An optimal "It ain't over till it's over" theorem

Ronen Eldan, Avi Wigderson, *Pei Wu*

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Some background

Influences

$f : \{0,1\}^n \to \{0,1\}$.

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Definition (influence and max influence)

$$\mathrm{Inf}_i[f] = \Pr_x[f(x) \neq f(x \oplus e_i)],$$

$$\mathrm{MaxInf}[f] = \max_{i \in [n]} \mathrm{Inf}_i(f)\ .$$

Examples

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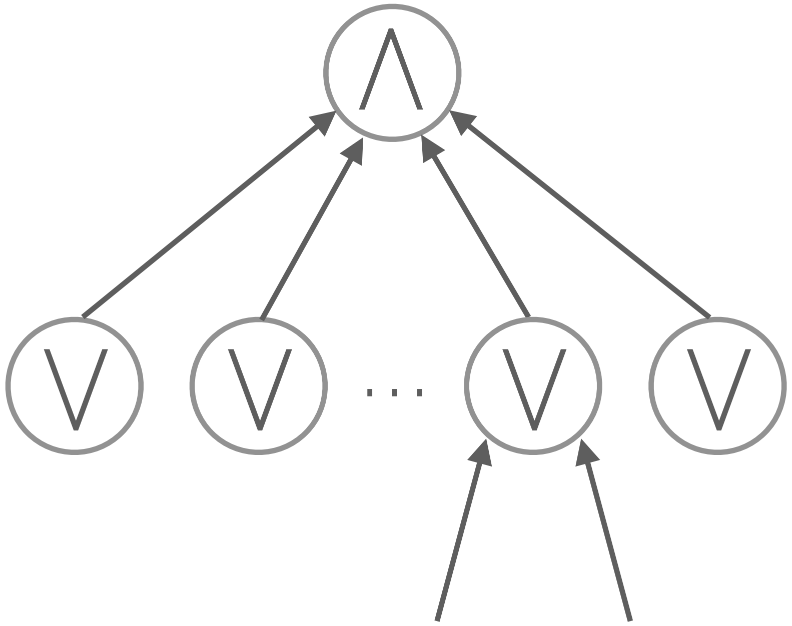
e.g. 01010101010, any 0 is sensitive

Examples

2. $\mathrm{TRIBES_n}(x) = \bigwedge_{i=1}^{s} \bigvee_{j=1}^{w} (x_{i,j}),$

$$s = n/w, w \approx \log n - \log\log n.$$



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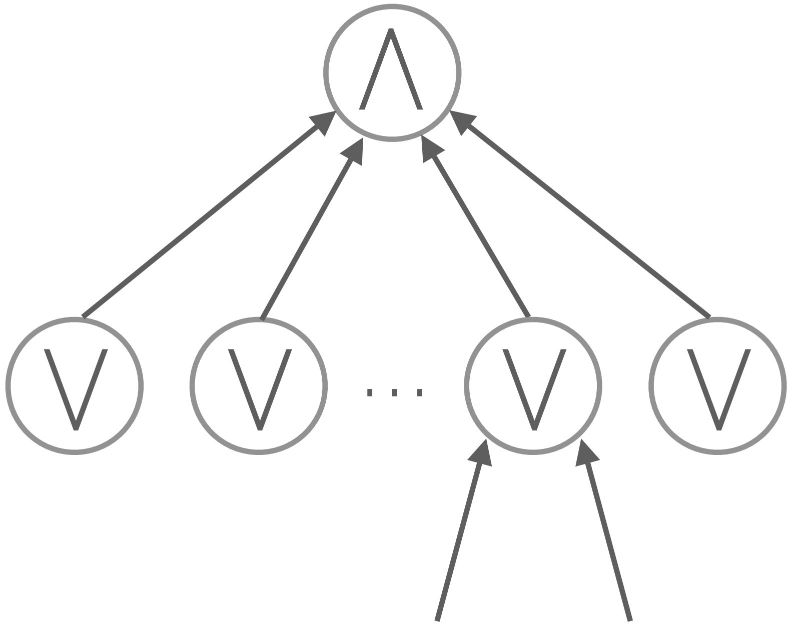

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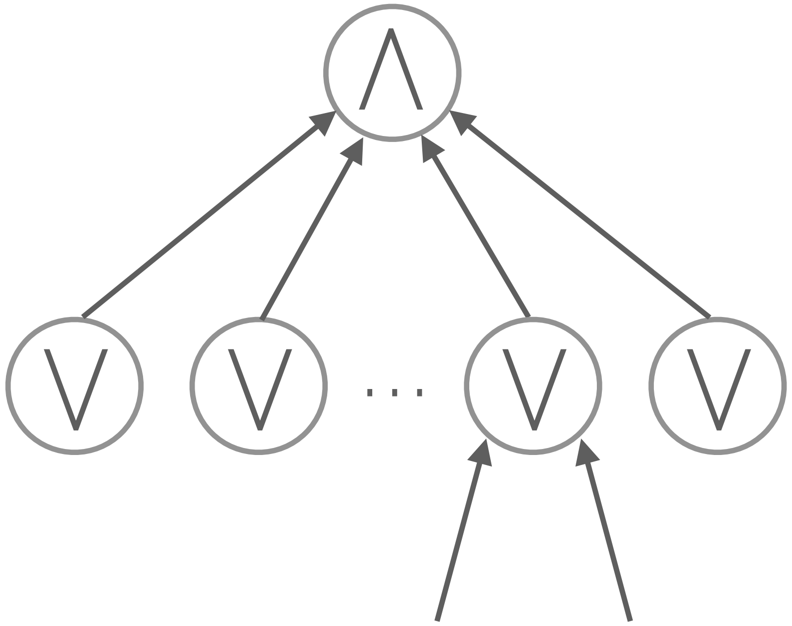

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Proof sketch:

Suffices to check the case when $\mathrm{TRIBES}(x) = 0$.

With $\Theta(1)$ probability, exactly one OR gate evaluates to 0.

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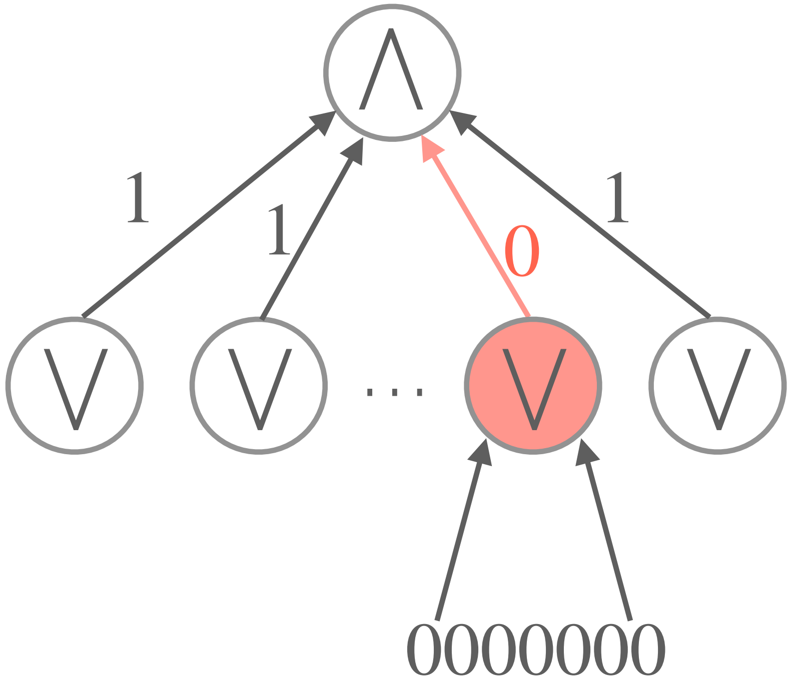

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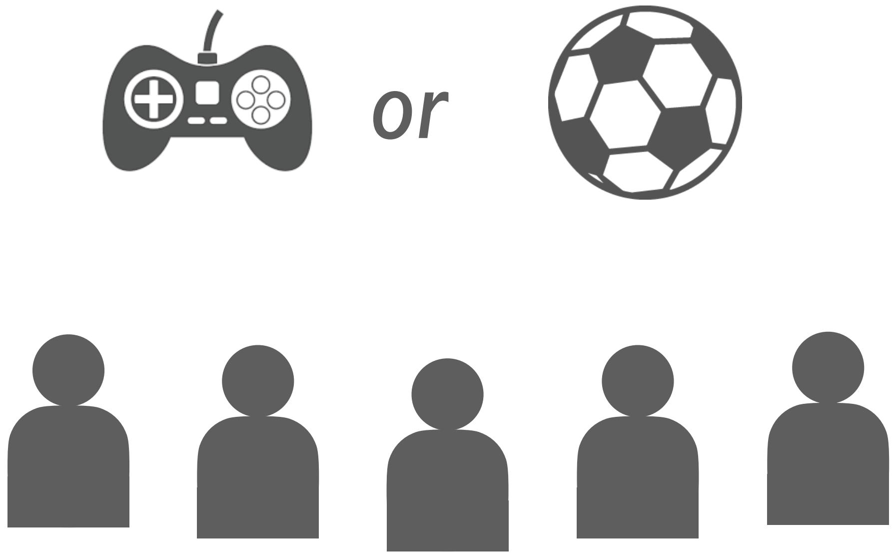

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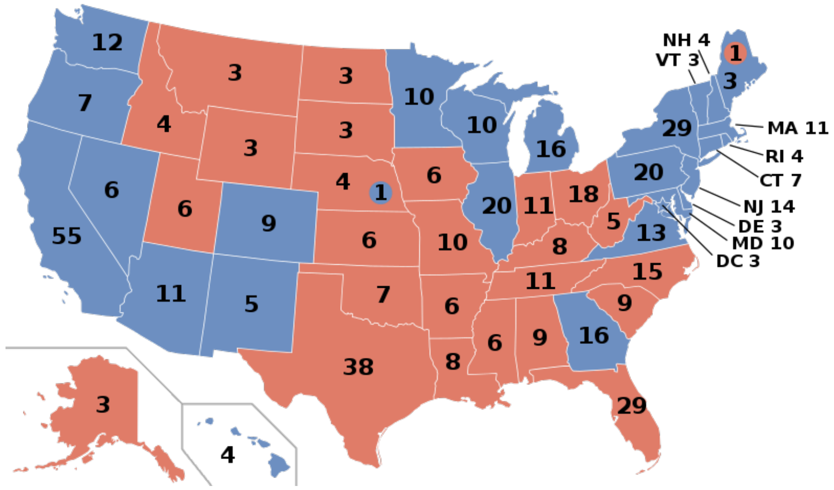

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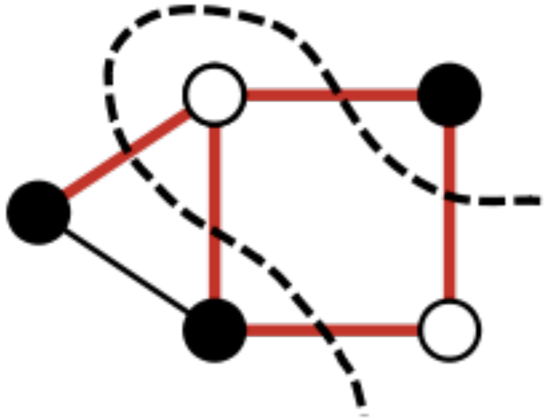

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3. Invariance principle

$\sum X_i \approx$ ***Gaussian***

Random restrictions

$f : \{0,1\}^n \rightarrow \{0,1\}$.

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0	*	1	0	*	1	0	*	0	1	…	…

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p-random restriction $f|_{R_p}$: fix pn random bits

Our results

Theorem (Eldan, Wigderson, W.)

For any $f : \{0,1\}^n \to \{0,1\}$, with $\mathrm{MaxInf}(f) = \tau = o(1)$, $\Omega(1)$ variance. Then for alive probability $\rho = \tilde{\Omega}\left(\frac{1}{\log 1/\tau}\right)$, we have

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Proof. by Hastad's Switching Lemma

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2. *Optimality (w.r.t.* Var*)*: the Majority function.

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- Koehler-Lifshitz-Minzer-Mossel have a different approach '22

Applications

One application

Corollary.

For any balanced function f, with $\tau = o(1)$ max influence, then

$$\Pr[\mathrm{bs}_f(x) \geq \tilde{\Omega}(\log(1/\tau))] = 1 - o(1)\,.$$

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$$s_f(x) := |\{i : f(x) \neq f(x \oplus e_i)\}|\,.$$

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Block sensitivity $\mathrm{bs}_f(x)$: max number of disjoint sensitive blocks

$$\mathrm{bs}_f(x) := \max |\{\text{disjoint } S_1, S_2, \ldots, S_k \subseteq [n] \\ : f(x) \neq f(x \oplus 1_{S_i})\}|\,.$$

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Remark: By KKL inequality,

$$\mathbf{E}[\mathrm{bs}_f(x)] \geq \mathbf{E}[\mathrm{s}_f(x)] \geq \Omega(\log(1/\tau))\,.$$

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Proof sketch.

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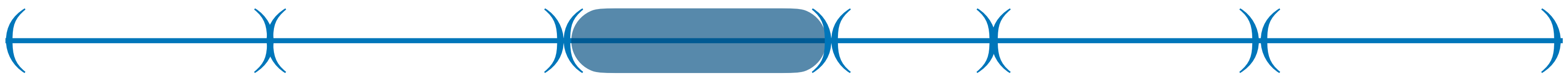

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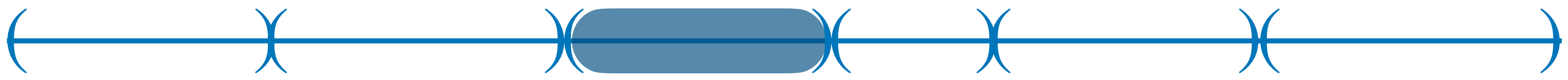

Random input x

Partition [n] into $M = \tilde{O}(\log(1/\tau))$ random blocks,

Any $M - 1$ blocks, induces a random restriction.

Condorcet Method

$k = 2$ candidates

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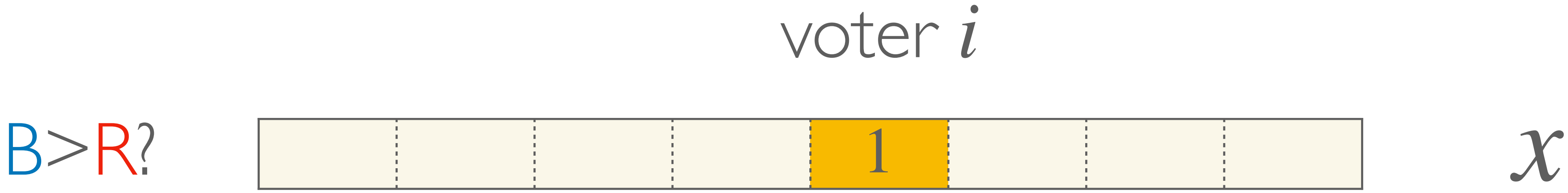

Condorcet Method

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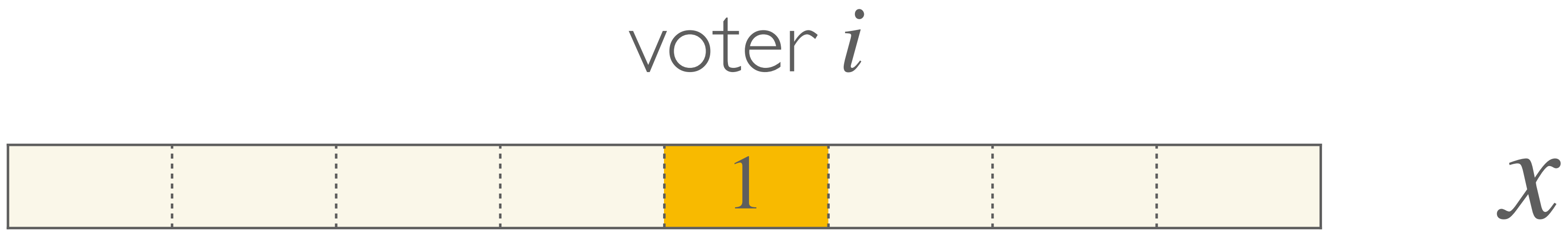
● ●

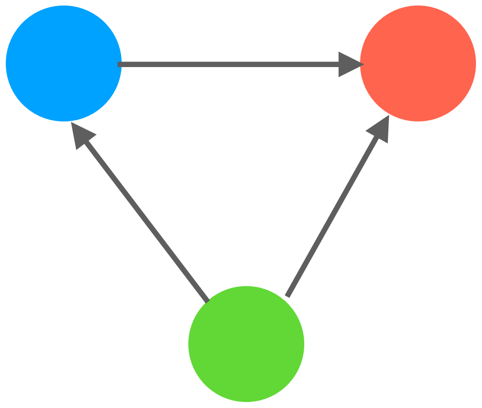
B>R?



