALL·E for Spanning Trees

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sampling in diffusion models [image by Andy Shih]
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stochastic localization [Eldan’13]
Sampling

Continuous

\( \mu : \mathbb{R}^n \to \mathbb{R} \geq 0 \)
Sampling

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\[ \mu : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \]

Tractable: \( \log \mu \) concave

\[ \beta I \preceq \nabla^2 \log \mu \preceq -\alpha I \]

and \( \beta/\alpha \) is small.
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- Even better (well-conditioned):
  
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Discrete

\[ \mu : \{\pm 1\}^n \rightarrow \mathbb{R}_{\geq 0} \]

- Tractable: ? (patchwork)
Sampling via counting

**Counting**

Sub-cube \( C \subseteq \{\pm 1\}^n \mapsto \sum_{x \in C} \mu(x) \).
Sampling via counting

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Sub-cube $C \subseteq \{\pm 1\}^n \mapsto \sum_{x \in C} \mu(x)$.

Spanning trees

- Matrix-tree thm.

Planar PMs

- [FKT] thm.

Eulerian tours*

- [BEST] thm.

Det. Point Process

- $\propto \det(L_S)$

$\det(L + I)$
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Polynomial-time counting $\implies$ polynomial-time sampling.

[Jerrum-Valiant-Vazirani'89]
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\[ P[e_1]? \quad P[e_2 | e_1]? \quad P[e_3 | e_1, e_2]? \]

Counting doable in parallel: \( \log(n)^{O(1)} \) time with \( n^{O(1)} \) processors (NC).

[Csanky’75]
Linear algebra is parallelizable.
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Question: Can we sample in parallel (RNC)?
Main result (informal)

We can sample spanning trees, DPPs, Eulerian tours, and more in parallel by moving to continuous space.

Note: list excludes planar perfect matchings.
Discrete to Continuous
- Exponential Tilts
- Interlude: Eulerian Tours
- Transport Stability

Sampling Algorithm
- Stochastic Localization
- Parallel Continuous Sampling
Discrete to Continuous

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Take convolution of $\mu$ with normal $\mathcal{N}(0, cI)$. 

The p.d.f. of $\nu$ at $w$ is $\propto \sum x e^{-\|w-x\|^2/2c} \mu(x) \propto e^{-\|w\|^2/2c} \cdot \sum x e^{\langle w/c, x \rangle} \mu(x)\$. 

$\nabla^2 \log \nu |_{w=0} = -I/c + \text{cov}(\mu)/c^2$

For larger variance, e.g., $\mu \ast \mathcal{N}(0, 2c_0 I)$, we have well-conditioned log-concavity (easy to sample).
From discrete to continuous

Take convolution of $\mu$ with normal $\mathcal{N}(0, cI)$.

**Main lemma**

$\nu := \mu \ast \mathcal{N}(0, cI)$ log-concave for $c \geq c_0 = O(1)$.
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Count of weighted $\mu$
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Take convolution of $\mu$ with normal $N(0, cI)$.

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Exponential tilts

For $\mu$ on $\{\pm 1\}^n$, an exponential tilt is $\tau_w \mu$ for $w \in \mathbb{R}^n$ defined as

$$\tau_w \mu(x) \propto e^{\langle w, x \rangle} \mu(x).$$
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Covariance bound

We just need all of these $\tau_w \mu$ to have bounded covariance (semi-log-concavity [Eldan-Shamir’20]):

$$\text{cov}(\tau_w \mu) \leq O(1) \cdot \mathbb{I}.$$
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Spectral independence [A-Liu-OveisGharan’20] is even stronger:
\[
\text{cov}(\tau_w \mu) \leq O(1) \cdot \text{diag}(\text{cov}(\tau_w \mu)).
\]

All except Planar PMs. 😞

[Alimohammadi-A-Shiragur-Vuong’21]
What are switching networks?
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General case reducible to \( \deg_{\text{in}} = \deg_{\text{out}} = 2 \).
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Replace each vertex by “switching network” gadget:
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Replace each vertex by “switching network” gadget:

Binary choice per vertex:

[Bouchet]: $\exists n \times n$ skew-symmetric $L$, such that

$$\det(L_S, S) = 1 [S \text{ indicates Eulerian tour}].$$
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![Diagram of switching network gadget]

Binary choice per vertex:

![Binary choice diagram]

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Exponential tilt becomes biased switching.
Want: random switching $\equiv$ uniformly random permutation.
Switching networks

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- Randomly constructed unbiased $\tilde{O}(\text{deg})$-sized network $\simeq$ uniform permutation [Czumaj’15].

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Main result 1
Approx. sampling ($\epsilon$ in $d_{TV}$) via weighted counting in $\text{polylog}(n/\epsilon)$ time and $\text{quasipoly}(n/\epsilon)$ processors, for $\mu$ spectrally independent under exponential tilts.

Spectral independence [A-Liu-OveisGharan'20] under exponential tilts is also known as "fractional log-concavity" [Alimohammadi-A-Shiragur-Vuong'21].
Weaker condition "semi-log-concavity" [Eldan-Shamir'20] is also enough.
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- Quasi-RNC sampling of DPPs on skew-symmetric matrix.

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The number of processors can be improved from \( \text{quasipoly} \left( \frac{n}{\epsilon} \right) \) to \( \text{poly} \left( \frac{n}{\epsilon} \right) \) if \( \mu \) is "transport-stable".

RNC sampling of DPPs on symmetric PSD matrix.