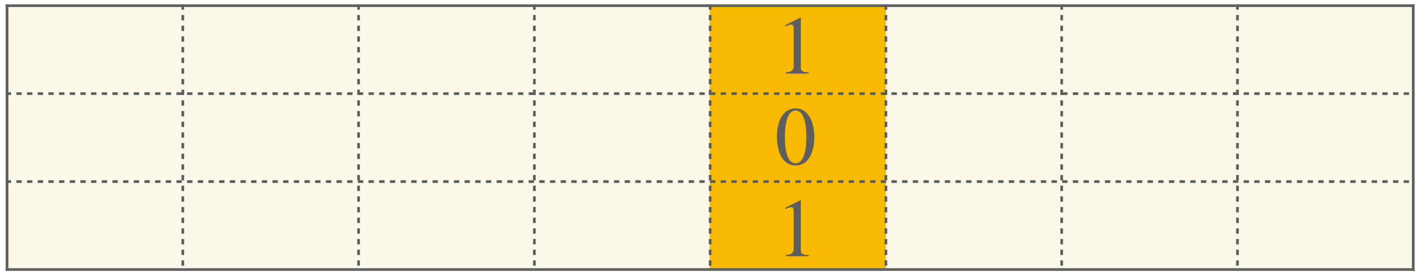
B is the winner if: $f(x) = 1$

Condorcet Method

$k = 3$ candidates



voter i



B>R?					1				$x^{\{B>R\}}$
R>G?					0				$x^{\{R>G\}}$
G>B?					1				$x^{\{G>B\}}$

Aggregate social choice pairwise — Condorcet

Condorcet Method

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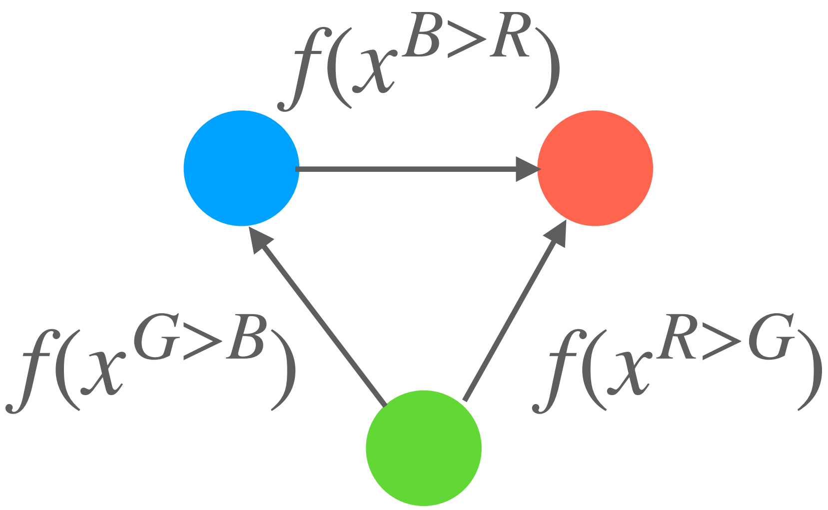

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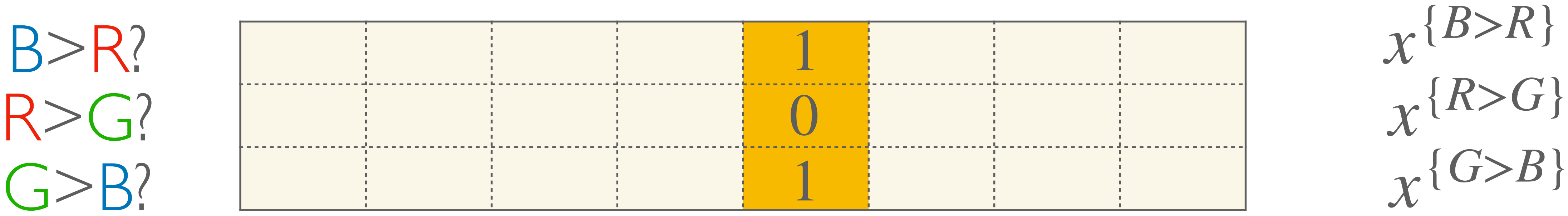

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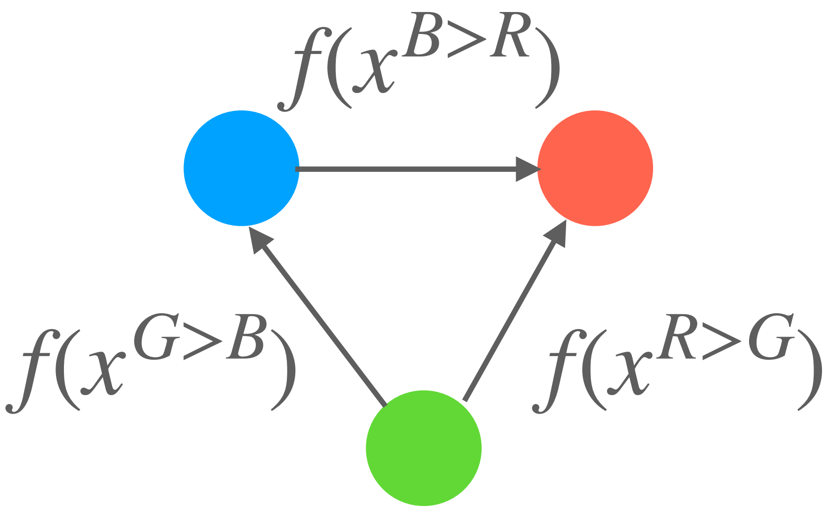

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Condorcet profile: the social preference between all pairs

Condorcet Method

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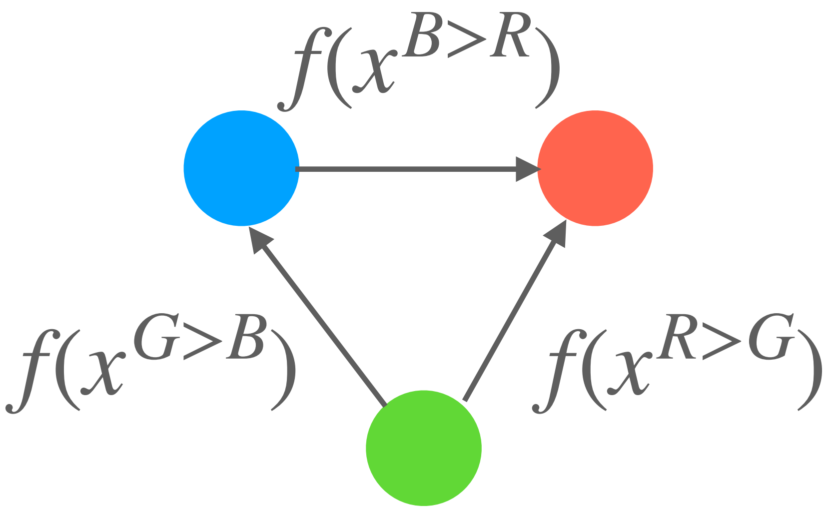

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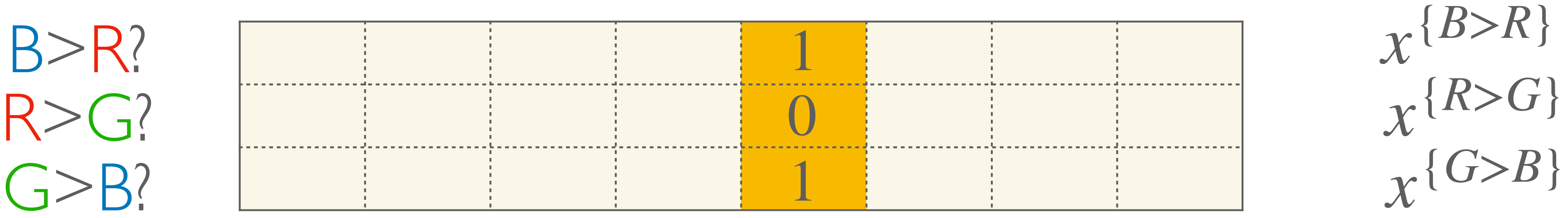

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B is the Condorcet winner if: $f(x^{B>R}) = 1$ and $f(x^{G>B}) = 0$

Condorcet Method

$k = 3$ candidates

Theorem.

Assume: f is fair, $\mathbf{E}[f] = 1/2$, $\mathrm{MaxInf}(f) = \tau = o(1)$
voters preferences are uniformly random (total order)

Then, any Condorcet profile happen with probability $\Omega(1)$.

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$\exp(-\tilde{O}(k^5))$.

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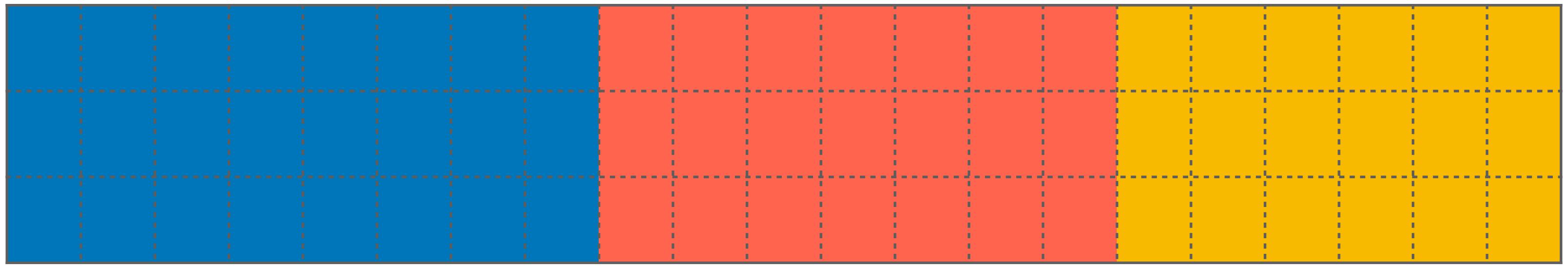

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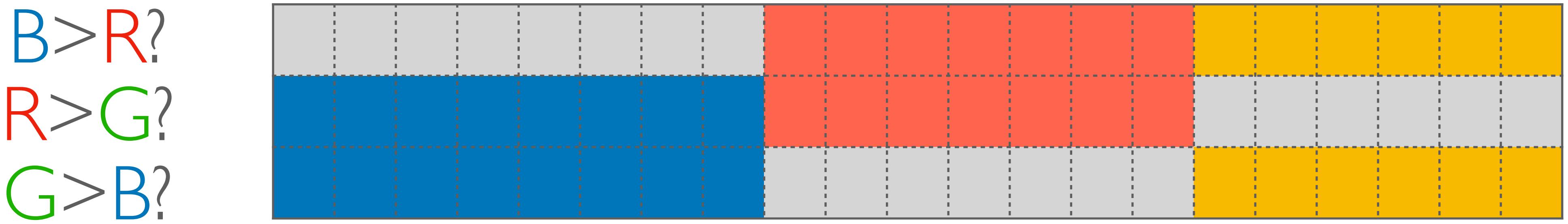

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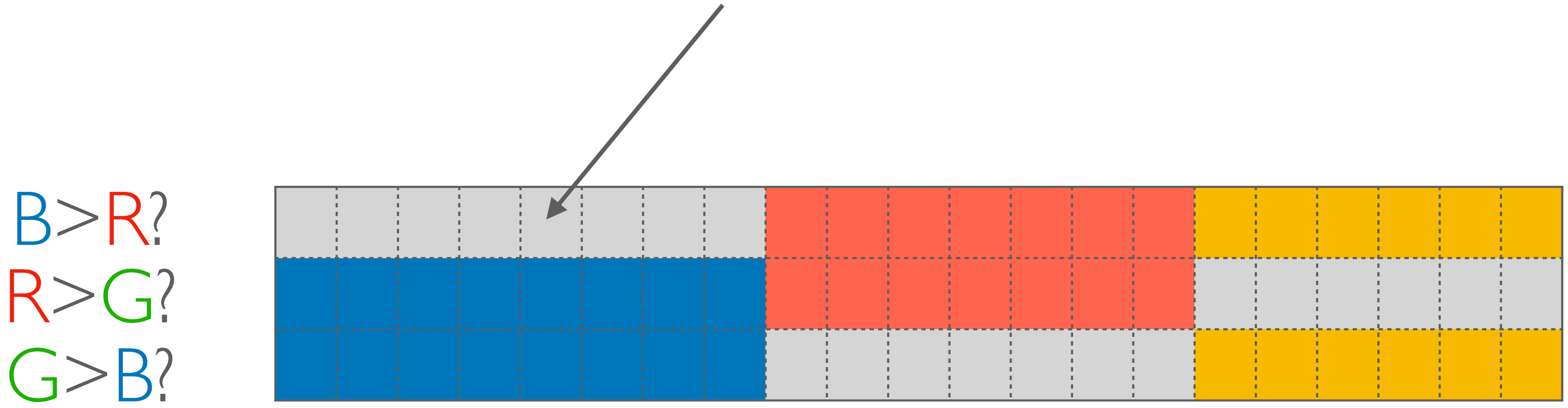
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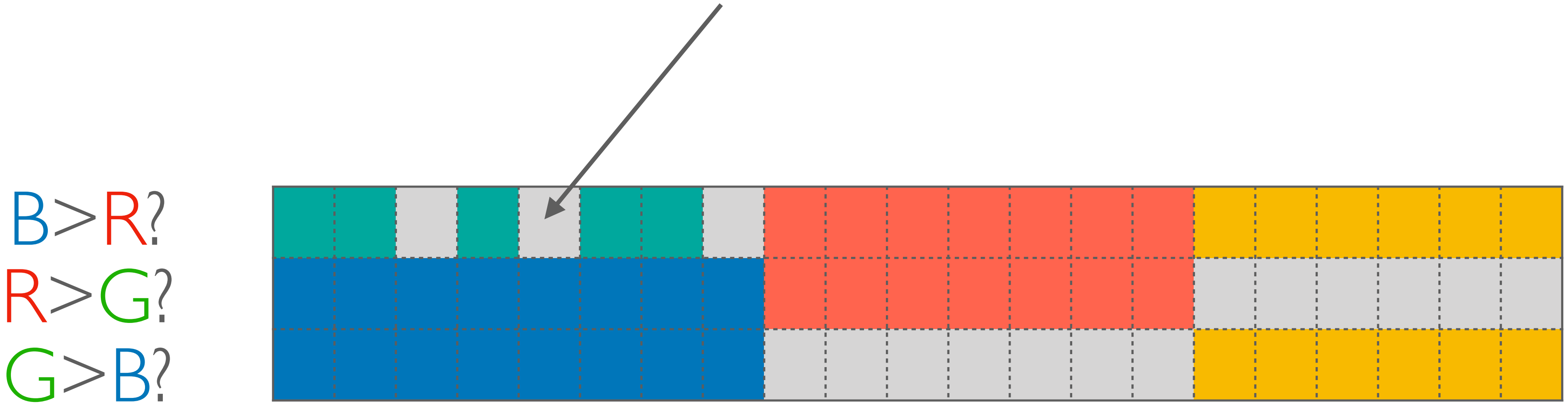

Unfixed entries induced a random restriction $\mathscr{R}^{B>R}$,
Thus, w.h.p., $\mathrm{Var}[f|_{\mathscr{R}^{B>R}}] = \Omega(1)$.

Condorcet Method

More on social choice:

“Probabilistic Aspects of Voting, Intransitivity and Manipulation”— Mossel

https://arxiv.org/abs/2012.10352

Proof of the main result

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Two random processes

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Uniform process

1. Random permutation π,
2. $X(0) = 0^n$,
3. $X_i(t) = X_i(t-1)$, for $i \neq \pi(t)$;
   $X_{\pi(t)}(t) \sim \{-1,1\}$, for $t = 1,2,\ldots,n$

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Observation.

i. $X(t)$ induces a random restriction $f|_{R(t)}$,

ii. $X(n)$ is a uniformly random element from $\{-1,1\}^n$,

iii. $\mathbf{E}[\partial_i f(X(n))^2] = \mathrm{\mathbf{Inf}}_i(f)$,

iv. $f(X(t)) = \mathbf{E}_{z \in \{-1,1\}^{n-t}}[f|_{R(t)}(z)]$ is a martingale.

Two random processes

$f : \{-1,1\}^n \to \{0,1\}$ (by multilinear extension $f : \mathbb{R}^n \to \mathbb{R}$)

Uniform process

1. Random permutation π,
2. $X(0) = 0^n$,
3. $X_i(t) = X_i(t-1)$, for $i \neq \pi(t)$;
   $X_{\pi(t)}(t) \sim \{-1,1\}$, for $t = 1,2,\ldots,n$

Observation.

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e.g. $f = 1 + x_1 + x_2 + x_1x_2$,
$f(0) = \mathbf{E}_{x\in\{-1,1\}^n}[f(x)]$

Two random processes

Conditioned process

1. Random permutation π,
2. $Y(0) = 0^n$,
3. for $t = 1,2,\ldots,n$, let $Y_i(t) = Y_i(t-1)$,

$$\Pr\{Y_{\pi(t)}(t) = \pm 1\} = \frac{1}{2} \pm \frac{\partial_{\pi(t)} f(Y(t-1))}{2f(Y(t-1))}\,.$$

Two random processes

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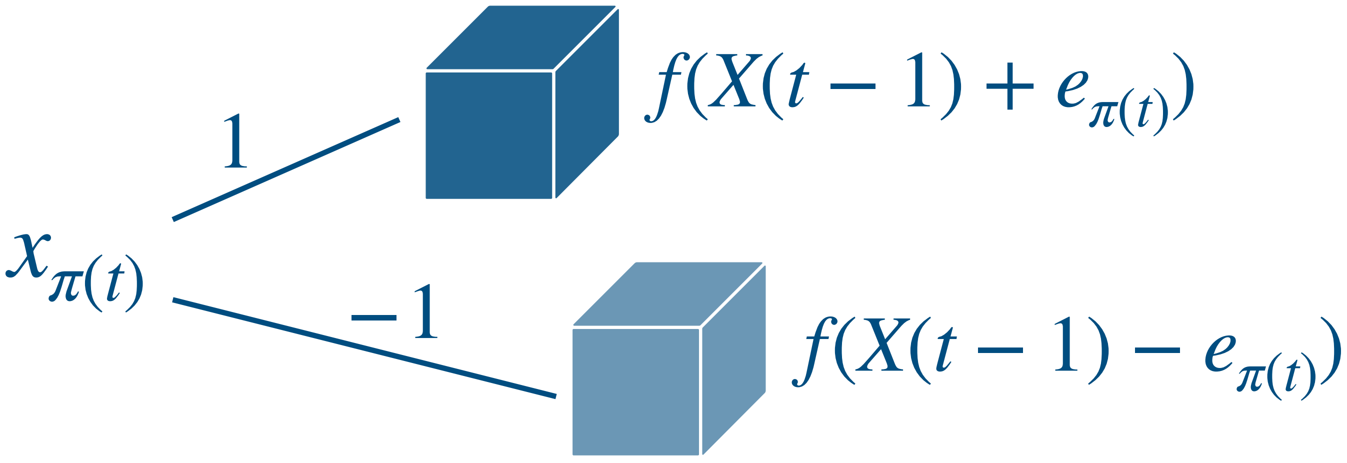

Two random processes

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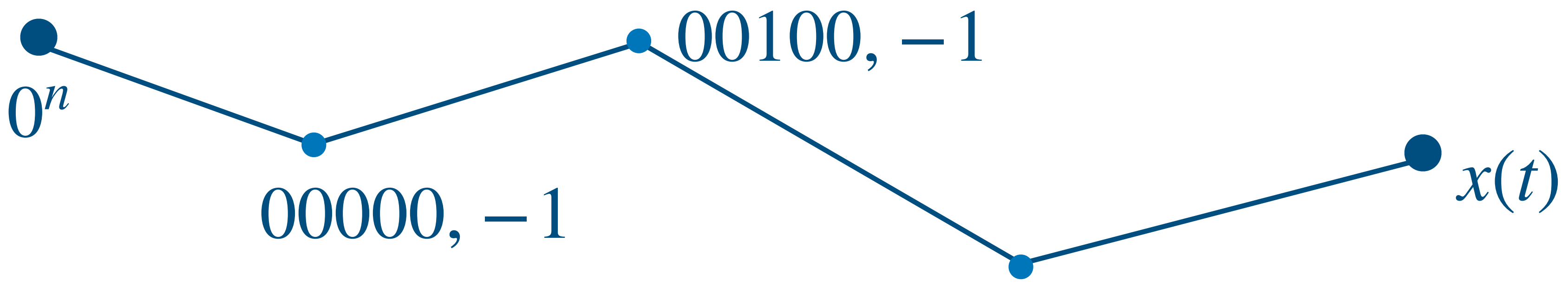
Fact. Fix some some path $x(0), x(1), \ldots, x(t) \in \{-1,0,1\}^n$,

$$\frac{\Pr[\forall i \in [t], Y(t) = x(t)]}{\Pr[\forall i \in [t], X(t) = x(t)]} = \frac{f(x(t))}{f(0)}.$$

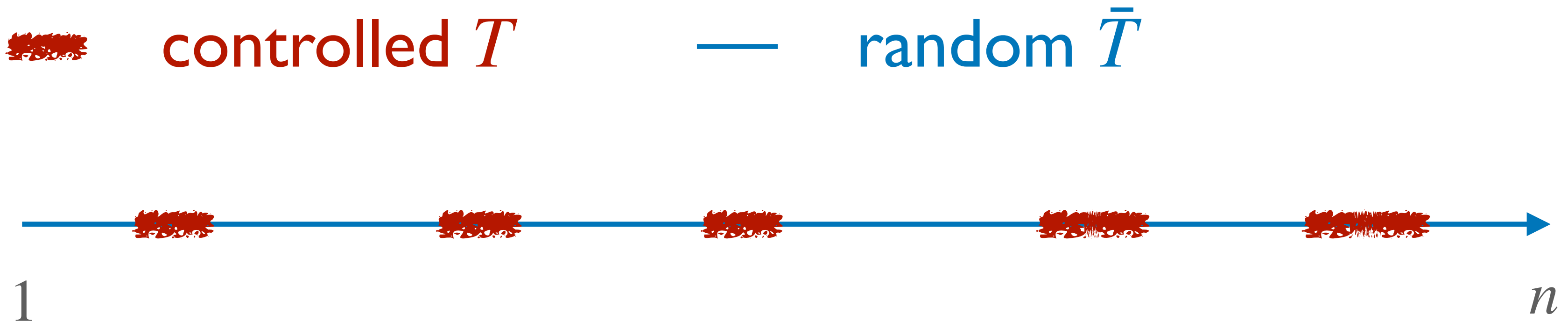


Controlled Process

▬ controlled T — random $\bar{T}$

Controlled process

Controlled Process



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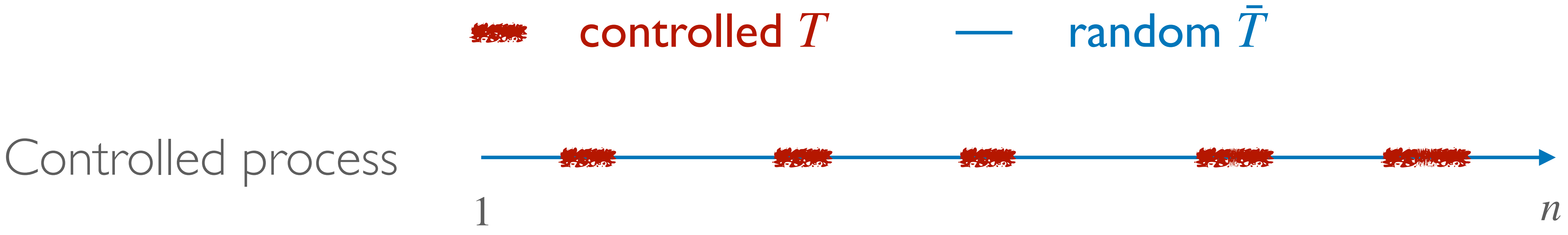
Controlled process

THE GAME: expose the variables in a random order:

(w.p. $1-\epsilon$) expose the variable uniformly randomly;

(w.p. ϵ) player gets to decide.

Controlled Process



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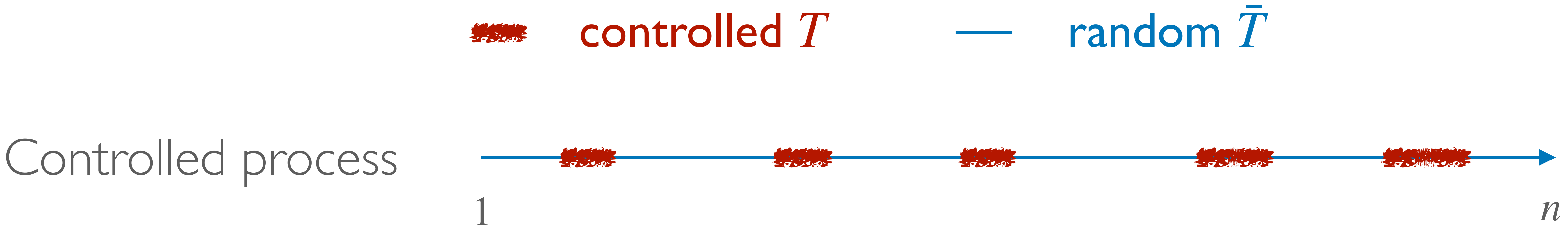
Controlled Process



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Controlled process

1 n

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Mission impossible!

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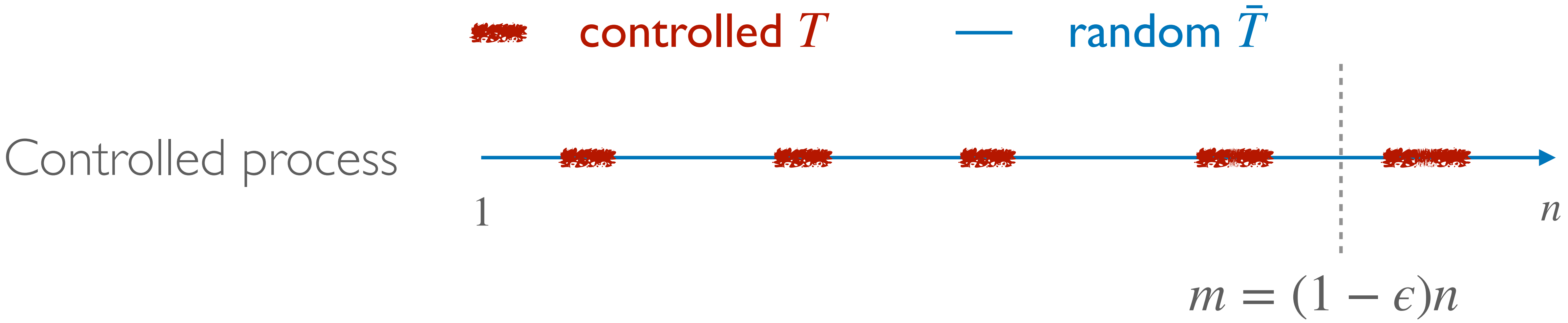

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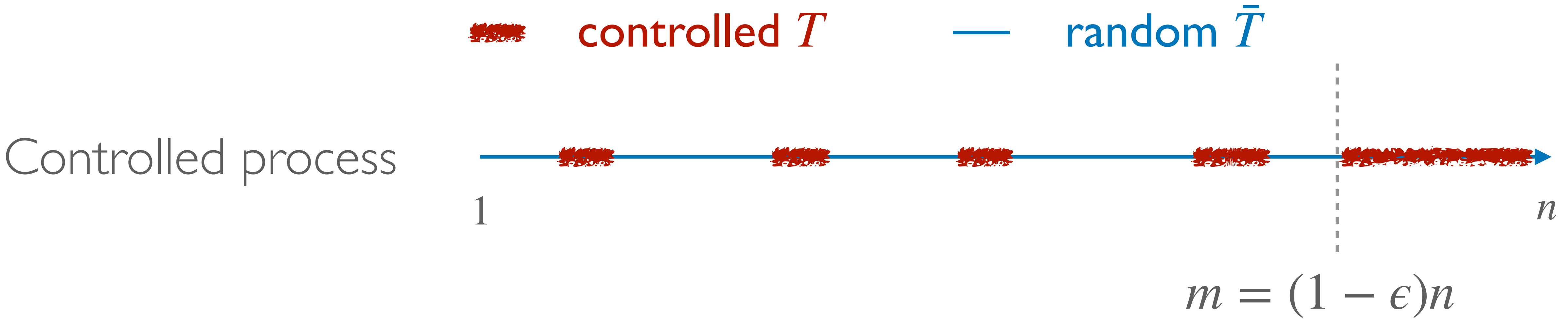

controlled T — random $\bar{T}$

Controlled process

1 n

$m = (1-\epsilon)n$

THE GAME: expose the variables in a random order:

(w.p. $1-\epsilon$) expose the variable uniformly randomly;

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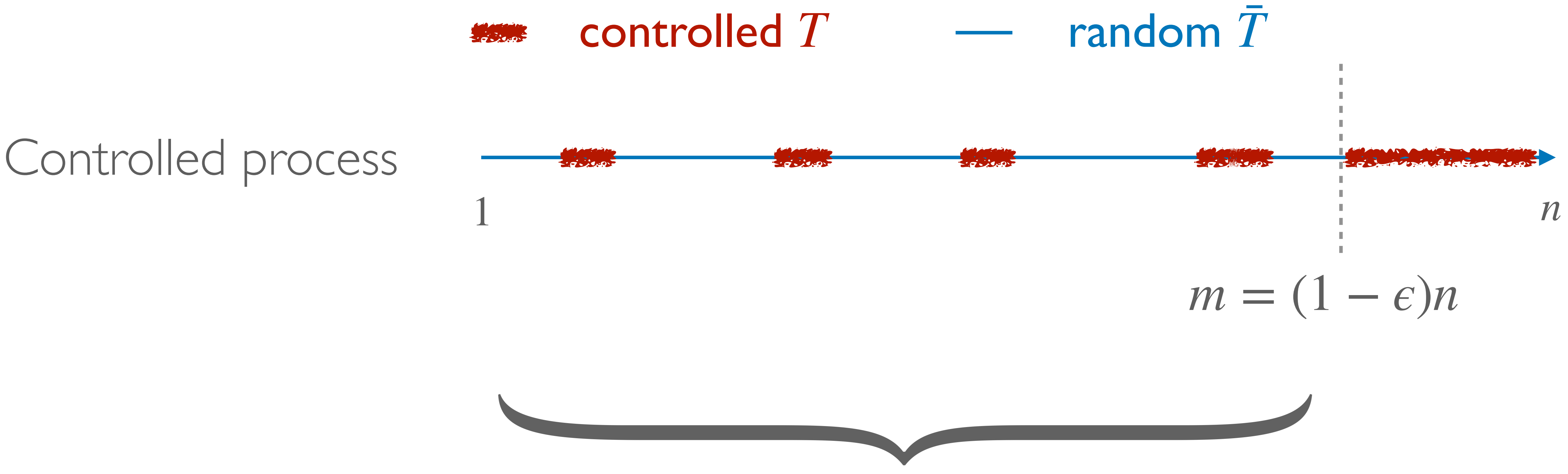
(after m) player gets to decide.