RNC sampling of spanning trees (already known via parallelization of Aldous-Broder alg. [Teng’95,A-Hu-Saberi-Schild’21]).

Conjecture: Eulerian tours and non-symmetric DPPs also "transport-stable".

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Transport stability

We call $\mu$ transport-stable if

$$W_1(\tau_w \mu, \tau_{w'} \mu) \leq C \cdot \|w - w'\|_1.$$ 

Wasserstein distance w.r.t. Hamming metric
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**Lemma**

$C = O(1)$ for spanning trees, etc.

**Fact**

$C = O(n)$ for any distribution.
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- Aside: \( \|\cdot\|_2 \) can be replaced by \( \|\cdot\|_1 \) in our dists.
Transport stability

[Feder-Mihail’92]

For edge $e$, $\exists$ random spanning trees $T, T'$, such that

- $T$ is uniformly random conditioned on $e \in T$.
- $T'$ is uniformly random conditioned on $e \notin T'$.
- Almost surely $|T \Delta T'| = 2$.
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- Transport stability $\Rightarrow$ semi-log-concavity.

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\|\text{each row of } \text{cov}(\mu)\|_1 \leq O(1).
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Transport stability $\implies$ semi-log-concavity.

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Discrete to Continuous

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How do we turn continuous samples into discrete ones?
Stochastic localization (i.e., DALL·E-for-theorists)


\[ w_0 \leftarrow 0 \]
\[
\text{for } i = 0, \ldots, T - 1 \text{ do} \\
\quad x \leftarrow \text{sample from } \tau_{w_i} \mu \ast N(0, cI) \\
\quad w_{i+1} \leftarrow w_i + x/c \\
\]
\[ \text{return } \text{sign}(w_T) \]

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Lemma [cf. ElAlaoui-Montanari’21]

\[
w_T / T \sim \mu \ast \mathcal{N}(0, cI/T)\
\]

Enough to stop at \( T \approx c \log(n) \).
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**Lemma [cf. ElAlaoui-Montanari’21]**

\[ cw_T / T \sim \mu \ast N(0, cI/T). \]

足够的停止条件是 \( T \approx c \log(n) \).
How do we sample from $\mu \star \mathcal{N}(0, cI)$ in parallel?
Parallel continuous sampling

- **Open:** For a well-conditioned log-concave $\nu$ on $\mathbb{R}^n$, what is the minimum number of $\nabla \log \nu$ we need to query to sample? We do not know if $\text{polylog}(n)$ is possible.

- Fortunately parallel time $\text{polylog}(n)$ is possible. We use randomized midpoint of [Shen-Lee'19], but others such as Lagenvin can be parallelized too [A-Chewi-Vuong]. Picard iterations change the sequential version:

  $$x_{t+dt} \leftarrow x_t + dt \nabla \log \nu(x_t) + \mathcal{N}(0, 2dt \cdot I)$$

  to iterations for $i = 1, \ldots, O(\text{poly log } n)$ of

  $$x_{t+dt}^i \leftarrow x_t^i + dt \nabla \log \nu(x_t^{i-1}) + \mathcal{N}(0, 2dt \cdot I).$$
Error propagation

Recall that $\mu$ transport-stable if

$$W_1(\tau_w \mu, \tau_{w'} \mu) \leq C \cdot \|w - w'\|_1.$$  

Wasserstein distance w.r.t. Hamming metric
Error propagation

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\[
\mathcal{W}_1(\tau_w \mu, \tau_{w'} \mu) \leq C \cdot \|w - w'\|_1.
\]

Wasserstein distance w.r.t. Hamming metric

\( \mathcal{W}_1\) is the Wasserstein distance.

- The sampling error in one step gets multiplied by \( C \) in every future step.

\[\text{(A-Chewi-Vuong)}\] we can get TV-accurate samples in parallel.
Recall that $\mu$ transport-stable if

$$\mathcal{W}_1(\tau_w \mu, \tau_{w'} \mu) \leq C \cdot \|w - w'\|_1.$$  

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The sampling error in one step gets multiplied by $C$ in every future step.

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**Lemma**  
$C = O(1)$ for spanning trees, etc.

**Fact**  
$C = O(n)$ for any distribution.

Wasserstein accuracy $\text{quasipoly}(n)^{-1}$ enough in continuous sampler.
Error propagation

Recall that $\mu$ transport-stable if

$$W_1(\tau_w \mu, \tau_{w'} \mu) \leq C \cdot ||w - w'||_1.$$ 

Wasserstein distance w.r.t. Hamming metric

- The sampling error in one step gets multiplied by $C$ in every future step.

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[A-Chewi-Vuong]: we can get TV-accurate samples in parallel.
Recall that $\mu$ transport-stable if

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**Lemma**

$C = O(1)$ for spanning trees, etc.

**Fact**

$C = O(n)$ for any distribution.

- Wasserstein accuracy $\text{quasipoly}(n)^{-1}$ enough in continuous sampler.
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Parallel reduction of sampling to counting for a class of distributions.
Parallel reduction of sampling to counting for a class of distributions.

Open: Planar perfect matchings.
Conclusion

- Parallel reduction of sampling to counting for a class of distributions.
- **Open**: Planar perfect matchings.
- **Open**: With no assumption on $\mu$, what is the parallel round complexity of sampling given $\text{poly}(n)$ queries of $\sum_x e^{\langle w, x \rangle} \mu(x)$?
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Thank you!