THE GOAL: sample $x \in f^{-1}(1)$ uniformly at random. ✓

Controlled Process



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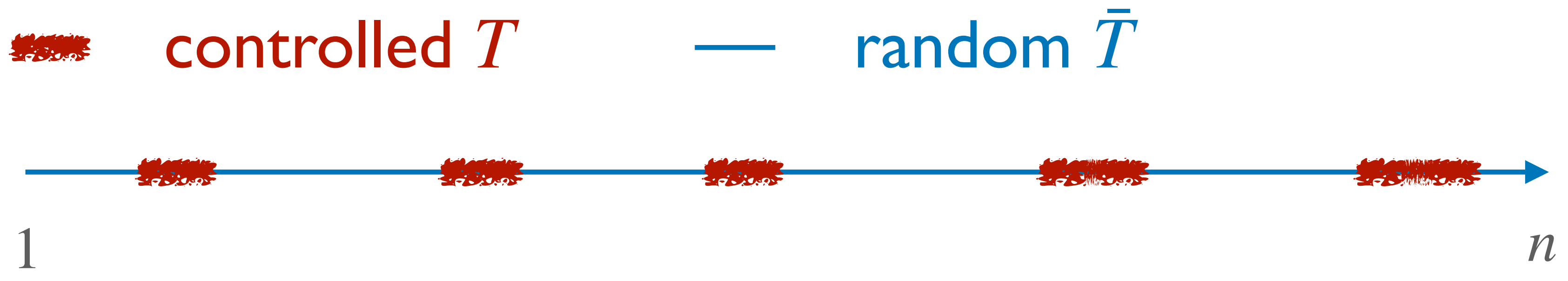
Controlled process

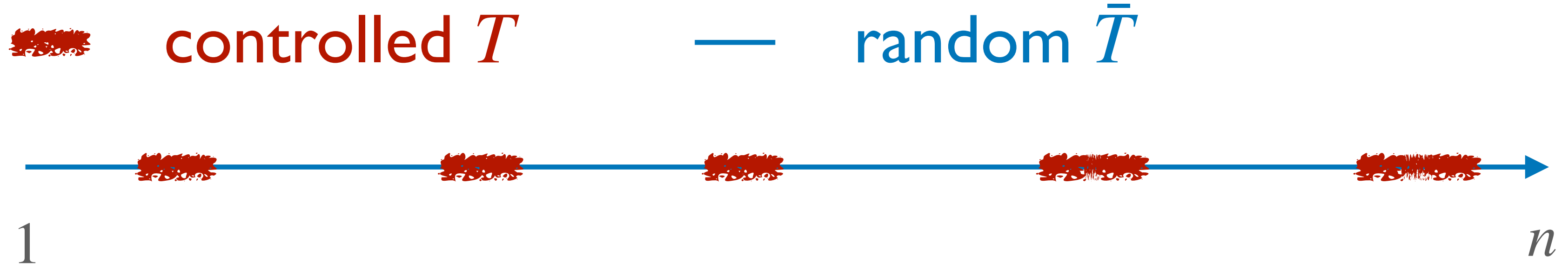
$m = (1 - \epsilon)n$

the blue variables = random restriction

Controlled Process



Controlled Process



$T \subseteq [n]$, a random $(1-\epsilon)$-set

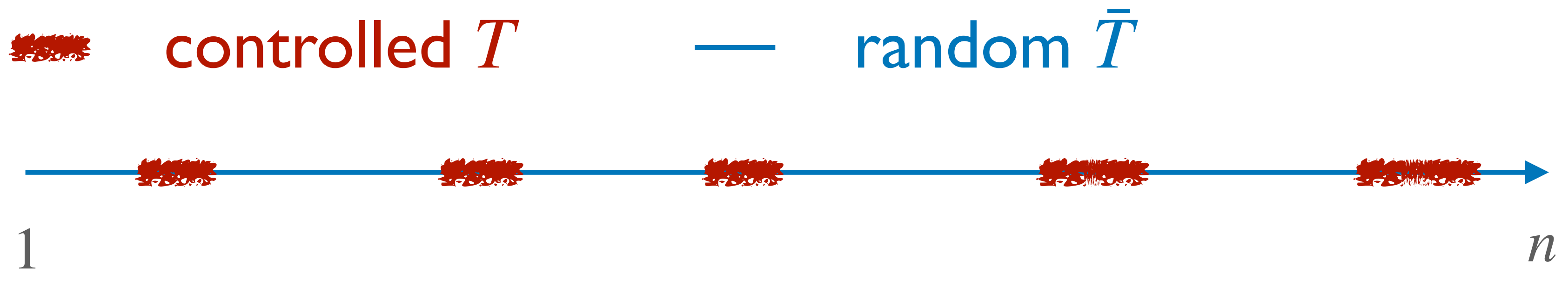
(Random coordinate $t \notin T$)

$$\Pr[Y_{\pi(t)}(t) = \pm 1] = 0.5$$

(Controlled coordinate $t \in T$)

$$\Pr[Y_{\pi(t)}(t) = \pm 1] = \frac{1}{2} \pm \frac{\partial_{\pi(t)} f(Y(t-1))}{2\epsilon f(Y(t-1))}.$$

Controlled Process



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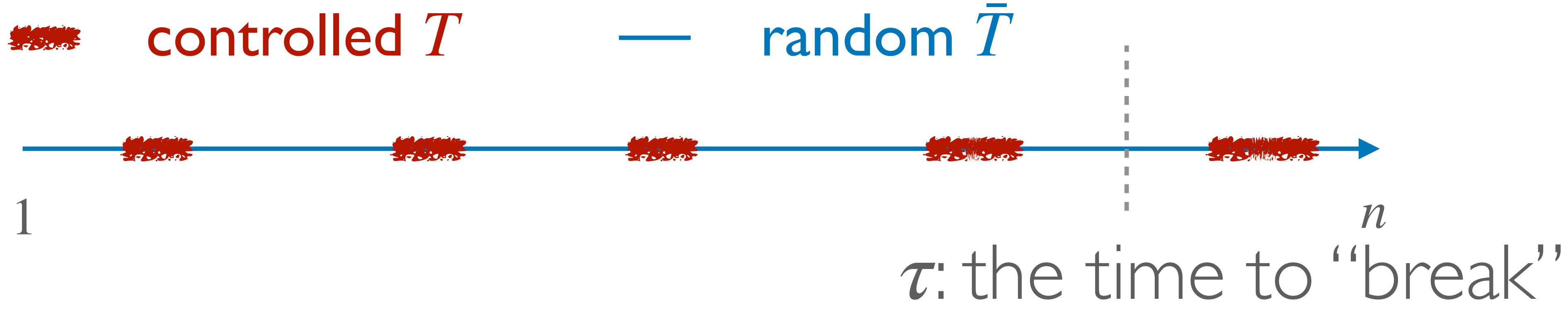

controlled T — random $\bar{T}$

1 n

τ: the time to "break"

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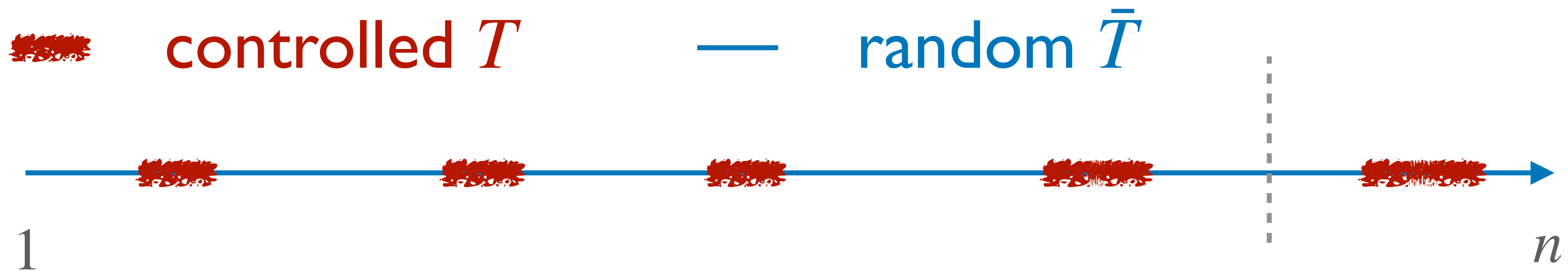

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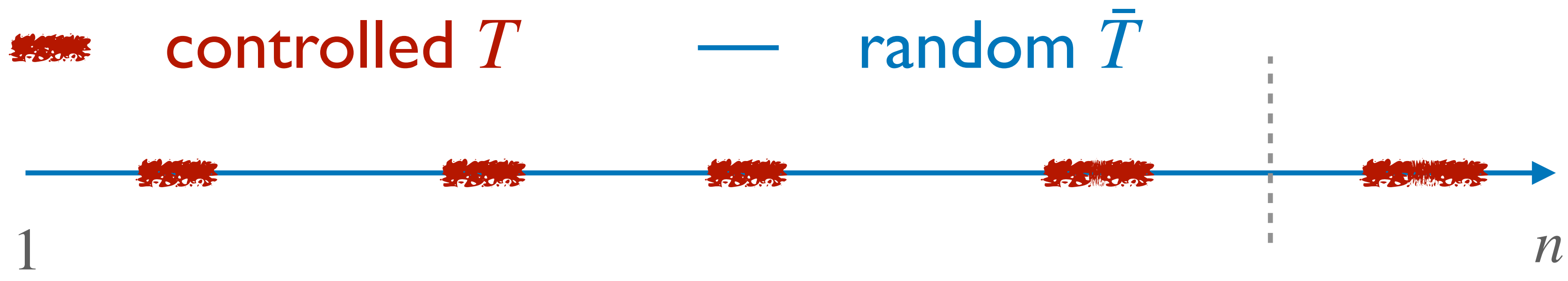

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(After m)

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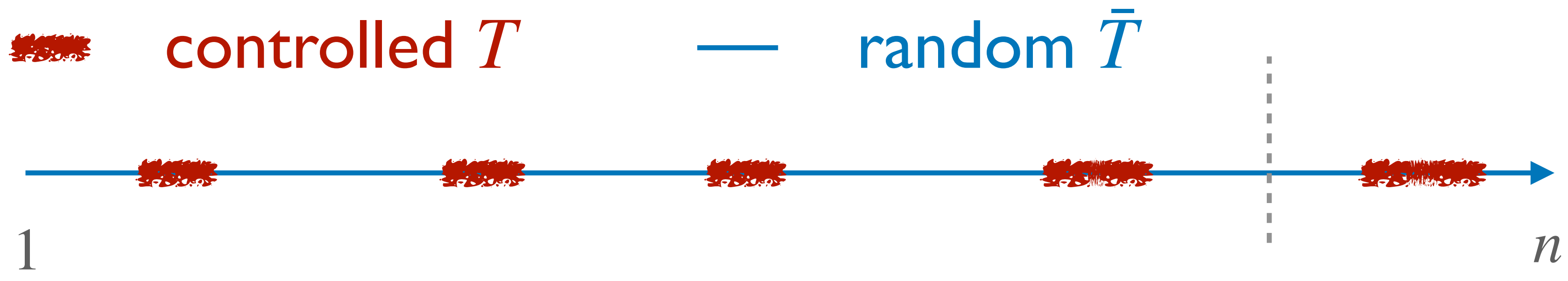
Controlled Process



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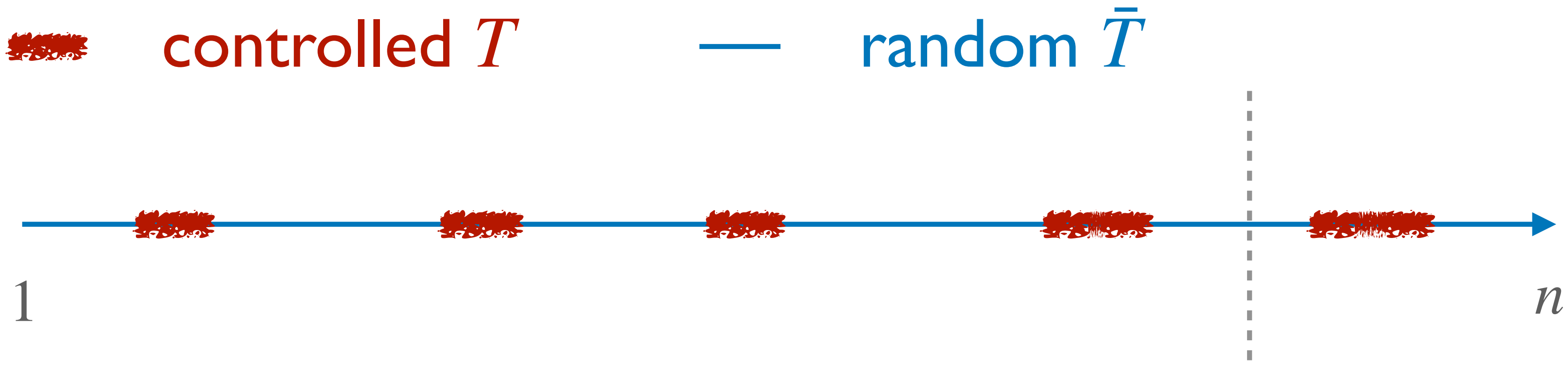
Goal accomplished,
as long as $\tau > m$!

Analysis



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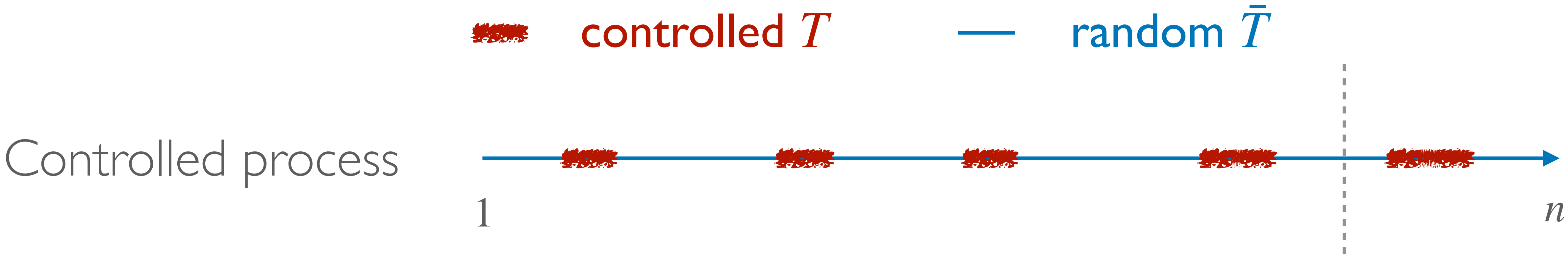

controlled T — random $\bar{T}$

Controlled process

1 n

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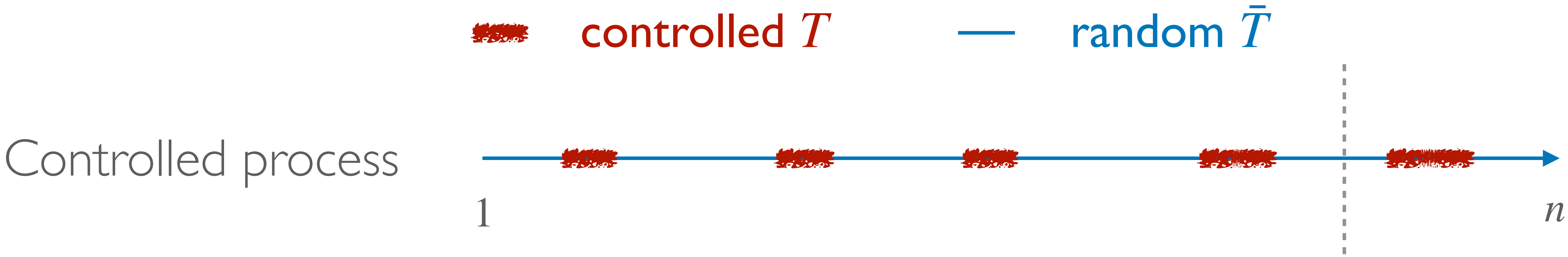
Lemma 1. w.h.p. $\tau_1 > (1-\epsilon)n$

Analysis



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Lemma 1. w.h.p. $\tau_1 > (1-\epsilon)n$

Lemma 2. w.h.p. $\tau_2 > (1-\epsilon)n$

Mean remains large

$$\tau_1 = \min_t \{ f(Y(t-1)) < \delta \}$$

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Lemma 1. w.h.p. $\tau_1 > (1-\epsilon)n$

Proof. $$\Pr_Y[\tau_1 \leq (1-\epsilon)n] \leq \Pr_X[\tau_1 \leq (1-\epsilon)n] \cdot \frac{\delta}{f(0)}$$

$$\leq \frac{\delta}{f(0)} .$$

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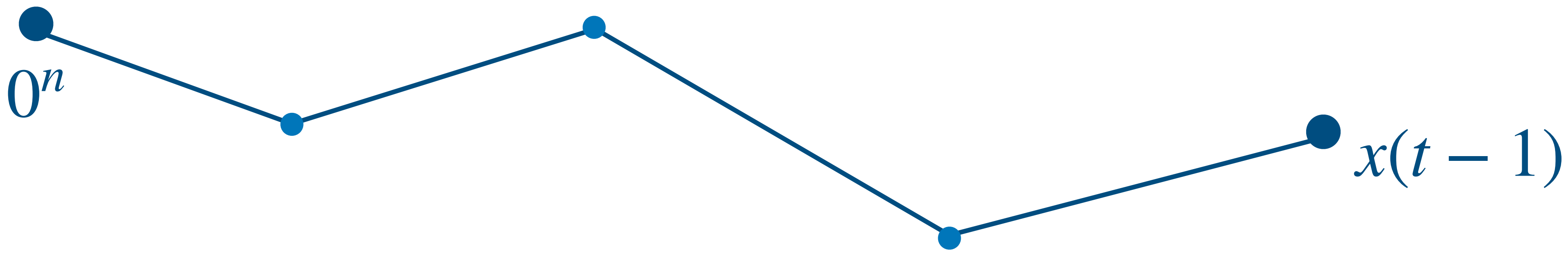

Partial derivatives remain small

$$\tau_2 = \min_t \{ \max_i |\partial_i f(Y(t-1))| > \epsilon\delta \}$$

Lemma 2. w.h.p. $\tau_2 > (1-\epsilon)n$

Proof. $\Pr_Y[\tau_2 \le (1-\epsilon)n] \le \Pr_X[\tau_2 \le (1-\epsilon)n] \cdot \dfrac{1}{f(0)}.$

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Switch to continuous process

Continuous uniform process

Sample $\eta(i) \in [0,1], i = 1,2,\ldots,n$,
Each $Z_i(t)$ is 0 until $t = \eta(i)$, set $Z_i(t) \sim \{-1,1\}$.

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Each $Z_i(t)$ is 0 until $t = \eta(i)$, set $Z_i(t) \sim \{-1,1\}$.

$$\tau_3 = \min_t \{\max_i |\partial_i f(Z(t))| > \epsilon\beta\}$$

Lemma 3. w.h.p. $\tau_3 > (1 - \epsilon)$

Partial derivatives under uniform process

$\tau_3 = \min_t \{ \max_i | \partial_i f(Z(t)) | > \epsilon\delta \}$

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Proof.

$$\Pr\left[\sup_{0 \le s \le t} \beta(s) \ge \theta\right] \le \Pr\left[\sup_{0 \le s \le t} \sum_{i=1}^{n} |\partial_i f(Z(s))|^{2+\eta} \ge \theta^{2+\eta}\right]$$

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$\partial_i f(Z(t))$ is a martingale.

Partial derivatives under uniform process

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$$\le \theta^{-2-\eta} \sum_i \mathbf{E}[| \partial_i f(Z(t)) |^{2+\eta}]$$

$$\le \theta^{-2-\eta} \sum_i (\mathbf{E}[\partial_i f(Z(T))^2])^{1+\eta/2}$$

A hypercontractivity inequality for random restrictions

Partial derivatives under uniform process

Theorem (Hypercontractive inequality for random restriction).

For any multilinear function $f : [-1,1]^n \to \mathbb{R}$, and $0 \le t \le T \le 1$. Then, for $\eta \le T - t$,

$$\mathbf{E}[\,|f(Z(t))|^{2+\eta}\,]^{\frac{1}{2+\eta}} \le \mathbf{E}[f(Z(T))^2]^{\frac{1}{2}},$$

Partial derivatives under uniform process

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c.f. the standard HC inequality

$$\mathbf{E}[(\mathbf{T}_{\epsilon(\eta)}f(x)^{2+\eta}])^{\frac{1}{2+\eta}} \leq \mathbf{E}[f(x)^2]^{\frac{1}{2}}$$

Partial derivatives under uniform process

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$$\le \theta^{-2-\eta} \sum_i \mathrm{Inf}_i(f)^{\eta/2} \mathbf{E}[\partial_i f(Z(T))^2]$$

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$$\le \theta^{-2-\eta} \sum_i \mathrm{Inf}_i(f)^{\eta/2} \mathbf{E}[\partial_i f(Z(T))^2]$$

Proposition. $\mathbf{E}[\partial_i f(Z(t))^2] \le \mathrm{Inf}_i(f)$

Partial derivatives under uniform process

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$$\begin{aligned}
\Pr\left[\sup_{0 \le s \le t} \beta(s) \ge \theta \right] &\le \Pr\left[\sup_{0 \le s \le t} \sum_{i=1}^{n} |\partial_i f(Z(s))|^{2+\eta} \ge \theta^{2+\eta} \right] \\
&\le \theta^{-2-\eta} \sum_i \mathbf{E}[|\partial_i f(Z(t))|^{2+\eta}] \\
&\le \theta^{-2-\eta} \sum_i (\mathbf{E}[\partial_i f(Z(T))^2])^{1+\eta/2} \\
&\le \theta^{-2-\eta} \sum_i \mathrm{Inf}_i(f)^{\eta/2} \mathbf{E}[\partial_i f(Z(T))^2] \\
&\le \theta^{-2-\eta} \frac{\mathrm{MaxInf}(f)^{\eta/2}}{1 - T}
\end{aligned}$$

Partial derivatives under uniform process

$$\tau_3 = \min_t \{ \underbrace{\max_i |\partial_i f(Z(t))|}_{\beta(t)} > \epsilon\delta \}$$

Lemma 3. w.h.p. $\tau_3 > (1-\epsilon)$

Proof.

$$\Pr\left[\sup_{0\le s\le t} \beta(s) \ge \theta\right] \le \Pr\left[\sup_{0\le s\le t} \sum_{i=1}^{n} |\partial_i f(Z(s))|^{2+\eta} \ge \theta^{2+\eta}\right]$$

$$\le \theta^{-2-\eta} \sum_i \mathbf{E}[|\partial_i f(Z(t))|^{2+\eta}]$$

$$\le \theta^{-2-\eta} \sum_i (\mathbf{E}[\partial_i f(Z(T))^2])^{1+\eta/2}$$

$$\le \theta^{-2-\eta} \sum_i \mathrm{Inf}_i(f)^{\eta/2} \mathbf{E}[\partial_i f(Z(T))^2]$$

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Proposition. $\sum_{i=1}^{n} \mathbf{E}[\partial_i f(Z(T))^2] \le \frac{1}{1-T}$

Partial derivatives under uniform process

$$\tau_3 = \min_t \{ \underbrace{\max_i |\partial_i f(Z(t))|}_{\beta(t)} > \epsilon\delta \}$$

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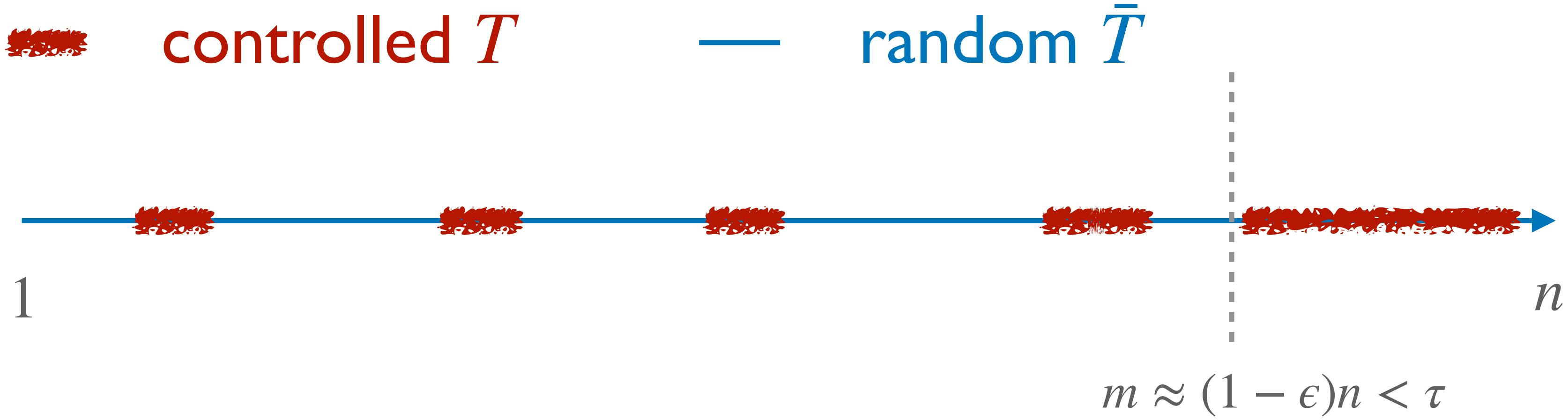
&\le \theta^{-2-\eta} \sum_i \mathbf{E}[\,|\partial_i f(Z(t))|^{2+\eta}] \\
&\le \theta^{-2-\eta} \sum_i (\mathbf{E}[\partial_i f(Z(T))^2])^{1+\eta/2} \\
&\le \theta^{-2-\eta} \sum_i \mathrm{Inf}_i(f)^{\eta/2} \mathbf{E}[\partial_i f(Z(T))^2] \\
&\le \theta^{-2-\eta} \frac{\mathrm{MaxInf}(f)^{\eta/2}}{1 - T} \le 2\theta^{-3+t/2} \frac{\mathrm{MaxInf}(f)^{(1-t)/4}}{1 - t}
\end{aligned}$$

set $T = (1 + t)/2, \eta = T - t$

Bound the variance

Entropy comparison



controlled T

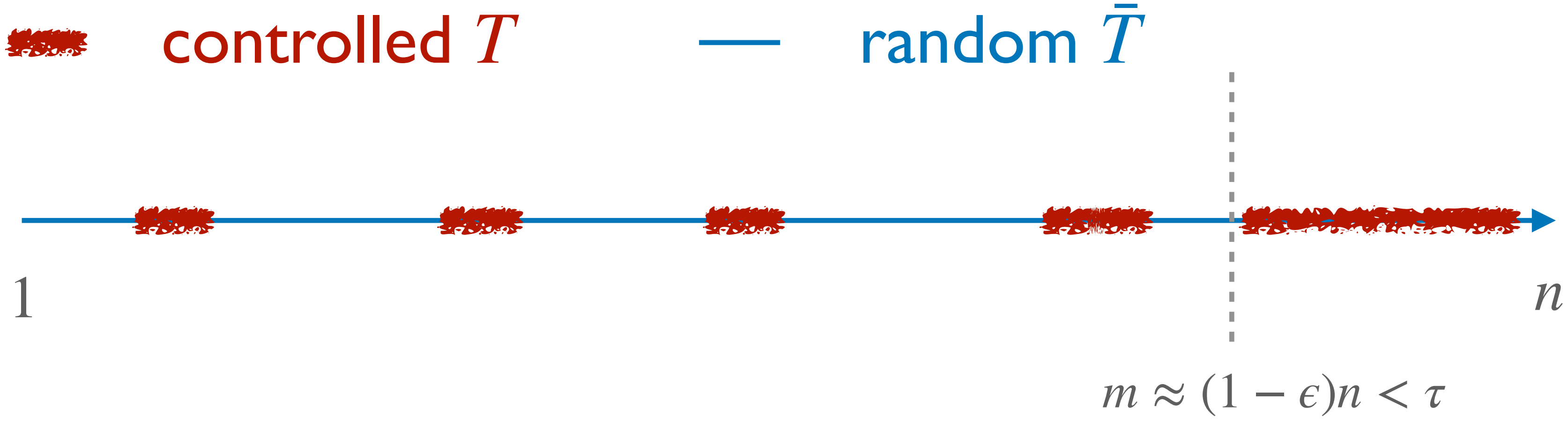
random $\bar{T}$

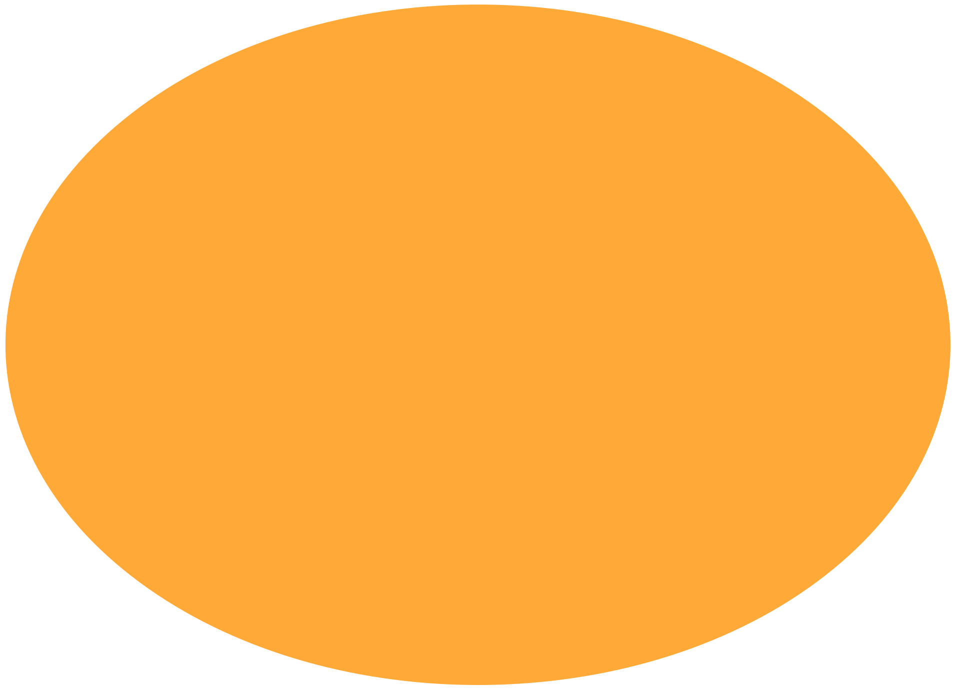
1

n

$m \approx (1 - \epsilon)n < \tau$

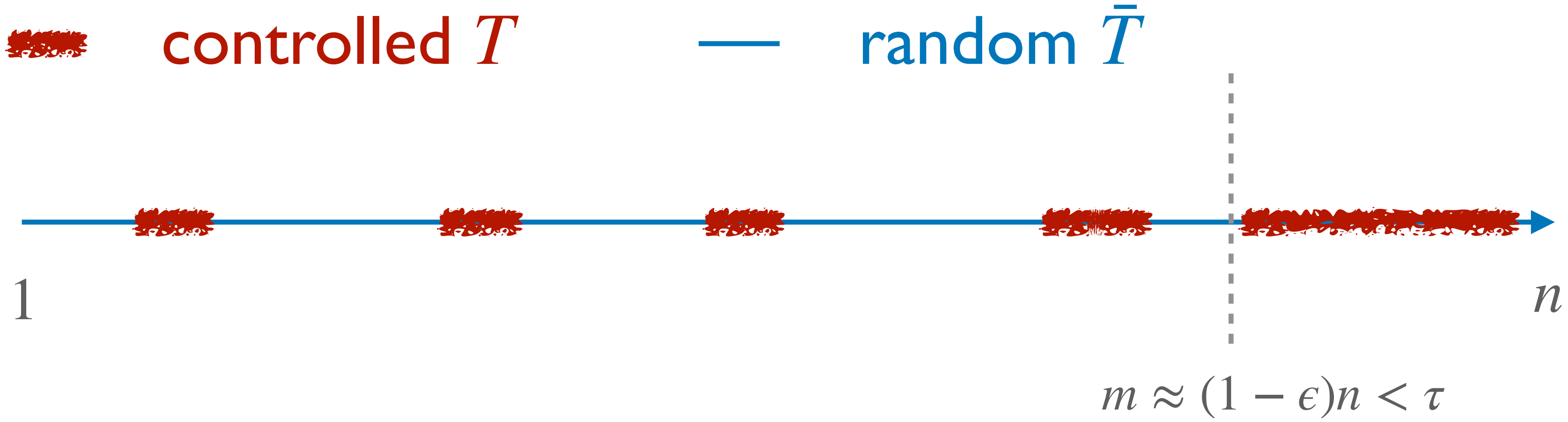
Entropy comparison





$\{-1,1\}^T$

Entropy comparison



🟥 controlled T — random $\bar{T}$

1 n

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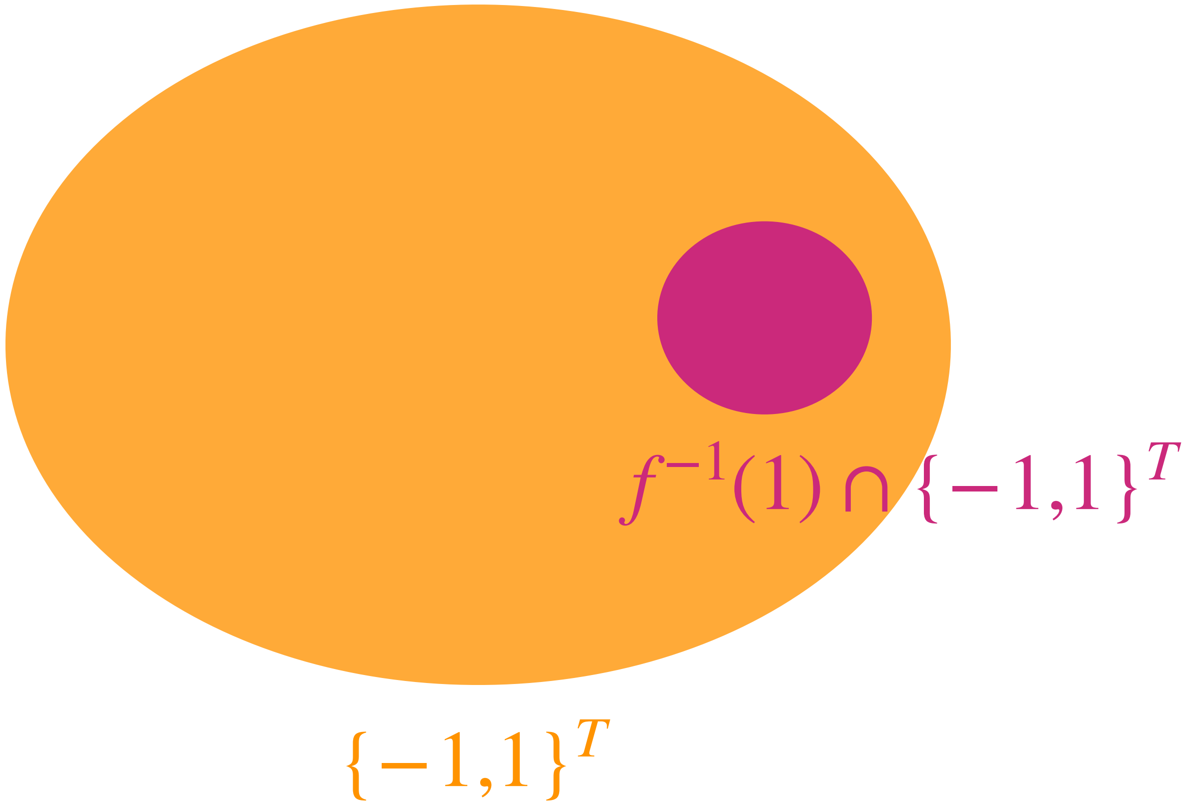
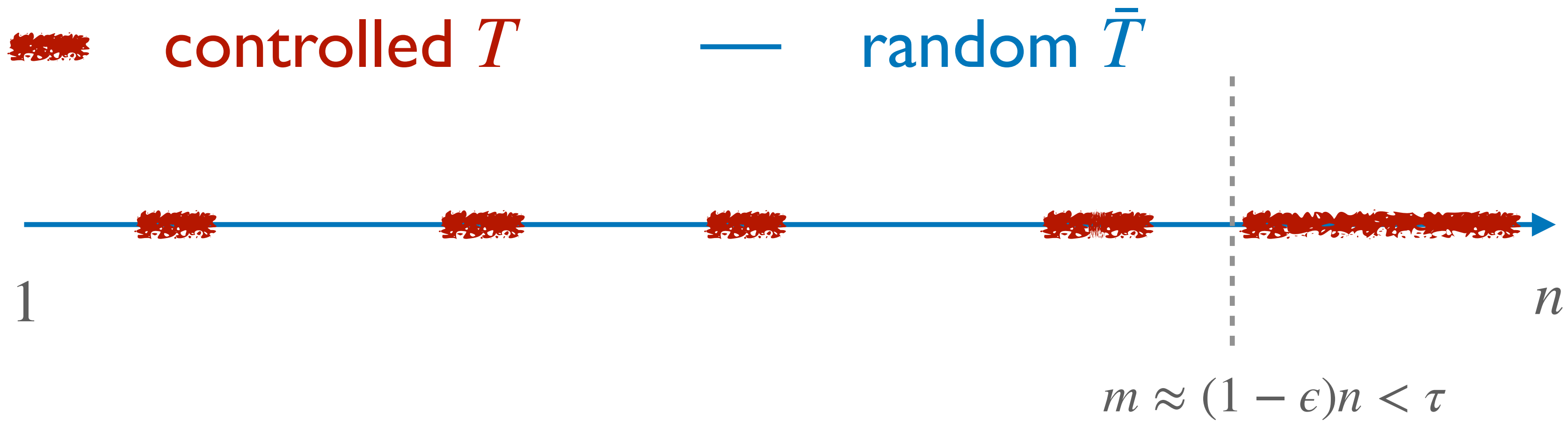
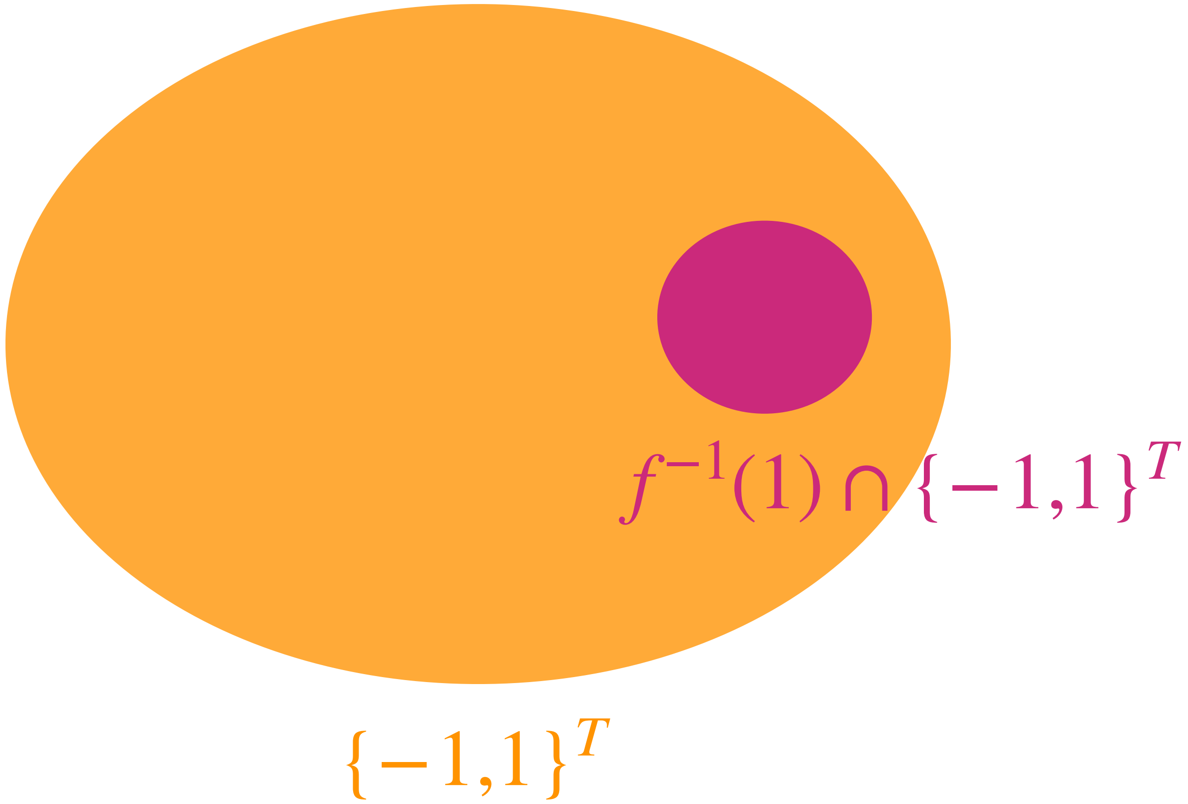


$f^{-1}(1) \cap \{-1,1\}^T$

$\{-1,1\}^T$

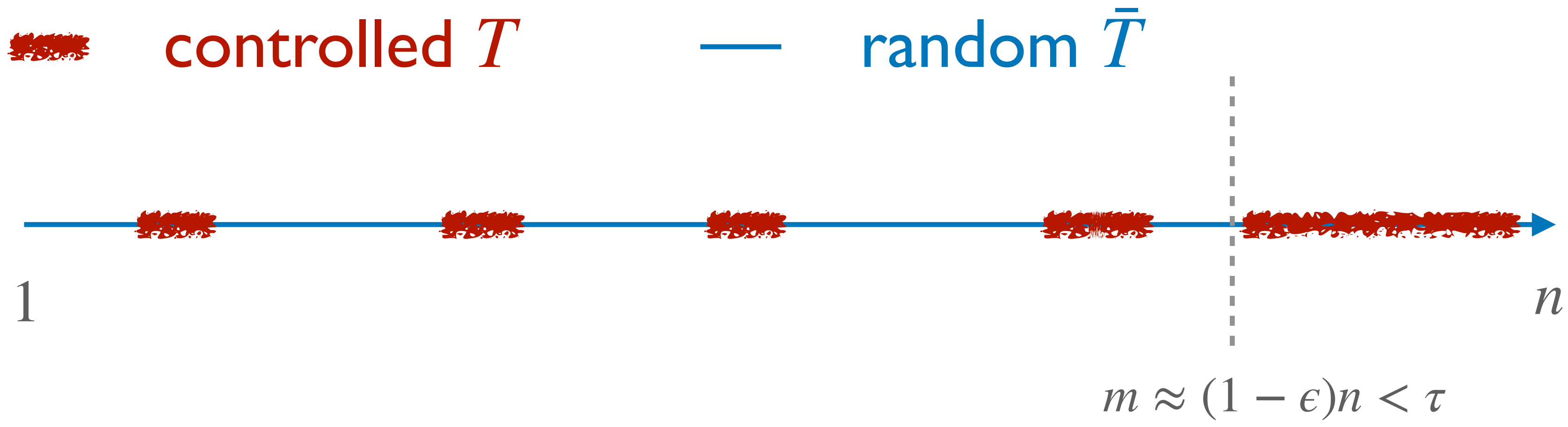
Entropy comparison

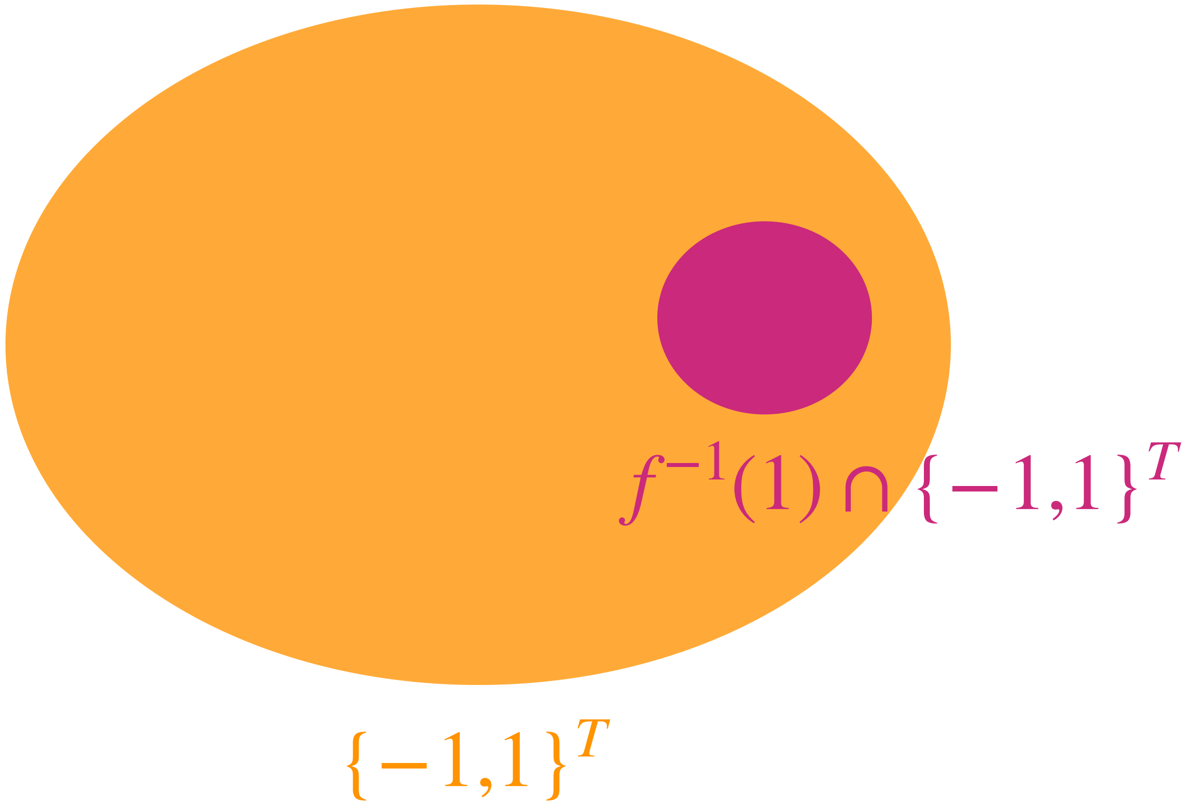




Strategy: to bound the KL-divergence between $Y(n)$ and $X(n)$ given $\pi, T,$ and the restriction.

Entropy comparison

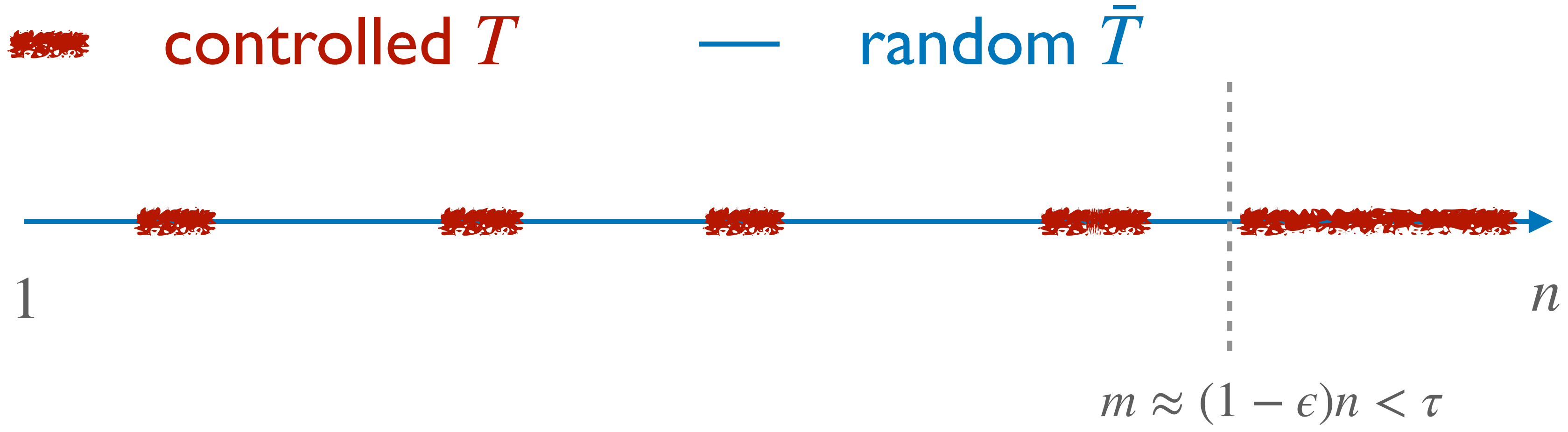




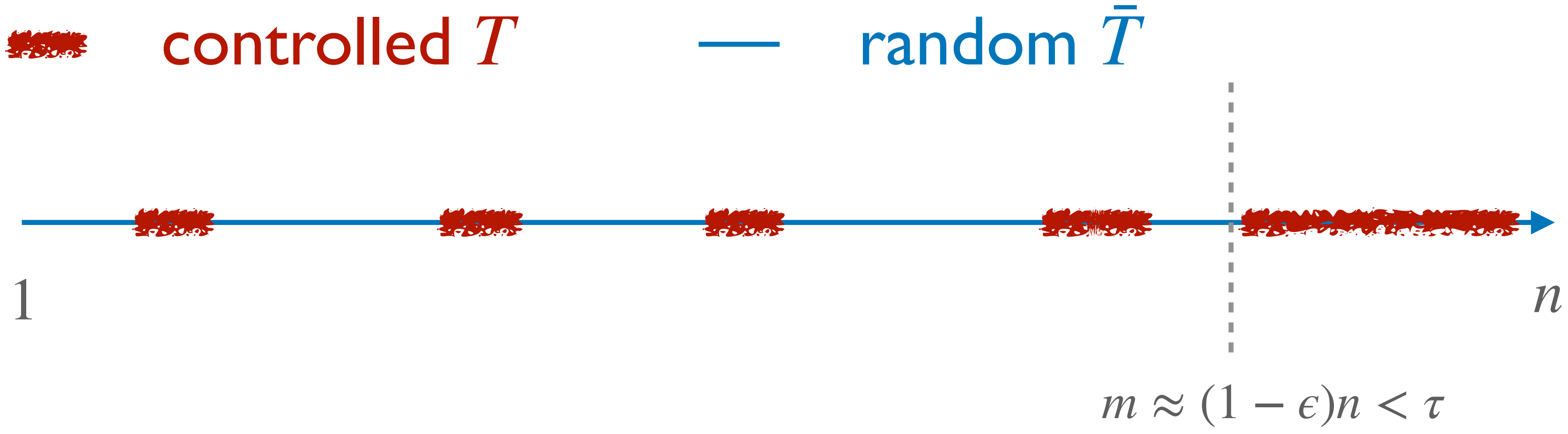
Strategy: to bound the KL-divergence between $Y(n)$ and $X(n)$ given π, T, and the restriction.

$$|T| - \log |f^{-1}(1) \cap \{-1,1\}^T|$$

KL-divergence

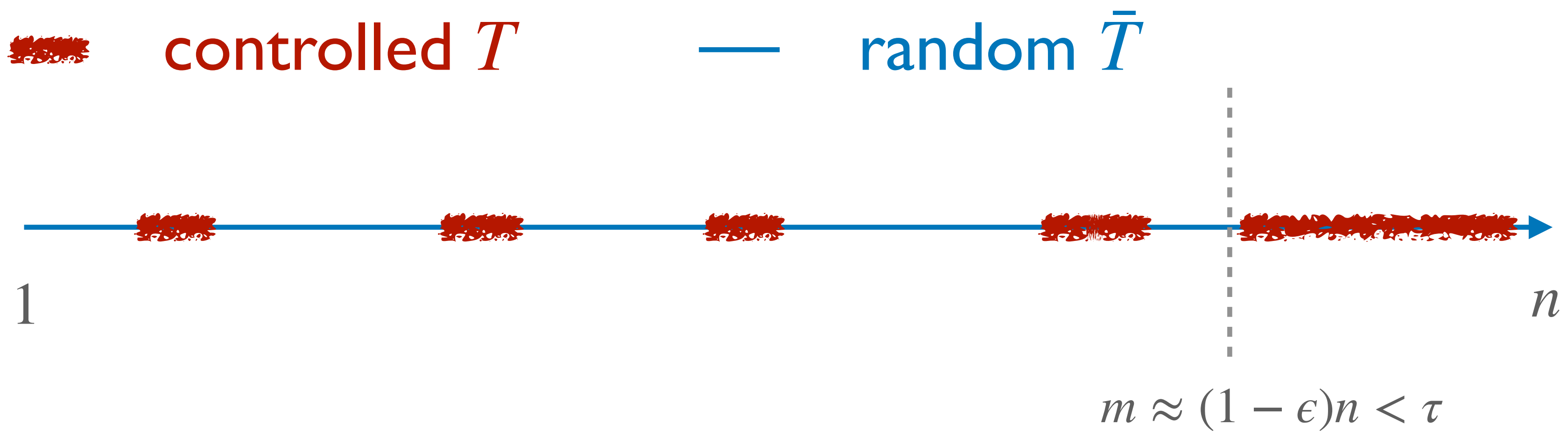


KL-divergence



$\mathcal{G}_m = (\pi, T, x(m)|_{\bar{T}})$: all the information we know about coordinates come before time m on $\bar{T}$.

KL-divergence



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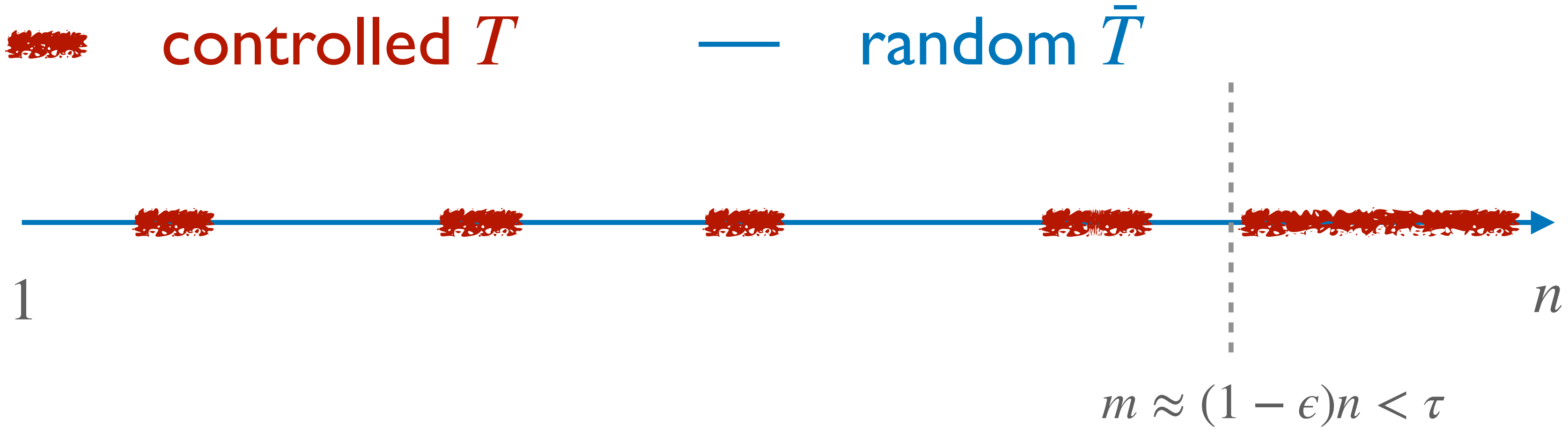
$$\mathbf{E}_{\mathscr{G}_m}[\mathbf{KL}(Y(n) \mid \mathscr{G}_m \| X(n) \mid \mathscr{G}_m)]$$

KL-divergence



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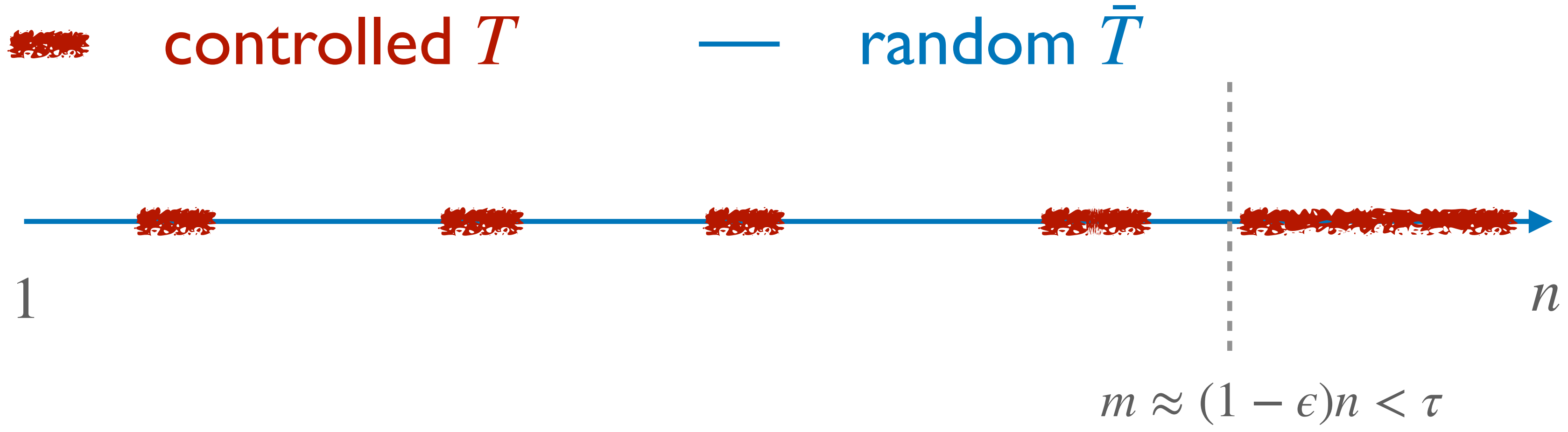
$$+\mathbf{E}[\mathrm{KL}((Y(n) \mid \mathcal{G}_m, Y(m)) \| (X(n) \mid \mathcal{G}_m, Y(m)))]$$

KL-divergence



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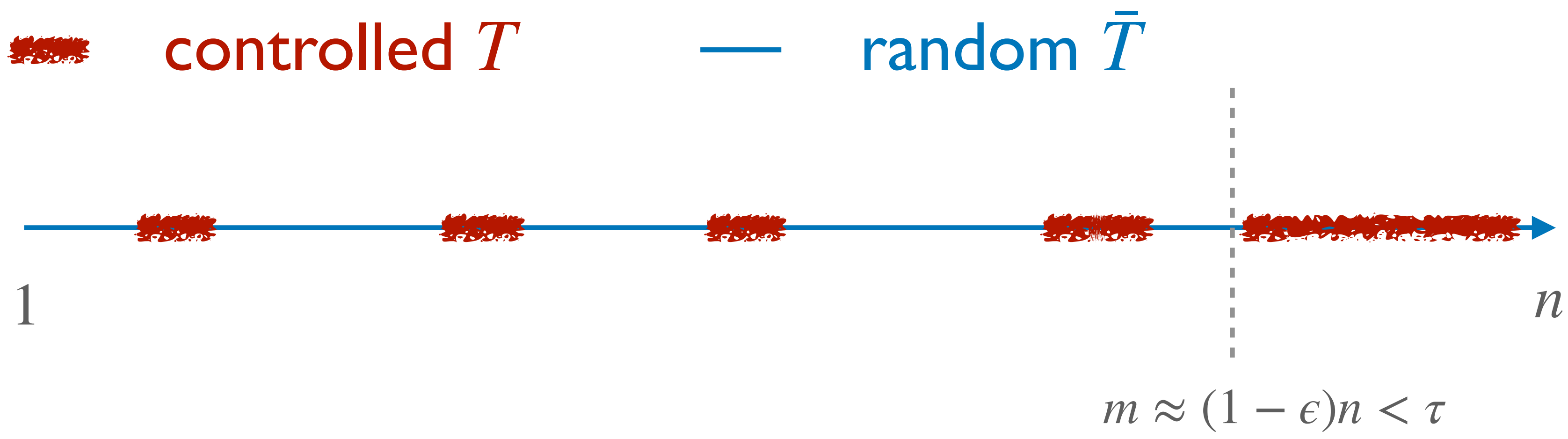
$$+\mathbf{E}[\boxed{\mathrm{KL}((Y(n) \mid \mathcal{G}_m, Y(m)) \,\|\, (X(n) \mid \mathcal{G}_m, Y(m)))}] \quad \le \log\frac{1}{\delta}$$

KL-divergence



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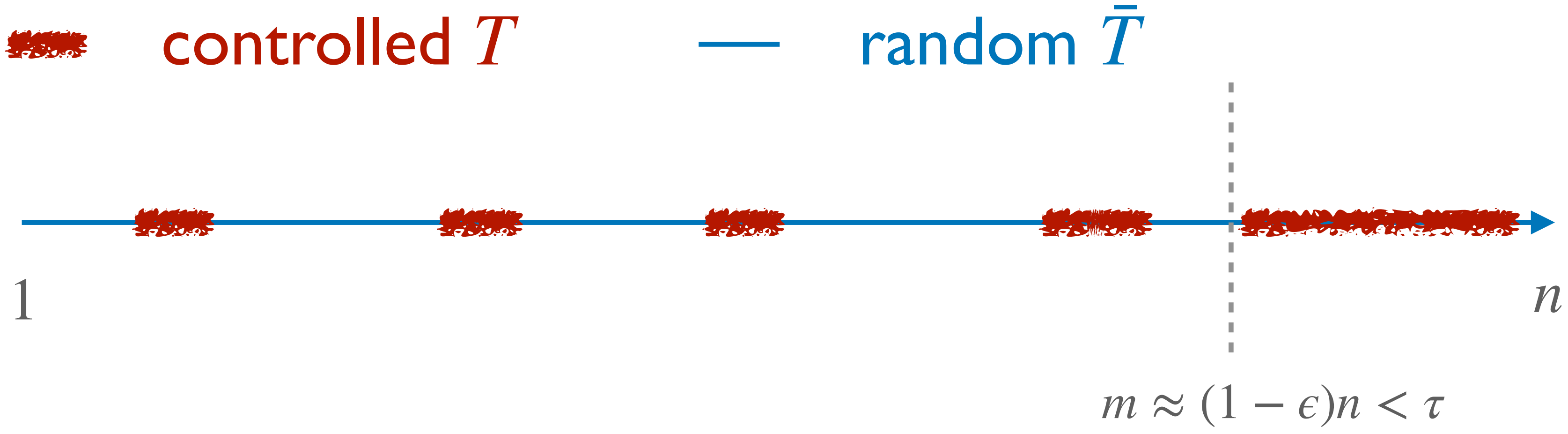
KL-divergence



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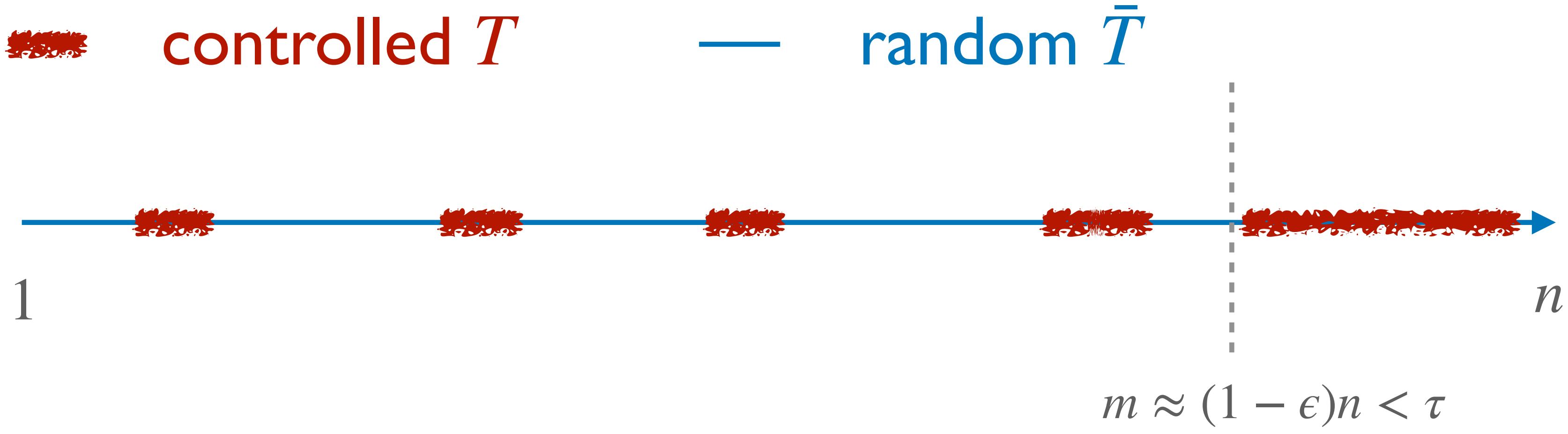
$$\leq \sum_{t=1}^{m} \frac{\epsilon}{n-t+1} \sum_{i:Y_i(t-1)=0} \left(\frac{\partial_i f(Y(t-1))}{2\epsilon f(Y(t-1))}\right)^2 + \log\frac{1}{\delta}$$

KL-divergence



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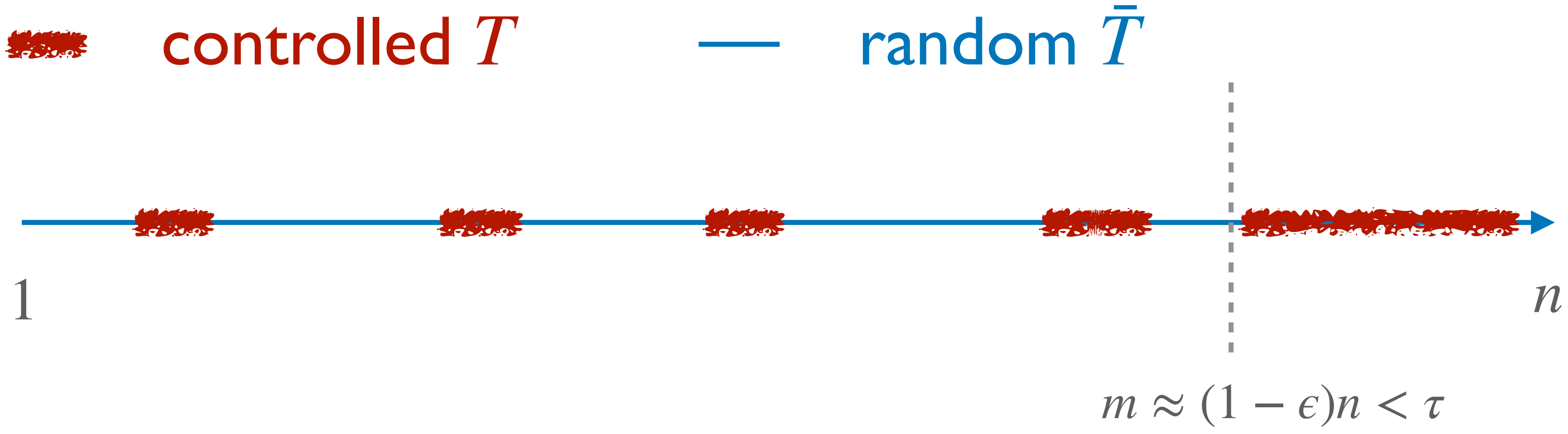
KL-divergence



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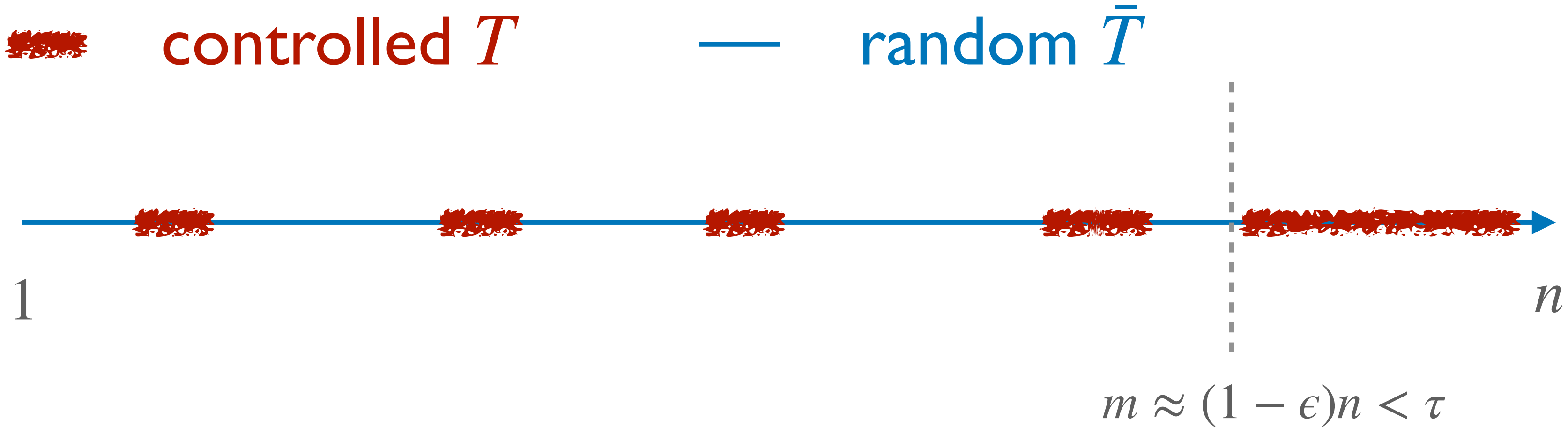
Theorem ([Talagrand 96])

For any $g : \{-1,1\}^n \to [0,1]$, we have $\sum \partial_i g(0)^2 \leq C g(0)^2 \log\dfrac{e}{g(0)}$.

KL-divergence



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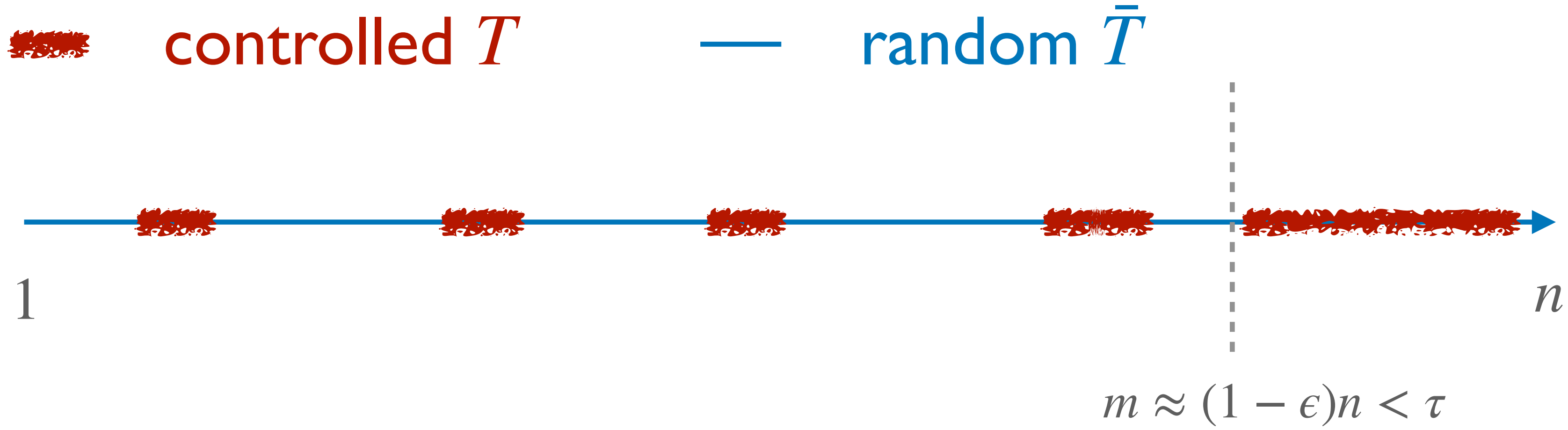
$$\leq \sum_{t=1}^{m} \frac{C}{4\epsilon(n-t+1)} \log \frac{e}{f(Y(t-1))} + \log \frac{1}{\delta}$$

KL-divergence



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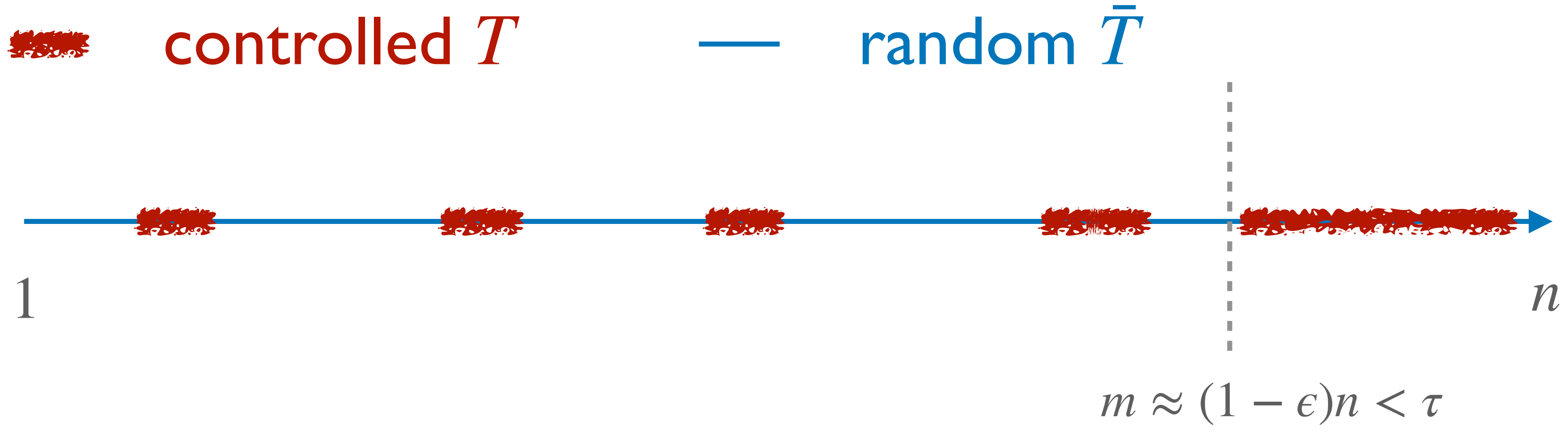

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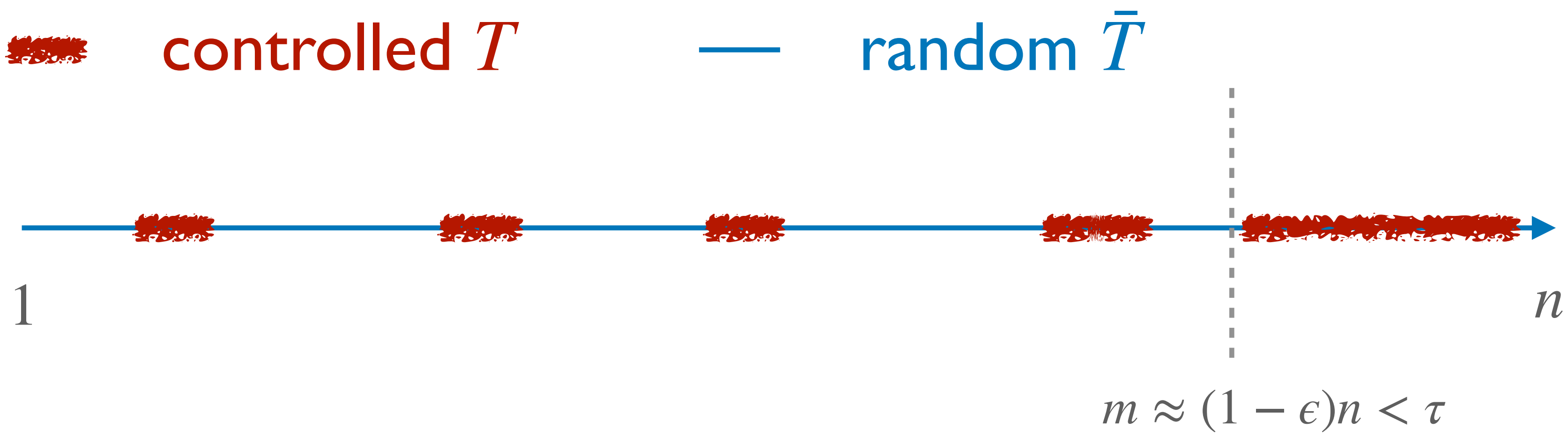
$$\leq \sum_{t=1}^{m} \frac{C}{4\epsilon(n-t+1)} \log \frac{e}{f(Y(t-1))} + \log \frac{1}{\delta}$$

$$\leq O\left(\frac{1}{\epsilon} \log \frac{1}{\delta}\right) \cdot \sum_{t=1}^{m} \frac{1}{n-t+1}$$

$$= O\left(\frac{1}{\epsilon} \log \frac{1}{\delta} \log \frac{n-m}{n}\right).$$

KL-divergence



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Thank you!